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New Examples of Singly-Periodic Minimal Surfaces and Their Qualitative Behavior

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1. Singly-Periodic Minimal Surfaces. I would like to describe some recent research, concerning properly embedded minimal surfaces with periodicity that I have been doing with Michael Callahan and Bill Meeks III. The work includes the construction of new examples and the characterization of the qualitative behavior of all examples in an important class. It is based, in turn, on an analysis of the geometric behavior of such surfaces at infinity. This talk is divided into two parts; the first discusses the examples and qualitative results; and the second gives a feeling for the analytical background.

To begin, it is easy to see that any connected, triply-periodic embedded surface in R^3 must have a single topological end. That is, the part of such a surface outside of any ball in R^3 is a connected set. For any doubly-periodic surface, Scherk's First Surface for example, the same is true, although it is by no means obvious. A proof of this fact will be described below.

For singly-periodic surfaces, things can get more complicated. Many singly-periodic minimal surfaces have a single topological end. (In order to avoid confusion, let me emphasize that we are counting ends of a surface M in R^3, not in R^3 mod T, where T is the cyclic group of symmetries. For example, Scherk's First Surface has *four* topological ends in R^3/T.) Scherk's Second Surface comes to mind. However, this is not always the case. Riemann discovered a 1-parameter family of connected, properly embedded minimal surfaces with an infinite number of flat ends.

The Riemann examples $\mathcal{R}$ possess a quite special set of properties:

(a) They have an infinite number of flat annular ends; (By an *annular end* of a surface we mean an end that has a representative homeomorphic to a punctured disk. Often such a representative is referred to as the end itself. For a properly embedded minimal surface, we may choose this representative to be the image of the punctured unit disk in the complex plane. An annular end is *flat* provided it is asymptotic to a plane in R^3.)

(b) They are invariant under a nontrivial translation T;

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(c) The surfaces $\mathcal{R}/T$ have genus 1 and 2 ends;

(d) The surfaces $\mathcal{R}/T$ have total curvature equal to -8π.

Inspired by these examples, we established the existence of an infinite family of properly embedded, periodic minimal surfaces, $\mathcal{M}_k$ each with an infinite number of flat annular ends [**1**].

These surfaces have the following properties:

(1) $\mathcal{M}_k$ has an infinite number of annular ends.

(2) $\mathcal{M}_k$ is invariant under the group of translations T generated by $T\colon \vec{x} \mapsto \vec{x} + (0,0,2)$.

(3) $\mathcal{M}_k/T$ has genus $2k+1$ and two ends.

(4) The symmetry group of $\mathcal{M}_k/T$ has order $8(k+1)$.

(5) Reflection in the plane $\{x_3 = n + 1/2\}$, $n \in Z$, is a symmetry of $\mathcal{M}_k$.

(6) $\mathcal{M}_k/T$ has finite total curvature $-4\pi(2k+2)$.

(7) All the ends of $\mathcal{M}_k$ are flat; they are asymptotic to the planes $x_3 = n$, $n \in Z$.

(8) $\mathcal{M}_k \cap \{x_3 = n\}$, $n \in Z$, consists of $k+1$ equally spaced straight lines meeting at $(0,0,n)$.

(9) $\mathcal{M}_k \cap \{x_3 = c\}$, $c \notin Z$, is a simple closed curve.

(10) The subgroup R of the symmetry group of $\mathcal{M}_k$ consisting of rotations about the x_3-axis has order $k+1$ and is generated by rotation by $2\pi/(k+1)$.

(11) $\mathcal{M}_k$ is symmetric under reflection through the $k+1$ vertical planes containing the x_3-axis and bisecting the lines of property 8.

(12) The full symmetry group of $\mathcal{M}_k$ is generated by T, R, one of the reflections in 5, rotation about one of the lines in 8, and reflection through one of the planes in 11.

We have proved that these surfaces are characterized by only a few of their properties. In particular, properties 1–5 above imply properties 6–12. Thus, if properties 1–5 are satisfied on a properly embedded minimal surface with a translational symmetry T and more than one end, the surface $\mathcal{M}_k/T$ must have finite total curvature [**2**].

There is strong computational evidence that the surfaces $\mathcal{M}_k$ are the unique in the sense that they are the only properly embedded minimal surface with a translational symmetry and more than one end on which properties 1–5 are satisfied. For each k, there is a one-parameter family of *immersed* minimal surfaces, which must contain any surface satisfying the conditions above. A surface in this family will be embedded and singly-periodic, and will satisfy these conditions, provided a period vanishes. This period is a smooth function of the parameter describing the family, and in [**2**], we show that this function possesses a zero and is asymptotic to a linear function. Computations indicate that it is also monotonic and hence its zero is unique. Nonetheless, there is the remote possibility that this is not the case and that there is more than one $\mathcal{M}_k$ satisfying the list of conditions.

We point out something quite interesting and (we believe!) nontrivial. Namely, property 9 is forced by properties 1-5. It is quite plausible that this should *not* be the case. Consider the following construction. Take a boundary curve consisting of the union of the positive $x-$ and $y-$ axes, which we shall label L_0, together with L_1, the vertical translation of L_0 by one unit. Suppose $L_0 \cup L_1$ bounds an embedded minimal annulus, which lies in the convex hull of this boundary and has all the symmetries of the boundary. Then this surface extends, by Schwarz reflection, to an example with Properties 1-8 and 10, but certainly not Property 9. To establish that Properties 1-5 forced Properties 6-12 we had to prove, among other things, that such a minimal annulus could not exist. We did this using a variant of the Alexandroff reflection principle, as developed for minimal surfaces by Rick Schoen [**9**] (see Lemma 3 of [**2**].)

A natural question to ask at this point is whether or not the examples $\mathcal{M}_k$ may be given a helical twist. To make this precise, we will define a *screw motion* of R^3 to be the composition of a nonzero translation, T, and a rotation,R, about the axis defined T. With this definition, we may consider a pure translation to be a (degenerate) screw motion. We were able to show [**2**] that:

For every positive integer k and angle θ, $0 < |\theta| < \frac{\pi}{k+1}$, there exists a properly embedded minimal surface $\mathcal{M}_{k,\theta}$ that has the following properties.

(i) $\mathcal{M}_{k,\theta}$ has an infinite number of flat annular ends.

(ii) $\mathcal{M}_{k,\theta}$ is invariant under the group of nondegenerate screw motions, $\mathcal{S}$, generated by $(r, \phi, z) \to (r, \phi + 2\theta, z + 2)$.

(iii) Properties 3 and 6–10 above hold for $\mathcal{M}_{k,\theta}$, with T replaced by $\mathcal{S}$.

(iv) The symmetry group of $\mathcal{M}_{k,\theta}/\mathcal{S}$, has order $4(k+1)$.

Remark 1. *When $\theta = 0$, the construction of $M_{k,\theta}$ yields a surface satisfying all the conditions characterizing the examples $\mathcal{M}_k$. If we knew that the surfaces $\mathcal{M}_k$ and $\mathcal{M}_{k,\theta}$ were unique, then we would have*

Conjecture 1. $\{\mathcal{M}_{k,\theta} \mid \theta \in (-\frac{\pi}{k+1}, \frac{\pi}{k+1})\}$ *is a smooth, one-parameter family of embedded minimal surfaces.*

The examples $\mathcal{M}_{k,\theta}$ and the Riemann examples $\mathcal{R}$ comprise all the known properly embedded minimal surfaces with more than one end and infinite symmetry group. They share many geometric properties. We have been able to show that this is not accidental.

Theorem 1. **(First Structure Theorem)** *Suppose M is a properly embedded minimal surface, with more than one end, whose symmetry group is infinite. Then either M is the catenoid or:*

(a) *M has an infinite number of ends;*

(b) *M is invariant under a screw motion $\mathcal{S}$;*

(c) *all annular ends of M are flat ends;* (d) *the total curvature of $\widetilde{M} = M/\mathcal{S}$ is*

$$C(\widetilde{M}) = 2\pi(\chi(\widetilde{M}) - r(\widetilde{M})),$$

where $r(\widetilde{M})$ is the number of ends of $\widetilde{M}$.

This theorem proved in [**2**] is a consequence of a more general structure theorem:

Theorem 2. **(Second Structure Theorem)** *Under the same hypotheses (M is a properly embedded minimal surface, with more than one end, whose symmetry group is infinite) then either M is the catenoid or:*

(a) *There exists a plane parallel to the limit tangent plane, whose intersection with M consists of a finite number of simple closed curves;*

(b) *The symmetry group of M contains an infinite cyclic, normal subgroup, generated by a screw motion $S = T + R$, where T is a translation and the index of this subgroup is finite;*

(c) *If $S = T + R$ and $R \neq 0$, the limit tangent plane is orthogonal to the axis of S.*

In the next section we shall give some indication of the analysis behind the proof of this theorem.

Note, as is evident from the picture of Riemann's example, that the axis of translation is *not* orthogonal to the limit tangent planes. This is not a contradiction to the result above because the Riemann examples are invariant under a degenerate screw-motion; that is, one without a rotational part.

For purposes of comparison, I should describe some very beautiful examples, recently-discovered by H. Karcher and J. Pitts. They are asymptotic to two coaxial helicoids and resemble two such helicoids, one twisted by a fixed angle from the other, with the intersection set replaced by a tower of tunnels resembling the core of Scherk's Second Surface. These surfaces have screw-motion symmetries and would appear to violate the structure theorems above. However, they have *one* topological end.

Using the Second Structure Theorem, we can establish the following topological result: *A doubly-periodic, properly embedded minimal surface in $\mathbf{R}^3$ has one end and infinite genus* (Corollary 2, Section 3 of [**2**]). The proof is relatively straightforward:

Since the symmetry group of a doubly-periodic minimal surface does not have a cyclic subgroup of finite index, the Second Structure Theorem above implies that a connected doubly-periodic minimal surface has one end. If a doubly-periodic minimal surface with translation group L has genus zero and one end, then, topologically, it is a plane and its quotient in $\mathbf{R}^3/L$ is a torus. Since a closed minimal torus in a flat 3-manifold is totally geodesic (by Gauss-Bonnet), we see that a nonplanar, doubly-periodic minimal surface must have one end and infinite genus.

2. Limit Tangent Planes and Canonical Annular Ends. In this section, I will describe a bit of the underlying theory on which the proofs of the Structure Theorems are based. Recall that the Second Structure Theorem in the previous section uses the existence of a unique limit tangent plane to the surfaces in question. While this is plausible, it is not obvious that such a plane exists in the generality that is required: we make no topological assumptions about the minimal surfaces in question, except that they have more than one end.

Some definitions are necessary. A subsurface $E \subset M$ is an *end-representative* if it is a closed noncompact subset of M with compact boundary. We say that a surface M has more than one end if it contains two or more pairwise-disjoint end representatives. Finally, a surface Σ in a region $R \subset \mathbf{R}^3$ is said to be a surface of least area in R if every compact subdomain $D \subset \Sigma$ has least area among surfaces in R with boundary ∂D.

An end of a complete minimal surface of finite total curvature has a well-defined limit normal vector and limit tangent plane [**2**]. Recall that a properly embedded minimal surface M of finite total curvature has the property that all of its limit tangent planes coincide and thus it makes sense to speak of *the* limit tangent plane to M. We have generalized the notion of limit tangent plane to apply to any properly-embedded minimal surface with more than one end, and have shown that in this case, there is a unique limit tangent plane.

LEMMA 1. *Suppose $M \subset \mathbf{R}^3$ is a properly embedded minimal surface. Let E be an end-representative of M with smooth boundary and having the property that $M - (E)$ is noncompact. Then $S = \partial E$ bounds a smooth, properly-embedded, noncompact, least-area surface Σ of finite total curvature given in Lemma 1 in the closure of one of the components of $\mathbf{R}^3 - M$.*

The limit tangent plane of Σ will be called a *limit tangent plane* to M.

THEOREM 3. *If M is a properly embedded minimal surface in $\mathbf{R}^3$ with more than one end, it has a unique limit tangent plane.*

We will refer to the limit plane defined by Theorem 3 as *the* limit tangent plane to M.

This tangent plane is the one referred to by the Second Structure Theorem. Its existence allows us to begin the analysis of that theorem. Beginning with a properly embedded minimal surface with more than one end and infinite symmetry group,we show first that one of the ends of a least-area surface Σ of finite total curvature in the closure of one of the components of $\mathbf{R}^3 - M$ must have a flat end. The tangent plane P to this flat end is, of course, parallel to the unique limit tangent plane. By using a version of the maximum principle at infinity [**7**] [**8**] we can perturb this tangent plane to the flat end to insure that it intersects M transversally in a compact set. This shows that there exists a plane parallel to the limit tangent plane, whose intersection with M consists of a finite number of simple closed curves.

Without loss of generality, we may assume that P is horizontal and equal to the (x, y)-plane. Let S be the orientation-preserving symmetries of M whose linear part fixes the vertical vector $(0, 0, 1)$. Because of the existence of a *unique* limit tangent plane, every element of the symmetry group of M either preserves or reverses $(0, 0, 1)$. Observe that S can contain no horizontal translations because $P \cap M$ must be compact. Therefore S consists of pure vertical translations, screw motions with vertical axis and rotations with vertical axis. It follows that the index of S in $Sym(M)$ is either 1, 2 or 4. We now concentrate on the orbit of P under S. Since $P \cap M$ is compact, the end of P is a positive distance from M, which means that the orbit of P under S consists of a family of horizontal planes, indexed by their third coordinate, z. Since we are assuming that M is not the catenoid, S must act discretely. Thus S cannot consist entirely of rotations. Therefore, there exists a minimum positive height, achieved by an element S of $\mathcal{S}$. This is the required screw-motion generator of a cyclic, infinite normal subgroup of isometries. It is evident from the previous paragraph that the cyclic subgroup generated by $\mathcal{S}$ has finite index, asserted in the second statement of the Second Structure Theorem.

Let L be the slab in R^3 between the plane P and the plane $\mathcal{S}(P)$. It is clear that $L \cap M$ is not compact; otherwise, M would have one end, a contradiction. It follows that the $\mathcal{S} - orbit$ of $L \cap M$ contains an infinite number of distinct ends, as claimed in Statement 1 of the First Structure Theorem.

In order to prove the aforementioned fact that if M has one annular end it has infinitely many, we need to use a technical result about the existence of canonical end representatives:

PROPOSITION 1. *(***Canonical Representation of Annular Ends***) Each annular end of a complete, nonsimply-connected, oriented minimal surface has a unique representative whose boundary is a closed geodesic. These representatives have pairwise-disjoint interiors. If the boundaries of two such annular ends touch, they coincide and M is an annulus.*

The cyclic subgroup generated by $\mathcal{S}$ acts freely on the collection of canonical annular ends of M. Suppose there is one canonical annular end. If there were not an infinite number of distinct canonical annular ends, then at least one of the annular ends would have to be fixed. But then the (compact) boundary of this end would be invariant under an infinite cyclic group of symmetries. It is easy to see that in this circumstance, that this end would be invariant under an action of S^1 by rotation, forcing M to be the catenoid. This proves that if M has one annular end, it has infinitely many, unless M is the catenoid. We can now evoke the Annular End Theorem of [**4**] which states that any properly embedded minimal surface M can have, at most, two annular ends of infinite total curvature. But the $\mathcal{S} - orbit$ of any annular end contains an infinite number of annular ends. Hence every annular end of M must have finite total curvature. An annular end of finite total curvature is asymptotic to either the plane or the catenoid. A straightforward argument using the existence of the plane P will show that the annular ends must be "flat", i.e. asymptotic to planes. This completes the proof of Statement 3 of the First Structure Theorem.

It is hoped that these few details give a feel of the technicalities involved in the proof of the Structure Theorem.

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