The Theory of Triply Periodic Minimal Surfaces

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Abstract

Working primarily within the conformal category, we develop complementary existence and rigidity theorems for periodic minimal surfaces in ^{n}. Applying this theory, we prove:

1. Every flat three-torus contains an infinite number of genus 3 embedded minimal surfaces;
2. Necessary and sufficient conformal conditions for a closed Riemann surface of genus g to conformally minimally immerse in a flat 3- or $(2g-1)$-torus;
3. The existence of distinct isometric minimal surfaces in flat tori;
4. Special results on the geometry of minimal surfaces of genus 3 and of classical examples of minimal surfaces in flat three-tori;
5. The determination of the group of symmetries of certain minimal surfaces in ^{3}.

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Contents

1 Introduction

In 1760, Lagrange extended to two dimensions the Euler theorems on simple integrals in the calculus of variations, and as an example, he proposed the following problem. *Given a closed curve C and a connected surface S bounded by the curve; to determine S so that the enclosed area shall be a minimum.* Sixteen years later, Meusier found that such a surface S is minimal only if the mean curvature of S is zero. Explicit physical solutions to this problem were found by Plateau, who showed that soap films on wires were sometimes least area among all surfaces having the wire as boundary. These studies marked the birth of the theory of minimal submanifolds in ^{3}.

Many of the more intricate and beautiful examples of minimal surfaces in ^{3} have the additional property of being preserved by a group of translations. The helicoid and Scherk's surfaces are well known examples of surfaces with this periodic behavior. During the middle of the nineteenth century, a thorough investigation of periodic minimal surfaces was carried out by Schwarz [55]. By extending Plateau's construction to polygonal curves and then extending them by repeated reflection across the line boundaries, he found an effective method for generating surfaces invariant under a lattice L of translations. The resulting quotient surfaces in $^3/L$ gave the first examples of compact minimal surfaces in flat three-tori.

We will call a closed Riemann surface M *periodic* if it conformally minimally immerses in a flat three-torus ^{3}. By lifting to the universal cover of ^{3}, these periodic surfaces become the proper triply-periodic minimal surfaces in ^{3}.

The compactness of a minimal surface M in ^{3} gives rise to restrictions on the conformal type of M. Frequently, these conformal restrictions give nontrivial geometric information about the lifted minimal surface in ^{3}. For these reasons, we consider the following fundamental questions:

1. *Which compact Riemann surfaces are periodic?*

2. *How does the conformal structure of a periodic surface influence its geometry?*

Our first result on these questions is that a surface of genus two is never periodic. Since every surface of genus two is hyperelliptic, this result follows from our more general result that a hyperelliptic Riemann surface of even genus is never periodic. We also find another family of nonperiodic Riemann surfaces: *Any nonsingular curve of degree four in* ^{2} *fails to be periodic.* Thus, the classical Fermat curve of degree four in ^{2}, given in homogeneous coordinates by $x^4 + y^4 + z^4 = 0$, provides a good example of a nonperiodic surface. The techniques of proof used here consist of a study of the relationship of the Gauss map of a periodic minimal surface to the canonical curve of the associated Riemann surface (see Definition 2.2).

Besides finding conformal obstructions to periodicity, we also begin the development of a general existence theory. We define a real five-dimensional family V of periodic hyperelliptic surfaces of genus three (see Theorems 7.1 and 10.1). The surfaces in this family are the ones that can be represented as two-sheeted covers of the sphere branched over four pairs of antipodal points. Since the minimal surfaces in the family V are embeddings, the following result takes on particular significance. (See Theorem 3.2 for the proof.)

Theorem 1.1 *If* $f\colon M_3 \to {}^3$ *is a minimal surface of genus three, then:*

1. M_3 *is hyperelliptic;*
2. *There exist eight zeros of Gauss curvature;*
3. *The hyperelliptic automorphism is an isometry and is induced by an inversion symmetry in* ^{3} *through any zero of Gauss curvature of* M_3*;*
4. *If* $f\colon M_3 \to {}^3$ *is an embedding, then, after a translation, the zeros of Gauss curvature of* M_3 *are the order two points of* ^{3}*, where we consider* ^{3} *to be an abelian group.*

We also prove other existence theorems for minimal surfaces in flat tori. Namely:

1. Every ^{3} contains, for every odd integer k, an infinite number of non-congruent embedded minimal surfaces of genus k. (Corollary 10.1).

2. If the genus g of M_g is greater than 3, then M_g will conformally minimally immerse fully in a flat $^{2g-1}$. A necessary and sufficient condition for M_3 to conformally minimally immerse fully in a flat ^{5} is for it to be hyperelliptic. (Theorem 9.1).

3. A closed surface M, orientable or not, will topologically minimally immerse in some flat ^{3} if and only if $M = {}^2$ or $\chi(M) \leq -2$ where $\chi(M)$ is the Euler characteristic of M. (Example 8.1 and Corollary 10.1).

4. Every flat ^{4} contains an embedded orientable minimal surface of every possible genus except genus 0. (Theorem 8.4).

Some of our deepest results are obtained by combining our rigidity and existence theories. For example, we show that the Schwarz diamond (or $\mathcal{D}$ surface) can be continuously deformed to its conjugate surface, the Schwarz primitive (or $\mathcal{P}$ surface), through minimal surfaces of genus 3 in flat three-tori.

The paper is organized as follows. In Section 2 we develop the background material necessary for discussing the geometry of triply-periodic minimal surfaces. In Section 3 we develop some of the basic results on the geometry of periodic surfaces with an emphasis on the geometry of the hyperelliptic examples. In Section 4 we discuss a fundamental restriction that the associated canonical curve of a periodic surface must satisfy. In Section 5 we develop the basic rigidity theory of periodic minimal surfaces that allows us to deal with symmetry questions and other theoretical discussions in latter sections. In particular, these rigidity results are the building block for the existence theory developed in the following two sections. In Section 6 we investigate the theory and examples of periodic minimal surfaces that have more than one isometric minimal immersion into flat three-tori. In Section 7 we construct the family V of periodic minimal surfaces of genus 3 that we discussed earlier. A variety of results of independent interest are developed in Section 8. In Section 9 we explore the theory of periodic minimal surfaces in ^{k}. In Section 10 we prove that: *Every flat 3-torus contains an infinite number of embedded genus 3 minimal surfaces.* J. Hass, J. Pitts and H. Rubenstein [17] have also proven this result.

The primary purpose of Section 11 is to bring the subject matter up to date as well as to give credits to others for their contributions. In particular, credit is given to T. Nagano and B. Symth for their research on triply-periodic minimal surfaces that, in minor ways, overlaps with some of our results. Some discussion of the recent appearances of periodic minimal surfaces in modeling liquid crystals as well as the surface interfaces in block copolymers is also given in this section. This section also includes a theorem, Theorem 11.1, on the rigidity of properly embedded minimal surfaces M with infinite symmetry group and whose quotient by their symmetry group have finite topology. This theorem generalizes our earlier rigidity theorem for triply-periodic minimal surfaces (Theorem 5.3).

Acknowledgements. Except for the material in Section 10, which was worked out around 1980, and Theorem 11.3, which was proved in 1988, this paper represents the research contained in my 1975 Ph.D. thesis at the University of California at Berkeley [26], [27]. I would like to express my sincere appreciation to my teachers Blaine Lawson and Mark Green for sharing their ideas and for taking a special interest in my thesis work. I offer a special thanks to Lawson who directed my thesis and whose constant encouragement and focus gave life to my own ideas during the writing of my thesis.

2 Background material

The main purpose of this section is to develop the notion of a minimal submanifold in Euclidean space, and to acquaint the reader with our notation for future discussions.

To begin with, let $(M^n, <>)$, or more conveniently, M^n, denote a C^∞ n-dimensional Riemannian manifold. We will assume that all mappings between manifolds are C^∞. If $f\colon M \to \overline{M}$ is a proper isometric immersion, then we will frequently identify M with its image, and refer to $M \subset \overline{M}$ as an immersed submanifold.

For the convenience of the reader, we give a summary of our notation for other standard objects in differential geometry:

1. T_pM = the tangent space at $p \in M$; TM = the tangent bundle of M;
2. Δ = the Laplacian; ∇ = the unique Riemannian connection on M;
3. $H^i(M,\ )$ = the i^{th} De Rham cohomology group of M;
4. $\mathcal{H}^i(M)$ = the harmonic i-forms on M;
5. $H_1(M,\ ) = H_1(M)$ = the first homology group of M;
6. $H^{1,0}(M)$ = the holomorphic one-forms on a complex manifold M;
7. M_g = a compact Riemann surface of genus g;
8. $\ ^k$ = a flat k-dimensional torus;
9. $f\colon M_g \to\ ^k$ = a conformal minimal immersion, unless otherwise stated.

The Riemannian connection $\overline{\nabla}$ on $\overline{M}$ induces the important second fundamental form on submanifolds. It is defined as follows: Assume that $M \subset \overline{M}$ is an embedded submanifold, and $\overline{X}$, $\overline{Y}$ are extensions to $\overline{M}$ of vector fields

X, Y on M. Then $B(X,Y)(p) = (\overline{\nabla}_{\overline{X}}\overline{Y})^N(p)$ gives rise to a well defined normal vector field on M. One can easily check that B is a symmetric tensor on M with values in the normal spaces, and hence, B has a well defined trace.

Definition 2.1 *The* mean curvature vector field *of* $M^n \subset \overline{M}$ *is* $K(p) = trace(B_p) = \sum_{i=1}^n B(e_i, e_i)$, *where* $\{e_1, \ldots, e_n\}$ *is an orthonormal basis for* T_pM. *If* $K \equiv 0$, *then we call* $M \subset \overline{M}$ *a* minimal submanifold.

There is a simple explanation as to why the word *minimal* is attached to a submanifold with zero mean curvature. Minimal submanifolds are characterized as surfaces that in a small neighborhood of any point on the surface minimize surface area with respect to local boundaries. Thus, they are the higher dimensional generalization of geodesics, which minimize arc length between any two close points. The proof of this geometric characterization of minimal submanifolds is by way of the first and second variation of area formulas. See [25] for details.

Well known examples of complete minimal surfaces in ^{3} include the catenoid and the helicoid. Physical realizations of compact minimal surfaces with boundary in ^{3} can be constructed by taking a bent circular piece of wire and then dipping it in a soapy water solution; surface tension forces an ideal soap film that forms on this wire to have zero mean curvature. In higher dimensions, any complex submanifold of ^{n}, ^{n}, or a complex torus is minimal with respect to the usual metrics on these spaces.

For the rest of this section, we will restrict our attention to minimal submanifold in Euclidean n-space or in a flat torus ^{n}. Frequently, we will assume that every ^{n} is canonically represented as $^n/L$, where L is a lattice in ^{n}. An easy calculation shows that if $f = (f_1, \ldots, f_n)\colon M \to {}^n$ is an isometric immersion, then the mean curvature vector field on M can be computed by the formula $K(p) = (\Delta f_1(p), \ldots, \Delta f_n(p))$. Thus we have:

Theorem 2.1 *A submanifold* $f\colon M \to {}^n$ *is minimal if and only if the coordinate functions are harmonic.*

In particular, the above theorem implies, after a possible translation, that f can be represented by $f(p) = \int_{p_0}^{p} (h_1, \ldots, h_n)$ where:

1. The $h_i = df_i$ are harmonic[1] one-forms with no periods on M;

2. The integration is carried out over any path joining the base point $p_0 \in M$ to p.

The above approach of integrating forms is useful in obtaining minimal immersions of compact manifolds into flat tori. In general, a compact M^k may have harmonic one-forms that can be integrated to obtain minimal immersions $f\colon M^k \to {}^n$, but it never has non-constant harmonic functions. More precisely, we have the following:

Theorem 2.2 *A submanifold $f\colon M \to {}^n = {}^n/L$ is minimal if and only if, after a translation, f can be represented as $f(p) = \int_{p_0}^{p}(h_1, \ldots, h_n)$, where $h_1, \ldots, h_n$ are harmonic one-forms and the periods $P = \{\int_\gamma (h_1, \ldots, h_n) \mid \gamma \in H_1(M, \;)\}$ are contained in the lattice L.*

One of the beautiful classical theorems on Riemann surfaces states that every Riemann surface of positive genus holomorphically embeds in a complex torus called its Jacobian. In particular, every closed Riemann surface of positive genus conformally embeds as a complex minimal submanifold in some flat complex torus.

Theorem 2.3 (Abel-Jacobi Embedding Theorem) *Let M be a closed Riemann surface of positive genus g and let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $H^{1,0}(M)$. Then $f(z) = \int_{z_0}^{z}(\omega_1 \ldots, \omega_g)\colon M \to {}^n/L = J(M)$ (= Jacobian of M) is a holomorphic embedding, where L is the lattice of period vectors $\{\int_\gamma(\omega_1, \ldots, \omega_g) \mid \gamma \in H_1(M, \;)\}$.*

[1] A real k-form ω on an n-dimensional Riemannian manifold is *harmonic* if it is both closed and co-closed; i.e., $d\omega = 0$ and $d * \omega = 0$, where $*$ is the Hodge star operator.

It is important to consider the conformal structure of a Riemann surface when searching for minimal surfaces in tori. In fact, on two-dimensional Riemannian manifolds, $(M, <>)$, there is usually much to be gained by using appropriate coordinate charts. When M is orientable, it is possible to pick coordinates so that the metric $ds^2 = F(dx^2 + dy^2)$, and under change of coordinates, angles are preserved. Such coordinates are called *isothermal coordinates* and give M a conformal or complex structure.

From now on, we will consider all two-dimensional orientable surfaces as Riemann surfaces and our mappings $f\colon M^2 \to \overline{M}$ as conformal immersions.

In local coordinates, $z = x + iy$ on M^2 we define $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. In these coordinates, the induced metric has the form: $ds^2 = 2F|dz|^2$, the Laplacian can be expressed by $\Delta = \frac{2}{F}\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$, and the Gauss curvature can be calculated by $K = \frac{-1}{F}\Delta \log(F)$.

For a more complete discussion of the following, we refer the reader to [25].

Suppose now that $f\colon M \to \ ^n$ is a minimal surface, and z is a local coordinate system on M. Then, by Theorem 2.1, $\Delta f = \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} f = (\Delta f_1, \ldots, \Delta f_n) = \vec{0}$, implying that $\phi = \frac{\partial}{\partial z} f$ is a local holomorphic $\ ^n$-valued function on M. Since $4(\phi \cdot \phi) = |f_x|^2 - |f_y|^2 - 2i\langle f_x, f_y\rangle$, and since the induced metric is given by $g_{xx} = |f_x|^2 = 2 \cdot F$, $g_{yy} = |f_y|^2 = 2 \cdot F$, $g_{xy} = \langle f_x, f_y\rangle = 0$, we arrive at the following equations:

$$\phi^2 = \sum_{k=1}^{n} \phi_k^2 = 0. \tag{2.1}$$

$$|\phi|^2 = \sum_{k=1}^{n} |\phi|_k^2 = F. \tag{2.2}$$

We get a well defined map $G\colon M \to \ P^{n-1}$ defined by $G(p) =$ the complex line through the point $(\phi_1(p), \phi_2(p), \ldots, \phi_n(p)) \in \ ^n$. Since $\sum_{i=1}^{n} \phi_i^2 = 0$, the image of G is contained in the quadric $Q_n \subset \ ^{n-1}$ given by $z_1^2 + z_2^2 + \ldots + z_n^2 = 0$ in homogeneous coordinates.

Q_n can be identified with the Grassman manifold $G^\circ(2,n)$ of oriented two-planes in $\mathbb{R}^n$ as follows: Let $\mathbb{R}^2 \subset \mathbb{R}^n$ be a two-dimensional subspace oriented by the pair of vectors (v_1, v_2), with $|v_1| = |v_2|$ and $v_1 \cdot v_2 = 0$, where $\cdot$ is the dot product. One can easily show that the mapping $I\colon G^\circ(2,n) \to Q_n$, defined by $I((v_1, v_2))=$ the complex line through $v_1 + iv_2 \in Q_n \subset \mathbb{C}P^{n-1}$, is an isometry in the standard metrics on these spaces. Furthermore, after this identification, the usual Gauss map $G\colon M \to G^\circ(2,n)$ given by translation of the tangent space to the origin, agrees with the definition of $G\colon M \to Q_n$ given in the previous paragraph.

One consequence of these identifications is that the Gauss map for a minimal surface in $\mathbb{R}^n$ becomes a holomorphic transformation. In $\mathbb{R}^3$, it is customary to define the Gauss map $G\colon M \to S^2$ by translation of the unit normal vector to the origin. We will always take S^2 oriented with respect to the *inward* normal vector field, the orientation induced from $\mathbb{C}$ under stereographic projection.

Proposition 2.1 *$f\colon M \to \mathbb{R}^3$ is a minimal surface if and only if the Gauss map $G\colon M \to S^2$ is holomorphic.*

Proof. Identify T_pM and $T_{G(p)}S^2$ by parallel translation. Then the *derivative matrix* G_* for G is a symmetric matrix. It follows that $f\colon M \to \mathbb{R}^3$ is minimal if and only if G is conformal if and only if G is holomorphic. □

Corollary 2.1 *If $f\colon M \to \mathbb{R}^3$ is a minimal surface, then the points of zero Gauss curvature are precisely the branch points of $G\colon M \to S^2$.*

Proof. Since $G\colon M \to S^2$ is holomorphic, the Gauss curvature at p given by $\det(G_*(p))$. Hence, $\det(G_*(p)) = 0$ if and only if $G_*(p)$ is the zero matrix if and only if p is a branch point of G. □

Weierstrass and Enneper found a canonical representation for a minimal surface in $\mathbb{R}^3$ in terms of meromorphic data on the surface. A further study of formulas 2.1, 2.2, gives the Generalized Weierstrass Representation for all minimal surfaces in $\ ^n$ [25] or [48].

Theorem 2.4 (Generalized Weierstrass Representation)

1. *After a translation, any minimal surface* $f\colon M \to \ ^n$ *can be represented by*
$$f(z) = \mathrm{Re}\int_{z_0}^{z} (\omega_1, \ldots, \omega_n),$$
where the $\omega_i = g_i(z)\,dz \in H^{1,0}(M)$ *have no real periods,* $\sum_{i=1}^n g_i^2 = 0$*, and* $\sum_{i=1}^n |g_i|^2 = F$*. Conversely, if* $f(z) = \mathrm{Re}\int_{z_0}^{z}(\omega_1, \ldots, \omega_n)$ *with* ω_i *as above, then* $f\colon M \to \ ^n$ *is an isometric minimal immersion.*

2. *When* $n = 3$*,* $\omega_1 = (1 - g^2)\eta$*,* $\omega_2 = (1 + g^2)i\eta$*,* $\omega_3 = 2g\eta$ *for some holomorphic one-form* η *and where* g *is the Gauss map composed with stereographic projection onto* $\ \cup \{\infty\}$*.*

3. *If* $\ ^n$ *is replaced by* $\ ^n$*, then the above statements hold if the period vectors* $P = \{\int_\gamma(\omega_1, \ldots, \omega_n) \mid \gamma \in H_1(M, \ )\}$ *are contained in the lattice of* $\ ^n$*.*

When looking for minimal immersions of a simply connected surface M into $\ ^n$, we need not worry about periods of the $\omega_i' s$. If $f(z) = \mathrm{Re}\int_{z_0}^{z}(\omega_1, \ldots, \omega_n)\colon M \to \ ^3$ is a simply connected minimal surface, then the Generalized Weierstrass Representation implies that the *associate surface* at angle θ, $f_\theta(z) = \mathrm{Re}[e^{i\theta}\int_{z_0}^{z}(\omega_1, \ldots, \omega_n)]$, is an isometric minimal immersion of M into $\ ^n$. Whether M is simply connected or not, if f_θ is well-defined, then f_θ is also called an *associate surface* at angle θ of f. The *conjugate surface* of f is the associate surface $f_{\pi/2}$ and its coordinate functions are the harmonic conjugates of the coordinate functions of f. Frequently, this conjugate surface has special geometric relationships with the original surface. If $f\colon M \to \ ^3$ is a minimal surface and $f_{\pi/2}$ is well defined, then $f = \sqrt{2}(f + i\, f_{\pi/2})\colon M \to \ ^3$ is a holomorphic isometric immersion.

By applying the Calabi Rigidity Theorem for holomorphic curves in ^{n}, Calabi gave the following important characterization of isometric minimal surfaces in ^{3} [7].

Theorem 2.5 (Calabi Rigidity Theorem) *If $f, g : M \to {}^3$ are isometric minimal immersions, then after a rigid motion, $f = g_\theta$ for some angle θ.*

In our study of closed minimal surfaces in flat tori, the canonical curve of a surface plays a fundamental role.

Definition 2.2 *Let M_g be a compact Riemann surface of genus g, and let $\omega_1, \ldots, \omega_g \in H^{1,0}(M_g)$ be a basis for the holomorphic one-forms on M_g. Suppose $z = x + yi$ in local coordinates. If $\omega_i(z) = f_i(z)dz$, then the vector $\widetilde{c}(z) = (\omega_1(z), \omega_2(z), \ldots, \omega_g(z)) = (f_1(z), \ldots, f_g(z))$ is well defined pointwise up to a scalar multiple. Hence, we get an induced map $c\colon M_g \to {}^{g-1}$ called the* canonical map *of M_g. The curve $c(M_g) \subset {}^{g-1}$ is the* canonical curve *of M_g.*

Definition 2.3 *A Riemann surface is* hyperelliptic *if it can be represented as a two-sheeted branched cover $H\colon M \to S^2$ of the sphere.*

The next theorem appears in Section 10 of [16].

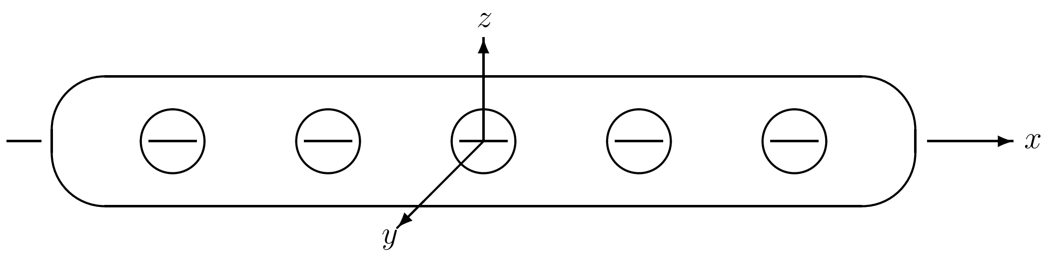
Theorem 2.6 *The canonical mapping $c\colon M_g \to {}^{g-1}$ is an embedding if and only if M_g is not hyperelliptic. If M_g is hyperelliptic, then c can be factored, up to a linear isomorphism, by:*

$$\begin{array}{ccc} & M_g & \\ H \swarrow & & \searrow c \\ {}^1 & \xrightarrow[V]{} & {}^{g-1} \end{array},$$

where $V(1,t) = (1, t, \ldots, t^{g-1})$ in homogeneous coordinates.

We now acquaint the reader with the elementary properties of hyperelliptic surfaces. Suppose that $H\colon M \to S^2$ is the two-sheeted covering of S^2 of a hyperelliptic surface M. The mapping $\Theta\colon M \to M$ which interchanges the points on the two sheets of H is called the *hyperelliptic automorphism*. The fixed points of Θ are the branch points of H and are called the hyperelliptic points.

The standard example of a hyperelliptic surface in $\mathbb{R}^3$ is a symmetric surface where Θ is rotation by $180°$ around the x-axis. See Figure 1 below for an example with genus $g = 5$.



Θ = rotation by $180°$

Figure 1:

It is clear that $M/\Theta = S^2$, and that the quotient map is a two-sheeted branched cover with $2(g+1)$ branch points. If $h \in H^1(M)$, then $h + \Theta^*(h)$ is invariant under the pull-back map Θ^*, and hence, descends to a harmonic one-form on S^2. Since the only harmonic one-form on S^2 is the zero form, we must have $\Theta^*(h) = -h$. This result is well known and we state it as a proposition.

Proposition 2.2 *Let M_g be a hyperelliptic Riemann surface, and let $h \in H^1(M)$ be a harmonic one-form. Then $\Theta^*(h) = -h$, where Θ is the hyperelliptic automorphism.*

The next proposition is also well known.

Proposition 2.3 *If $f\colon M_g \to \mathbb{C}^k$ is a full holomorphic mapping, then f factors through $J(M_g)$; i.e., there exist a lift $\widetilde{f}\colon M_g \to J(M_g)$ and a "$\mathbb{C}$-linear" map $A\colon J(M_g) \to \mathbb{C}^k$ such that*

$$\begin{tikzcd} & J(M_g) \arrow[d, "A"] \\ M_g \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & \mathbb{C}^k \end{tikzcd}.$$

In particular, $J(M_g)$ contains a codimension k complex subtorus contained in the kernel of A.

Proof. Suppose $f\colon M_g \to \mathbb{C}^k$ is given by $f = \int(\omega_1, \ldots, \omega_k)$, where $\omega_i \in H^{1,0}(M_g)$. After completing the $\{\omega_i\}$ to an ordered basis, define $\widetilde{f} = \int(\omega_1, \ldots, \omega_g)$, and define A to be projection on the first k factors. $\square$

3 The geometry of hyperelliptic minimal surfaces

Recall from the Introduction that a closed Riemann surface M is called periodic if it conformally minimally immerses in a flat three-torus. In this section, we begin our study of the geometry and conformal structure of these periodic surfaces. The geometric tools of this investigation are the Gauss map and the Gauss-Bonnet Theorem. Note that the Gauss map of an orientable minimal surface in a flat three-torus is well defined. Some of the results in this section were found independently by Nagano and Smyth [40] [41] [44]. See Section 11 for further credits.

Theorem 3.1 (Gauss-Bonnet Theorem) *If $f\colon M_g \to \mathbb{R}^3$ is a minimal surface of genus g, then the Gauss map[2] $G\colon M_g \to S^2$ represents M_g as a $(g-1)$-sheeted conformal branched cover of S^2.*

[2]The Gauss or normal map is well defined on an oriented surface in a flat three-torus by lifting the surface to $\mathbb{R}^3$.

Proof. By Proposition 2.1, the Gauss map $G\colon M \to S^2$ for a minimal surface in $\mathbb{R}^3$ is holomorphic. Similarly, when $f\colon M_g \to \mathbb{T}^3 = \mathbb{R}^3/L$ is a minimal surface, G is also holomorphic, and hence, exhibits M as a conformal branched cover of S^2. The usual Gauss-Bonnet Theorem states that the degree of G is $g-1$, where g is the genus of M_g. $\square$

Straightforward applications of the Gauss-Bonnet Theorem give rise to the following restrictions on the topological and conformal type of minimal surfaces in $\mathbb{T}^3$.

Corollary 3.1 *A surface of genus two is never periodic.*

Proof. If a surface of genus 2 was periodic, then, by the Gauss-Bonnet Theorem, the Gauss map would represent the surface as a one-sheeted branched cover of S^2. But any one-sheeted cover of S^2 is again S^2. $\square$

Corollary 3.2 *A periodic surface of genus three is hyperelliptic.*

Proof. By definition, a Riemann surface is hyperelliptic if it can be represented as a two-sheeted covering of S^2. In the case of genus three, the Gauss map provides this representation. $\square$

Corollary 3.3 *If $f\colon M_g \to \mathbb{T}^3$ is a minimal surface of genus g, then M_g has $4(g-1)$ zeros of Gauss curvature, counted with multiplicity.*

Proof. Since the Gauss map $G\colon M_g \to S^2$ has degree $g-1$, the Riemann-Hurwitz formula implies there are $4(g-1)$ branch points counted with multiplicity. By Corollary 2.1, we can identify the zeros of Gauss curvature with the branch points. The corollary now follows. $\square$

The rest of this section is devoted to the study of minimal immersions of hyperelliptic surfaces in flat tori.

Proposition 3.1 *If $f\colon M \to \mathbb{T}^n$ is a hyperelliptic minimal surface, then:*

1. *The hyperelliptic automorphism is an isometry that is induced by an inversion symmetry in $\mathbb{T}^n$ through any hyperelliptic point of $f(M)$;*

2. *After a translation, the hyperelliptic points are contained in the set of order two points in $\mathbb{T}^n$. (Note 0 trivially has order two in the abelian group $\mathbb{T}^n$.)*

Proof. After a translation, suppose that f is represented by $f(p) = \int_{p_0}^{p}(h_1, \dots, h_n)$, where $h_i \in H^1(M)$ and p_0 is a hyperelliptic point. Then, by change of variables, we have

$$f(\Theta(p)) = \int_{p_0}^{\Theta(p)}(h_1, \dots, h_n) = \int_{\Theta(p_0)}^{\Theta(p)}(h_1, \dots, h_n) = \int_{p_0}^{p}\Theta^*(h_1, \dots, h_n) = -f(p),$$

since Θ^* is multiplication by (-1) on the harmonic one-forms (Proposition 2.2). Hence, $(-1)\colon \mathbb{T}^3 \to \mathbb{T}^3$ leaves M invariant and fixes the hyperelliptic points.

If x is a hyperelliptic point, then the above equation shows that $f(x) = f(\Theta(x)) = -f(x)$. Hence, every hyperelliptic point has order two or is $0 \in \mathbb{T}^n$. $\square$

One can easily verify that the following theorem holds on the four Schwarz surfaces of genus three.

Theorem 3.2 *If $f\colon M_3 \to \mathbb{T}^3 = \mathbb{R}^3/L$ is a minimal surface of genus three, then:*

1. *M_3 is hyperelliptic;*

2. *The hyperelliptic automorphism is an isometry and is induced by inversion symmetry in $\mathbb{T}^3$ through any hyperelliptic point;*

3. *If f is an embedding, then after a translation, the set of zeros of Gauss curvature can be identified with $\frac{1}{2}L$ = order two points of $\mathbb{T}^3$. (Note $0 \in \frac{1}{2}L$ trivially has order two.)*

Proof. By Theorem 3.1, the Gauss map $G\colon M_3 \to S^2$ represents M_3 as a two-sheeted cover of S^2 with simple branch points. Hence, by the Riemann-Hurwitz formula, we get eight branch points or zeros of Gauss curvature. Since there are precisely eight order two points in ^{3}, the theorem follows from the previous proposition. $\square$

Theorem 3.3 *A hyperelliptic surface of even genus is never periodic.*

Proof. Let $f\colon M \to {}^3$ be a hyperelliptic minimal surface translated so as to have a hyperelliptic point at the origin. By the proof of Proposition 3.1, we have $f(\Theta(p)) = -f(p)$. We also have by Proposition 3.1 that $\Theta\colon M \to M$ is orientation-preserving, and is induced from an isometry in ^{3} that leaves invariant parallel lines. Hence, we can factor the Gauss map through the hyperelliptic quotient:

$$\begin{array}{ccccc} & & M & & \\ & \Pi \swarrow & & \searrow G & \\ M/\Theta = & S^2 & \underset{\widetilde{G}}{\longrightarrow} & S^2 & . \end{array}$$

Since the degree of Π is two, elementary degree theory implies that the degree of the Gauss map is even. Hence, by the Gauss-Bonnet Theorem, the genus of M must be odd. $\square$

Another application of Proposition 3.1 yields the following obstruction to minimal embeddings in higher-dimensional tori.

Theorem 3.4 *If $f\colon M \to {}^n$ is an embedded hyperelliptic minimal surface of genus g, then $2(g+1) \leq 2^n$.*

Proof. We know by the Riemann-Hurwitz formula that there are $2(g+1)$ hyperelliptic points on M. By Proposition 3.1, these hyperelliptic points must lie halfway from the origin to the corners of the fundamental region of ^{n} in ^{n}, or they must lie at the origin; i.e., they are the order two points in ^{n}. Hence, there are 2^n such positions for the hyperelliptic points. If $f\colon M \to {}^n$ is an embedding, then $2(g+1) \leq 2^n$ (since f is one-one on the hyperelliptic points). □

The above formula easily proves our next corollary.

Corollary 3.4 *A minimally embedded hyperelliptic surface in* ^{3} *has genus three.*

Since any holomorphic immersion of a surface in a complex torus is minimal, the above theorem gives conformal obstructions to holomorphically embedding a hyperelliptic surface into a complex torus. We list one of these instances in the next corollary.

Corollary 3.5 *A hyperelliptic surface of genus larger than seven will never holomorphically embed in a complex two-dimensional torus.*

The following theorem gives a natural interpretation of the classical Abel's Theorem for periodic minimal surfaces.

Theorem 3.5 (Abel's Theorem for Periodic Surfaces) *Let* $f\colon M_g \to {}^3$ *be a minimal surface and let* $G\colon M_g \to S^2$ *be its Gauss map. Then for all* $s \in S^2$, $q = \sum_{p \in G^{-1}(s)} p \in {}^3$ *(summed with multiplicity) is independent of* $s \in S^2$.

Proof. Since G is holomorphic, the continuous map $\widetilde{G}\colon S^2 \to {}^3$ defined by $\widetilde{G}(s) = \sum_{p \in G^{-1}(s)} p$, where the sum is taken with multiplicity, is locally a vector sum of harmonic coordinate functions and hence is itself a harmonic map. Since S^2 is simply connected, $\widetilde{G}$ lifts to the universal cover ^{3} of ^{3} with harmonic coordinate functions. Since a harmonic function on a closed Riemann surface is constant, the lifted map is constant. Hence, $\widetilde{G}$ is constant, which proves the theorem. □

Remark 3.1 *The above theorem holds for any meromorphic function $F\colon M_g \to S^2$, not just the Gauss map.*

Corollary 3.6 *After a fixed translation, the three points of a periodic surface of genus 4 with the same unit normal are coplanar, i.e. they lie on the quotient of a plane passing through the origin.*

Proof. Suppose M_4 is a genus 4 minimal surface in a flat three-torus ^{3}. After a fixed translation of M_4, we may assume by Theorem 3.5 that $p_1 + p_2 + p_3 = 0 \in {}^3$ for $\{p_1, p_2, p_3\} \subset G^{-1}(s)$ for every $s \in S^2$ not in the branch locus of G. This implies p_1, p_2, and p_3 are coplanar. $\square$

Our next theorem clarifies somewhat the extent to which the obstruction theory for periodic surfaces may be pursued. In this direction, we conjecture:

Conjecture 3.1 *If $g > 2$ is even, then there exists a full branched minimal hyperelliptic surface of genus g in every flat ^{3}, and when g odd, this branched immersion can be choosen to be free of branch points.*

Conjecture 3.2 *For every flat tori ^{3} and $k > 1$, there exists an embedded orientable minimal surface with Euler characteristic $\chi = -2(k-1)$, and an embedded non-orientable minimal surface with $\chi = -2k$ in ^{3}.*

Theorem 3.6

1. *For every $g = 2^k + 1$, $k \geq 1$, there exists a periodic hyperelliptic surface of genus g.*

2. *For every odd genus g, there exists an embedded minimal surface M_g in some flat three-torus.*

3. *For every even Euler characteristic less than -1, there exists an embedded non-orientable minimal surface M in some flat three-torus.*

Proof. Let M_5 be the hyperelliptic surface of genus 5 described in Figure 1 at the end of Section 2. Let $R_y\colon M_5 \to M_5$ be rotation by $180°$ around the y-axis. Since R_y commutes with the rotation Θ around the y-axis and Θ is the hyperelliptic automorphism on M_5, $M_3 = M_5/R_y$ is a hyperelliptic surface of genus three. Since any two hyperelliptic Riemann surface M_g, M_g' are diffeomorphic by a diffeomorphism h satisfying $h\Theta = \Theta' h$, one can extend the method used in the above example to show that every hyperelliptic surface M_3 has a two-sheeted cover by another hyperelliptic surface. Replacing the surface M_5 in the above figure by a surface with $2k+3$ holes, the above argument shows that every hyperelliptic Riemann surface of odd genus has a two-sheeted cover that is hyperelliptic. Hence, there exists a 2^k-sheeted cover of M_3 by a hyperelliptic surface. Thus, by taking appropriate covers of the Schwarz $\mathcal{P}$ surface, we get periodic hyperelliptic surfaces of genus $g = 2^k + 1$ for every $k \geq 1$. This proves Part 1.

Part 2 follows by taking appropriate covers and lifts of the Schwarz $\mathcal{P}$ surface into three-tori, or more concretely, by taking the quotient of the surface in $\mathbb{R}^3$ by appropriate sublattices.

Before proving Part 3, we outline the construction of a simple example of a non-orientable periodic surface with Euler characteristic $\chi = -2$, which we call the $\mathcal{S}$ surface. *Construction:* Take a polygonal curve with $90°$ angles with all sides of length L except the center lines of length $2L$. Now solve the Plateau problem for this curve.

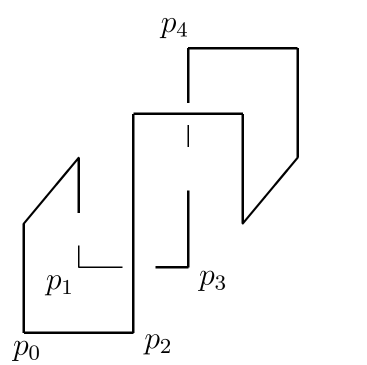


Figure 2:

This surface represents an embedded non-orientable minimal surface of Euler characteristic $\chi = -2$ in the quotient torus ^{3} with lattice L generated by the vectors $\{p_3 - p_0,\, p_2 - p_1,\, p_4 - p_0\}$. By taking appropriate sublatices L' of L, one obtains the examples $\mathcal{S}/L'$ described in Part 3 of the theorem. $\square$

Remark 3.2 *A closed nonorientable surface of odd Euler characteristic will not embed topologically in a three-torus. See [6] for this result. Also, the Klein bottle K can not be minimally immersed in a flat ^{3}, since this would imply that K has two linearly independent one-forms, which it fails to have.*

4 The canonical curve of a periodic surface

In the last section we found that the Gauss-Bonnet Theorem gives rise to restrictions on the conformal type of compact minimal surfaces in ^{3}. In Section 2, we briefly discussed how the usual Gauss map $G: M \to S^2$ for a minimal surface $f(p) = \mathrm{Re} \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3): M \to {}^3$ can be identified with the mapping $G: M \to Q_3 \subset {}^2$, where Q_3 is the standard quadric in ^{2} and $G = (\omega_1(p),\, \omega_2(p),\, \omega_3(p))$ in homogeneous coordinates.

By applying standard techniques of algebraic geometry to study G, we shall develop some further theory for minimal surfaces in flat tori. This theory includes both existence theorems and conformal obstructions for a Riemann surface to be periodic, and we will prove theorems concerning the existence of minimal surfaces in flat tori of dimension greater than three.

Throughout the rest of this section, we will assume that our minimal immersions $f: M \to {}^k$ are *full*, i.e., the image surface is not contained in a proper flat subtorus.

Recall that the canonical curve of M_g in ${}^{g-1}$, denoted by $c(M_g)$, is given in homogeneous coordinates by $c(p) = (\omega_1(p), \ldots, \omega_g(p))$, where $\{\omega_i\}$ is a fixed basis of $H^{1,0}(M_g)$.

Theorem 4.1 *Let $f(p) = \mathrm{Re}\int_{p_0}^{p}(\omega_1, \ldots, \omega_n)\colon M \to {}^n$ be a conformal branched minimal immersion. Suppose that $k = \mathrm{rank}\ \{\omega_1, \ldots, \omega_n\}$ in $H^{1,0}(M)$ is not equal to $n/2$. Then $c(M)$ is contained in a quadric of rank r, for some integer $r \geq 3$ with $2k - n \leq r \leq k$.*

In particular, the canonical curve of a periodic surface is contained in a quadric of rank 3.

Proof. After a possible reordering, assume that $f(p) = \mathrm{Re}\int_{p_0}^{p}(\omega_1, \ldots, \omega_n)$, with $\omega_1, \ldots, \omega_k$ linearly independent holomorphic one-forms. Suppose that the remaining one-forms $\omega_{k+1}, \ldots, \omega_n = \omega_{k+s}$ can each be expressed as a linear combination of the first k forms: $\omega_{k+1} = \sum_{j=1}^{k} a_{1j}\omega_j,\ \ldots,\ \omega_n = \sum_{j=1}^{k} a_{sj}\omega_j$. Note: $s = k$ if and only if rank $\{\omega_1, \ldots, \omega_n\} = n/2$. Suppose from now on that $k \neq n/2$.

Let $Q(x) = Q(x_1, \ldots, x_k)$ be the quadratic form defined by $Q(x_1, \ldots, x_k) = x_1^2 + x_2^2 + \ldots + x_k^2 + (\sum_{j=1}^{k} a_{1j}x_j)^2 + \ldots + (\sum_{j=1}^{k} a_{sj}x_j)^2$.

Claim 4.1 *Rank $Q(x) \geq k - s > 0$. In particular, $Q \not\equiv 0$.*

Proof of Claim 4.1. The possibility of $Q(x) = 0$ is readily seen to be equivalent to finding a solution x to the quadratic matrix equations $A^tA = -I_{k\times k}$, where $I_{k\times k}$ is the $k \times k$ identity matrix, and $A = (a_{ij})$ is the $s \times k$ coefficient matrix defined above. Clearly, rank $(A^tA) \leq s$. Therefore, rank $(I_{k\times k} - A^tA) \geq k - s$, and hence the rank $Q(x) \geq k - s > 0$. $\square$

After an orthogonal change of basis to $y_1, \ldots, y_n$, Q will be diagonalized, so we may assume that $Q(x_1, \ldots, x_k) = \overline{Q}(y_1, \ldots, y_r) = y_1^2 + \ldots + y_r^2$, where $k - s \leq r \leq k$. Since the mapping $f(p) = \mathrm{Re}\int_{p_0}^{p}(\omega_1, \ldots, \omega_n)\colon M \to {}^n$ is minimal, the generalized Weierstrass representation implies $Q(\omega_1, \ldots, \omega_k) = 0$. From the above diagonalization procedure, we know that there exist linearly independent $\alpha_1, \ldots, \alpha_r \in H^{1,0}(M)$, with $\overline{Q}(\alpha_1, \ldots, \alpha_r) = \alpha_1^2 + \ldots + \alpha_r^2 = 0$. This implies that the canonical curve of M is contained in a quadric of rank r, with $2k - n = k - s \leq r \leq k$. Rank $Q = 2$ implies the canonical curve of

M_g lies in a linear subspace of $\mathbb{CP}^{g-1}$. Since this never occurs, $r > 3$, which completes the proof of Theorem 4.1. $\square$

For every genus greater than 2, there are Riemann surfaces whose canonical curve fails to lie in a quadric of rank three, and hence, by the last theorem, they are not periodic. In the case of genus 3, the surfaces whose canonical curves are contained in a quadric are precisely the hyperelliptic surfaces.

Recall from the previous section that the Gauss map for a periodic surface M_3 represents M_3 as a two-sheeted cover of S^2; this implies that M_3 is hyperelliptic. Our next theorem generalizes this result from $\mathbb{R}^3$ to $\mathbb{R}^5$ and proves the characterization of hyperelliptic surfaces M_3 referred to in the last paragraph.

Theorem 4.2

1. *M_3 is a nonsingular curve of degree four in $\mathbb{CP}^2$ if and only if M_3 is not hyperelliptic.*

2. *If M_3 conformally minimally immerses in some $\mathbb{R}^3$ or $\mathbb{R}^5$, then M_3 is hyperelliptic.*

Proof. (We refer the reader to Section 10 of [16] for further details in the following proof.) Let $V^{d-3}\colon \mathbb{CP}^2 \to \mathbb{CP}^{[(d-1)(d-2)/2]-1}$ be the Veronese map given in homogeneous coordinates by monomials of degree $(d-3)$. It is well-known that $V^{d-3}(M)$ is the canonical mapping, where M is a non-singular curve of degree d in $\mathbb{CP}^2$. Thus, a non-singular curve of degree four in $\mathbb{CP}^2$ is always canonical.

On the other hand, the canonical curve of a Riemann surface is singular if and only if the surface is hyperelliptic. Therefore, M_3 is a nonsingular curve of degree four in $\mathbb{CP}^2$ if and only if M_3 is not hyperelliptic. Furthermore, the canonical curve of a hyperelliptic surface of genus 3 is the unique (up to isomorphism) quadric of rank three in $\mathbb{CP}^2$. Since the quadrics in $\mathbb{CP}^2$ of rank three are homeomorphic to S^2, it follows from the above discussion that the canonical curve of M_3 is contained in a quadric if and only if M_3 is hyperelliptic. Hence, by Theorem 4.1, any M_3 which conformally minimally immerses in $\mathbb{R}^3$ or $\mathbb{R}^5$ is hyperelliptic. $\square$

Example 4.1 *By Theorem 4.2, the Fermat curve of degree* 4 *in* $\mathbb{CP}^2$ *given in homogeneous coordinates by* $x^4+y^4+z^4=0$ *will never conformally minimally immerse in* $\mathbb{R}^3$ *or* $\mathbb{R}^5$*.*

If the canonical curve of M is contained in a quadric of rank k, then after a change of basis, we may assume that it lies in the standard quadric of rank k. This implies the existence of k linearly independent holomorphic one-forms $\omega_1,\dots,\omega_k$ with $\sum_{i=1}^k \omega_i^2 = 0$. From the generalized Weierstrass Representation, we can define a branched minimal immersion, $\widetilde{f}(p) = \operatorname{Re}\int_{p_0}^{p}(\widetilde{\omega}_1,\dots,\widetilde{\omega}_k)\colon \widetilde{M} \to \mathbb{R}^k$ on the universal cover $\widetilde{M}$ of M. If $\widetilde{f}(\widetilde{M}) \subset \mathbb{R}^k$ is a closed subset of $\mathbb{R}^k$, then the periods $L = \{\operatorname{Re}\int_\gamma(\omega_1,\dots,\omega_k) \mid \gamma \in H_1(M,\mathbb{Z})\}$ form a lattice in $\mathbb{R}^k$. Therefore, when the closure $\overline{\widetilde{f}(\widetilde{M})} = \widetilde{f}(\widetilde{M})$, we get an induced branched minimal surface $f = \operatorname{Re}\int(\omega_1,\dots,\omega_k)\colon M \to \mathbb{T}^k = \mathbb{R}^k/L$.

Much of the remaining work in this paper is an elaboration of the problem discussed above: If $c(M) \subset Q_k$, then do there exist conformal or geometric conditions on M that guarantee the existence of holomorphic forms $\omega_1,\dots,\omega_k$ satisfying

1. $\sum_{i=1}^k \omega_i^2 = 0$;

2. $L = \{\operatorname{Re}\int_\gamma(\omega_1,\dots,\omega_k) \mid \gamma \in H_1(M,\mathbb{Z})\}$ is a lattice in $\mathbb{R}^k$?

From this point of view, the next proposition gives us two families of candidates for periodic surfaces.

Proposition 4.1 *The canonical curve of a hyperelliptic surface is contained in a quadric of rank three, as is the canonical curve of a non-singular curve of degree* > 4 *in* $\mathbb{CP}^2$*.*

Proof. Theorem 2.6 states that the canonical curve of a hyperelliptic surface M_g can be factored up to isomorphism of ${}^{g-1}$, as $V \circ H$ where $H\colon M_g \to S^2 = {}^{1}$, and $V((1,t)) = (1, t, t^2, \ldots, t^{g-1})$ in homogeneous coordinates. Therefore, $c(M_g)$ is contained in the quadric of rank three defined by the equation $z_0 z_2 = z_1^2$ in homogeneous coordinates. This proves the first part of the proposition.

If $F\colon M_g \to {}^{2}$ is a nonsingular curve of degree d, then the canonical curve is obtained from the composition $M_g \xrightarrow{F} {}^{2} \xrightarrow{V^{d-3}} {}^{g-1}$, where V^{d-3} is the Veronese mapping given by monomials of degree $d-3$: $V^{d-3}(x,y,z) = (x^{d-3}, x^{d-4}y, x^{d-5}y^2, \ldots)$. Then $V^{d-3} \circ F(M_g) \subset V^{d-3}({}^{2})$ is contained in the quadric of rank three defined by the equations $z_0 z_2 = z_1^2$. $\square$

The above proposition indicates that there might not be any general conformal obstructions for a hyperelliptic surface M_g to be a branched minimal surface in some ^{3}. On the other hand, by pursuing the study of the canonical map for a hyperelliptic surface M_g, we can prove that when the genus is even, branch points always occur. This approach gives an algebric-geometric proof of Theorem 3.3 that a hyperelliptic surface of even genus is never periodic.

5 The rigidity theory for minimal surfaces

The rigidity theory for minimal surfaces in ^{3} is quite varied and interesting. If one is dealing with a simply connected minimal surface M, then the set of associate surfaces, parameterized by the circle, constitute the collection of all non-congruent isometric minimal immersions of M in ^{3}. But if M is not simply connected, then a minimal immersion of M might be rigid, i.e., any other isometric minimal immersion is obtained by composition with a symmetry of ^{3}. In fact, it is not difficult to show that the standard embedding of the catenoid is the unique isometric minimal immersion of this surface in ^{3} up to congruence.

An important extrinsic invariant of the congruence class of a minimal immersion is its space group. The *space group* of a submanifold $M \subset {}^3$ is the group of isometries of M induced by symmetries of ^{3}. For example, in Section 3, we showed that the space group of any hyperelliptic periodic surface contains the hyperelliptic automorphism.

In this section we will investigate conditions on a periodic surface M that insure the existence of more than one distinct isometric minimal embeddings of M into flat tori. Our first theorem gives restrictions on the conformal type of such surfaces.

Definition 5.1 *Two minimal immersions* $f\colon M \to {}^3_1$ *and* $g\colon M \to {}^3_2$ *are* distinct *if their lifted mappings to* ^{3} *differ by a symmetry of* ^{3}*. Note that the lifted point set surfaces in* ^{3} *may be congruent without implying that the mappings differ by a symmetry of* ^{3}*.*

Theorem 5.1 *If there exist two distinct isometric immersions of M into three-tori, then:*

1. *These immersions lift to holomorphic isometric immersions of M into a complex three-torus;*
2. *The Jacobian of M contains a codimension-three complex torus.*

Proof. After a translation, suppose that the first immersion $f_1(p) = \mathrm{Re}\int_{p_0}^{p}(\omega_1, \omega_2, \omega_3)$. To show that f_1 lifts to an isometric holomorphic immersion, it is sufficient to check that the periods $P = \{\frac{1}{\sqrt{2}}\int_\gamma(\omega_1, \omega_2, \omega_3) \mid \gamma \in H_1(M, \ )\}$ form a lattice in ^{3}. Since $\omega_1, \omega_2, \omega_3$ are linearly independent over , showing P is a lattice is equivalent to proving that the closure of P, denoted by $\overline{P}$, does not contain a linear subspace $S \subset \overline{P}$. By Theorem 2.5, we may assume that the second immersion, f_2, is given by $f_2(p) = \mathrm{Re}\,[e^{i\theta}\int_{p_0}^{p}(\omega_1, \omega_2, \omega_3)]$.

We know that the periods of f_1 and f_2 are discrete. This implies that $\mathrm{Re}(S) = \{\vec{0}\} = \mathrm{Re}(e^{i\theta}S)$, or equivalently, S is contained in the kernels of both of the projections Re: ${}^3 \to {}^3$ and $\mathrm{Re} \circ e^{i\theta}$: ${}^3 \to {}^3$. Since the intersection of these kernels is $\{\vec{0}\}$, $S = \{\vec{0}\}$. This shows that P is a lattice, and therefore the map $f(p) = \frac{1}{\sqrt{2}} \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3) \colon M \to {}^3 = {}^3/P$ is an isometric holomorphic immersion.

The second part of the theorem follows from Proposition 2.3. $\square$

The same proof as above proves the following:

Theorem 5.2 *If $f \colon M \to {}^k_1$ and $g \colon M \to {}^k_2$ are distinct associate surfaces, then f and g lift to a holomorphic immersion into a complex r-dimensional torus, for some $r \leq k$.*

To the best of my knowledge, all known classical examples of triply-periodic minimal surfaces in ^{3} isometrically and minimally immerse into two or more distinct tori. In fact, for most of the classical examples, $f \colon M \to {}^3_1$, the conjugate surface induces a minimal immersion $f_{\pi/2} \colon M \to {}^3_2$. Theorem 5.1, 2. motivates the question: *Is the Jacobian of a periodic surface ever simple, i.e., does not contain a complex subtorus?*

As the above discussion indicates, there might be many isometric minimal immersions of a closed Riemannian surface into three-dimensional tori. By Theorem 8.1, 3. there exist distinct isometric minimal embeddings of a surface M into a fixed ^{3}. However, in ^{3} the following strong rigidity theorem holds.

Theorem 5.3 (Rigidity Theorem) *Any two proper triply-periodic isometric non-planar minimal immersions in ^{3} are congruent.*

Proof. Let $\widetilde{f} \colon \widetilde{M} \to {}^3$ be a proper non-planar triply-periodic minimal surface and let $f \colon M \to {}^3$ be the quotient compact minimal surface, where $f(p) = \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3)$. Note that there always exists a cycle γ on M with $\int_\gamma (\omega_1, \omega_2, \omega_3)$ being a purely imaginary non-zero period vector. We can choose the cycle γ to be a simple closed curve on M. Since $\mathrm{Re}[\int_\gamma (\omega_1, \omega_2, \omega_3)] = \vec{0}$, $[\gamma] = 0 \in H_1({}^3) = \pi_1({}^3) =$ the fundamental group of ^{3}. Hence, we can lift γ to $\widetilde{\gamma} \subset \widetilde{M}$. Therefore, $\int_{\widetilde{\gamma}}(\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$ is a purely imaginary non-zero period vector on $\widetilde{M}$. Since this period vector is imaginary, none of the associate mappings $\widetilde{f}$ are well defined. Theorem 2.5 implies $\widetilde{f}\colon \widetilde{M} \to \mathbb{R}^3$ is the unique isometric minimal immersion of $\widetilde{M}$ up to congruence. $\square$

We conjecture that the above theorem holds in the following greater generality:

Conjecture 5.1 *A properly embedded, nonsimply connected, minimal surface in $\mathbb{R}^3$ is rigid in the sense that any other isometric minimal immersion is congruent to it.*

We would now like to discuss the geometry of triply-periodic minimal surfaces that have a infinite number of distinct isometric minimal immersions in flat three-tori. To this end, we make the following definition.

Definition 5.2 *A minimal surface $f\colon M \to \mathbb{R}^3$ satisfies* Property **P** *if for a countable dense set of angles $\theta \subset S^1$, the associate surfaces f_θ in $\mathbb{R}^3$ induce minimal immersions of M into flat three-tori.*

Definition 5.3 *A lattice $L \subset \mathbb{R}^3$ is $\mathbb{Q}$-simple if for some $c, d \in \mathbb{Q}$, there exists a basis of $(d \cdot L) \otimes \mathbb{C}$ consisting of $\{v_1, cv_1, v_2, cv_2, v_3, cv_3 \mid v_i \in \mathbb{Q}^3\}$.*

Theorem 5.4 *If $f\colon M \to \mathbb{R}^3$ is a minimal surface, then the following are equivalent:*

1. *f is the real part of a holomorphic immersion into a complex three-torus with a $\mathbb{Q}$-simple lattice.*

2. *f satisfies Property* **P**.

3. *There exist three distinct isometric minimal immersions of M into flat three-tori. (Note $f_\pi = -f$ is not considered to be distinct from f.)*

Proof. Assume Statement 1 holds and we shall prove Statement 2. After a translation, assume that the lift $\widetilde{f}\colon M \to \mathbb{T}^3 = \mathbb{C}^3/L$ is given by $\widetilde{f}(p) = \int_{p_0}^{p}(\omega_1, \omega_2, \omega_3)$ and $f = \mathrm{Re}(\widetilde{f})$. Suppose $\{v_1, cv_1, \ldots, v_3, cv_3\}$ is an $\mathbb{Q}$-simple basis for $(d\cdot L)\otimes \mathbb{Q}$. Then for any $\frac{n}{m} \in \mathbb{Q}$, we have $nv_i + mcv_i = (n+mc)v_i \in (d\cdot L)\otimes \mathbb{Q}$. By varying $\frac{n}{m} \in \mathbb{Q}$, it follows that $e^{i\theta}(L\otimes \mathbb{Q})$ contains imaginary three-dimensional rational subspaces for a dense set of $\theta \in S^1$. By projecting along these subspaces, we get isometric minimal immersions of M into flat three-tori.

Statement 2 implies Statement 3 by the definition of Property **P**.

It remains to show that Statement 3 implies 1. If there exist three distinct isometric minimal immersions f_1, f_2, f_3, then by Theorem 5.1 there exists a lift $\widetilde{f}\colon M \to \mathbb{T}^3 = \mathbb{C}^3/L$ of f. We may assume that $f_2 = (f_1)_{\theta_1}$, $f_3 = (f_1)_{\theta_2}$ and that $0 < \theta_1,\, \theta_2 < \frac{\pi}{2}$.

Let $K_j \subset L \otimes \mathbb{Q}$ denote the kernels of the projections Π_j restricted to $L \otimes \mathbb{Q}$, which give rise to $f_j, j = 1, 2, 3$. Then $K_1 \subset i\mathbb{R}^3, K_2 \subset e^{i\theta_1}\mathbb{R}^3$, and $K_3 \subset e^{i\theta_2}\mathbb{R}^3$. Let $B = \{v_1, v_2, v_3\}$ be a basis for $\Pi_1(L \otimes Q)$. Note that $\Pi_1(K_2) = \Pi_1(K_3) = \Pi_1(L \otimes Q)$, since the dimension of each of these vector spaces is three. For $j = 1, 2$, let $d_j = e^{i\theta_j}/\cos(\theta_j)$, respectively. It follows that $B_2 = \{d_1v_1,\, d_1v_2,\, d_1v_3\}$ is a basis for K_2 and $B_3 = \{d_2v_1,\, d_2v_2,\, d_2v_3\}$ is a basis for K_3. Let $d = 1/d_1$ and $c = d_2/d_1$. Then $\{v_1,\, c\,v_1, \ldots, v_3, c\,v_3\}$ is an $\mathbb{Q}$-simple basis for $(d \cdot L) \otimes Q$.

$\square$

Definition 5.4 *A complex torus $\mathbb{T}^n$ is said to be* isogenous *to $\widetilde{\mathbb{T}}^n$ if $\widetilde{\mathbb{T}}^n$ is a finite cover of $\mathbb{T}^n$.*

Corollary 5.1 *If $f\colon M \to \mathbb{R}^3$ is a minimal surface satisfying Property* **P**, *then the Jacobian of M is isogenous to $\mathbb{E}^1 \times \mathbb{E}^1 \times \mathbb{E}^1 \times \mathbb{T}^{g-3}$, where the first three elliptic curves are isomorphic.*

Proof. If M satisfies Property **P**, then by Theorem 5.4, there is a full holomorphic immersion of M into a complex three-torus $\mathbb{T}^3 = \mathbb{C}^3/L$ with

$\{v_1, cv_1, v_2, cv_2, v_3, cv_3\}$ an $\mathbb{C}$-simple rational basis for L. Clearly, $\mathbb{T}^3$ contains three elliptic curves spanned, respectively, by the pairs $\{v_i, cv_i\}$ for $1 \leq i \leq 3$. Hence, $\mathbb{T}^3$ is isogenous to $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1$. By the universality property of $J(M)$, we get the following commutative diagram:

$$\begin{array}{ccc} J(M) & \xrightarrow{g} & \mathbb{T}^3 \\ \uparrow & \nearrow \widetilde{f} & \downarrow \\ M & \xrightarrow[f]{} & \mathbb{T}^3 \end{array},$$

where g is linear, and $\widetilde{f}$ is the lift of f.

If $K \subset J(M)$ is the kernel of g, then $J(M)$ is finitely covered by $\mathbb{T}^3 \times \mathbb{T}^{g-3}$, by a property of abelian varieties (see [24]). Since $\mathbb{T}^3$ is covered by $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1$, $J(M)$ is isogenous to $\mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^{g-3}$. $\square$

In the next section we will discuss some classical examples of minimal surfaces satisfying Property **P**. We now will investigate the space group of an orientable periodic minimal surface.

Definition 5.5 *Let $f: M \to \mathbb{R}^3$ be a minimal surface. Then the* space *or* symmetry group *of f, denoted by $S_f(M)$, is the group of isometries of M induced by symmetries of M in $\mathbb{R}^3$. We make the same definition of $S_f(M)$ if $f: M \to \mathbb{T}^3$. We will denote the orientation-preserving subgroup of $S_f(M)$ by $S_f^{\circ}(M)$ and the orientation-reversing elements by $S_f^{r}(M)$. Here, orientation-preserving means that the isometry is orientation-preserving as a mapping from M to M.*

Remark 5.1 *Later, when we study $S_f(M)$ for $f: M \to \mathbb{T}^3$, we will always assume that the induced map $f_*: \pi_1(M) \to \pi_1(\mathbb{T}^3)$ is onto and that M is orientable. This can be accomplished by taking a lift of M to a finite cover of $\mathbb{T}^3$.*

Since pulling back forms is linear, we have:

Lemma 5.1 *Suppose $f\colon M \to \overline{M}$ is a submanifold. If $(\alpha_1, \alpha_2, \alpha_3)^t$ is a vertical vector valued one-form and A is a 3×3 matrix, then $f^*(A(\alpha_1, \alpha_2, \alpha_3)^t) = A(f^*\alpha_1, f^*\alpha_2, f^*\alpha_3)^t$.*

Lemma 5.2 *Let $f\colon M \to \mathbb{R}^3$ be a minimal surface represented by $f(p) = \operatorname{Re}\int_{p_0}^{p} \omega$, where $\omega = (\omega_1, \omega_2, \omega_3)^t$ is the transpose of the horizontal vector valued one-form $(\omega_1, \omega_2, \omega_3)$. If $g\colon M \to M$ is an isometry, then there is an orthogonal matrix O and angle θ such that:*

1. *$g^*(\omega) = e^{i\theta}O(\omega)$, when g is orientation-preserving;*
2. *$g^*(\omega) = e^{i\theta}O(\overline{\omega})$, when g reverses orientation.*

Proof. Since $f \circ g\colon M \to \mathbb{R}^3$ is a minimal surface isometric to f, Part 1 follows from Theorem 2.5. Part 2 follows by a similar argument. $\square$

Lemma 5.3 *Let $g\colon M \to M$ be an element of $S_f(M)$ induced from an isometry $\widetilde{g}\colon \mathbb{R}^3 \to \mathbb{R}^3$, where $\widetilde{g}$ is given by an orthogonal matrix O followed by composition with a translation. If $f(p) = \operatorname{Re}\int_{p_0}^{p} \omega$, where $\omega = (\omega_1, \omega_2, \omega_3)^t$, and $\omega_j = \alpha_j + i * \alpha_j \in H^{1,0}(M)$, then:*

1. *$g \in S_f^{\circ}(M)$ implies $g^*(\omega) = O(\omega)$;*
2. *$g \in S_f^{r}(M)$ implies $g^*(\omega) = O(\overline{\omega})$.*

Proof. Let $dx = (dx_1, dx_2, dx_3)^t$ and let $\alpha = (\alpha_1, \alpha_2, \alpha_3)^t$. Since $g\colon M \to M$ is induced from $\widetilde{g}\colon \mathbb{R}^3 \to \mathbb{R}^3$, we have $(f \circ g) = (\widetilde{g} \circ f)$. Applying Lemma 5.1, we have $g^*(\alpha) = g^*(f^*(dx)) = (f \circ g)^*(dx) = f^*(\widetilde{g}^*(dx)) = f^*(O(dx)) = O\, f^*(dx) = O(\alpha)$. Note that $g^*(*\alpha_i) = *g^*(\alpha_i)$ if g is orientation-preserving, and $g^*(*\alpha_i) = - * g^*(\alpha_i)$ if g is orientation-reversing. The lemma follows directly from these calculations. $\square$

Lemma 5.4 *Let $f\colon M \to \mathbb{R}^3$ be a minimal surface given by $f(p) = \operatorname{Re}\int_{p_0}^{p} \omega$, where $\omega = (\omega_1, \omega_2, \omega_3)^t$. If $g\colon M \to M$ is a map such that $g^*(\omega) = O(\omega)$ or $g^*(\omega) = O(\overline{\omega})$, then $g \in S_f(M)$.*

Proof. Suppose that $g^*(\omega) = O(\omega)$. Then $f(g(p)) = \operatorname{Re}\int_{p_0}^{g(p)} \omega = \operatorname{Re}\int_{p_0}^{g(p_0)} \omega + \int_{g(p_0)}^{g(p)} \omega = t + \operatorname{Re}\int_{p_0}^{p} g^*\omega = t + Of(p)$. A similar argument proves the case $g^*(\omega) = O(\overline{\omega})$. $\square$

Weierstrass proved that a straight line on a minimal surface gives rise to a symmetry of the surface by reflection through the line. Furthermore, this orientation-reversing isometry of the surface also extends to a symmetry on the conjugate surface by reflection through a plane orthogonal to the line. More generally, the next theorem explains the relationships between the space groups of the different associate surfaces. We will say that a $g \in S_f(M)$ has *linear part O* to mean O is the linear part of the extended symmetry in $\mathbb{R}^n$.

Theorem 5.5 *Let $f\colon M \to \mathbb{R}^3$ be a simply connected minimal surface. Then*

1. *$S_f^{\circ}(M) = S_{f_\theta}^{\circ}(M)$ for all associate surfaces f_θ.*
2. *If $g \subset S_f^r(M)$, then $g \in S_{f_\theta}^r(M)$ for $\theta \neq 0, \pi$ if and only if f_θ is the conjugate surface, i.e., $\theta = \pm\frac{\pi}{2}$. Furthermore, if $g \in S_f^r(M)$ has linear part O, then $g \in S_{f_{\pi/2}}^r(M)$ has linear part $-O$.*
3. *If $g\colon M \to M$ is an orientation-reversing isometry, then $g \in S_{f_\theta}(M)$, for some θ.*
4. *If $g\colon M \to M$ is an isometry of order 2, then $g \in S_{f_\theta}(M)$, for some θ.*
5. *If $g\colon M \to M$ is a commutator of isometries, then $g \in S_{f_\theta}(M)$ for all θ.*
6. *$S_f^{\circ}(M)$ is a normal subgroup of the isometry group of M. However, $S_f(M)$ is not, in general, a normal subgroup of the isometry group of M.*

Proof. Throughout this proof, let $\omega = (\omega_1, \omega_2, \omega_3)^t$, $\alpha = \mathrm{Re}(\omega)$ and $\beta = \mathrm{Im}(\omega)$.

Suppose $f_\theta(p) = \mathrm{Re}(e^{i\theta} \int_{p_0}^{p} \omega)$ and $g \in S_f^\circ(M)$ has linear part O. Then $f_\theta(g(p)) = \mathrm{Re}(e^{i\theta} \int_{p_0}^{g(p)} \omega) = \mathrm{Re}[e^{i\theta} \int_{p_0}^{g(p_0)} \omega + e^{i\theta} \int_{g(p_0)}^{g(p)} \omega] = t + \mathrm{Re}[e^{i\theta} \int_{p_0}^{p} g^*(\omega)]$ $= t + \mathrm{Re}[e^{i\theta} O \int_{p_0}^{p} \omega] = t + O f_\theta(p)$, where $t = f_\theta(g(p_0))$. This proves Part 1.

If $g \in S_f^r(M)$, then $f_\theta(g(p)) = t + O\,\mathrm{Re}(e^{i\theta} \int_{p_0}^{p} \overline{\omega})$ by Lemma 5.4. When $\theta = \pi/2$, $f_\theta(g(p)) = t - O f_\theta(p)$, and hence g extends to a symmetry of $\ ^3$. Suppose now that $\theta \neq 0, \pi, \pm\pi/2$. Then $g^*(e^{i\theta}\omega) = e^{i\theta} O(\overline{\omega})$. The linear independence of the one-forms $\omega_1, \omega_2, \omega_3$ implies that if $g^*(e^{i\theta}\omega) = \widetilde{O}(\overline{\omega})$ for some orthogonal matrix, then $\widetilde{O} = \pm O$. This leads to a contradiction since we assumed that $\theta \neq 0, \pi, \pm\pi/2$. Thus, we have proved that if $g \in S_f^r(M)$, then $g \in S_{f_\theta}^r(M)$ if and only if f_θ is the conjugate surface.

Suppose $g\colon M \to M$ is an orientation-reversing isometry and that $g^*(\omega) = e^{i\theta} O(\overline{w})$. If $\sigma = (\sigma_1, \sigma_2, \sigma_3)^t = e^{-i\theta/2}\omega$, then

$$g^*(\sigma) = e^{-i\theta/2} e^{i\theta} O(\overline{\omega}) = O(e^{i\theta/2}\overline{\omega}) = O(\overline{\sigma}).$$

Hence, by Lemma 5.4, g extends to a self-congruence of the associate surface $f_{-\theta/2}$ and this proves Part 3.

Suppose $g\colon M \to M$ is an orientation-preserving isometry of order 2, with $g^*(\omega) = e^{i\theta} O(\omega)$. Since $g^2 = id$, $\omega = (g^*)^2(\omega) = e^{2i\theta} O^2(\omega)$, which implies that $e^{2i\theta} O^2 = I$. Note that $e^{2i\theta} = \pm 1$, since both O and I are real matrices. In the case $e^{2i\theta} = 1$, we must have $e^{i\theta} = \pm 1$ and hence $g \in S_f(M)$ by Lemma 5.4. The possibility that $-1 = e^{2i\theta}$ cannot occur: By taking the determinant of both sides of the equation $e^{2i\theta} O^2 = I$, we get $(-1)(\det(O))^2 = 1$, which is impossible. If $g\colon M \to M$ is orientation-reversing, then we have $g \in S_{f_\theta}(M)$ for some θ by Part 3. This proves Part 4.

Suppose an isometry $g\colon M \to M$ can be expressed as a commutator of isometries f and h. For convenience, suppose f and h are orientation-preserving. Then if $f^*(\omega) = e^{i\theta_1} O_1(\omega)$ and $g^*(\omega) = e^{i\theta_2} O_2(\omega)$, where O_i is an orthogonal matrix, we have

$$(fgf^{-1}g^{-1})^*(\omega) = e^{i\theta_1} e^{i\theta_2} e^{-i\theta_1} e^{-i\theta_2} O_1 O_2 O_1^{-1} O_2^{-1}(\omega) = O(\omega)$$

where $O = O_1 O_2 O_1^{-1} O_2^{-1}$. Hence, by Lemma 5.4 and Part 1, $g \in S_{f_\theta}(M)$ for all θ. This proves Part 5.

The proof that $S_f^\circ(M)$ is normal in the isometry group of M is a straightforward calculation similar to the proof of Part 5. For the classical Enneper's surface of total curvature 4π, one can check that the symmetry group is not a normal subgroup of the isometry group. This completes the proof of the theorem. $\square$

Corollary 5.2 *Let $f\colon M \to \mathbb{R}^n$ be a minimal surface and $f_{\pi/2}\colon M \to \mathbb{R}^n$ be the conjugate surface. Then $g \in S_f^r(M)$ has linear part $-I_{n\times n}$ if and only if $g \in S_{f_{\pi/2}}^r(M)$ extends to a pure translation in $\mathbb{R}^n$.*

Proof. By the proof of Theorem 5.5, 2., $g \in S_f^r(M)$ has linear part $-I_{n\times n}$ if and only if $g \in S_{f_{\pi/2}}^r(M)$ has linear part $I_{n\times n}$. This proves the corollary. $\square$

6 Existence of non-congruent isometric periodic minimal surfaces

This section is a continuation and culmination of the work of Section 5. Here we study classical examples of minimal surfaces in $\mathbb{R}^3$ satisfying Property **P** defined in Section 5. We recall that a minimal surface $f\colon M \to \mathbb{R}^3$ satisfies Property **P**, if for a countable dense set of angles $\theta \in S^1$, the associate surfaces f_θ induce minimal immersions of M into flat tori. Before discussing these examples, we first extend our rigidity study to minimal surfaces in flat three-tori.

Definition 6.1 *If $f\colon M \to \mathbb{R}^n$ is a minimal surface, then the* space group *of $M = S_f(M)$ is the set of symmetries of M in $\mathbb{R}^n$.*

We note that whenever only one minimal surface $f\colon M \to \mathbb{R}^n$ is being considered, we will assume f does not lift to a cover of $\mathbb{R}^n$ and that M is orientable. Geometrically, this lift condition means that the lattice of ^{n} is the natural lattice for the lifted periodic surface in ^{n}. We will denote this second lattice by $L_f(M)$ in cases of ambiguity.

Proposition 6.1 *Let $\widetilde{f}: M \to {}^3 = {}^3/L$ be a holomorphic lift of the minimal surface $f = \mathrm{Re}(\widetilde{f}): M \to {}^3$.*

1. *If $g \in S_f^\circ(M)$ has linear part O, then $g \in S_{\widetilde{f}}^\circ(M)$ has linear part O.*
2. *If $g \in S_f^r(M)$ has linear part O, then $g \in S_{\widetilde{f}}^r(M)$ has linear part which is complex conjugation followed by O.*

Proof. Let $\omega = (\omega_1, \omega_2, \omega_3)^t$. Let $g \in S_f^\circ(M)$ and suppose $g^*(\omega) = O(\omega)$. Then a calculation similar to one in the proof of Theorem 5.5, 1. shows $\widetilde{f}(g(p)) = \widetilde{f}(g(p_0)) + O\widetilde{f}(p)$, where $\widetilde{f}(p) = \int_{p_0}^p \omega$. To complete the proof of Part 1, we need only show $O: L \to L$. We check this for $\ell = \int_\gamma \omega \in L$, where $\gamma \in H_1(M, \ )$. $O(\int_\gamma \omega) = \int_\gamma O(\omega) = \int_\gamma g^*(\omega) = \int_{g_*(\gamma)} \omega \in L$. If $g \in S_f^r(M)$ has linear part O, then $g^*(\omega) = O(\overline{\omega})$. A calculation, similar in spirit to the proof of the orientation-preserving case, also works to prove Part 2. □

Proposition 6.2 *Let $f: M \to {}^3 = {}^3/L_f(M)$ and $g: M \to {}^3$ be isometric minimal immersions. If the Jacobian of M is simple (contains no complex subtori) and the genus of M is different from three, then f and g are congruent in ^{3}. In particular, the space group of f equals the isometry group of M.*

Proof. Note that if $J(M)$ is simple, then for the lifted surfaces in ^{3}, $\widetilde{f} = t + O\,\widetilde{g}$ by Theorem 5.1, 2. Arguing as in the proof of Proposition 6.1, O leaves $L_f(M)$ invariant. Therefore, $t + O: {}^3 \to {}^3$ is a congruence between f and g. □

Remark 6.1 *Using Theorem 8.1, 3., it is easy to check that the conclusions of Proposition 6.2 fail to hold when the genus of M is three or the Jacobian of M is not simple.*

Through techniques similar to those used above, we can prove that the number of isometric minimal immersions $f\colon M \to {}^3$ is finite. The next theorem shows this result holds for n-dimensional minimal varieties in flat tori.

Theorem 6.1 (Finiteness Theorem) *The number of noncongruent isometric minimal immersions of $(M^n, <>)$ into a fixed flat k-torus ^{k} is finite.*

Proof. Let $H(M) = H^1(M)$ denote the harmonic one-forms on M^n, and let $(H(M))^k$ denote the k-fold Cartesian product $H(M) \times \ldots \times H(M)$. We now define the norm of $h \in (H(M))^k$ by $|h| = |(h_1, \ldots, h_k)| = \max\{|\int_{\gamma_i} h| \mid 1 \leq i \leq p\}$, where $\gamma_1, \ldots, \gamma_p$ are fixed representatives for a basis of $H_1(M, \ )$.

Claim 6.1 *There exist $r_1, r_2 > 0$ such that whenever $f = \int h\colon M \to {}^k = {}^k/L$, $h \in (H(M))^k$, is an isometric minimal immersion, then $h \in B(r_1, r_2) = \{h \in (H(M))^k \mid \ r_1 \leq |h| \leq r_2\}$.*

Proof. If r_1= length of the shortest $\ell \in L$, then $r_1 \leq |\int_{\gamma_i} h| \leq |h|$ for all i. If r_2 is the length of the longest γ_i, then for some j, $|h| = |\int_{\gamma_j} h| \leq r_2$. $\square$

Now assume ^{k} has fixed coordinates induced from the representation ${}^k/L$. Note that $I_{p_0} = \{h \in (H(M))^k \mid f(p) = \int_{p_0}^{p} h\colon M \to {}^k/L$ is an isometric minimal immersion$\}$ is a discrete subset of $(H(M))^k$. This can be shown by the following argument. When $f, g \in I_{p_0}$ are close in $(H(M))^k$, then they are homotopic in ${}^k/L$. Since two homotopic isometric minimal immersions of $(M, <>)$ into ^{k} are obtained by integrating the same harmonic one-forms, they differ by a translation. This implies that $f = g$, and hence I_{p_0} is discrete. Since I_{p_0} is a discrete closed subset of the compact set $B(r_1, r_2)$, I_{p_0} must be finite. $\square$

Many classical examples of periodic minimal surfaces have cubical lattices for their tori, and have the standard linear symmetries of a cube in their space group. We will say that a minimal surface $f\colon M \to \mathbb{R}^3/\mathbb{Z}^3$ has the *symmetries of a cube* if $S^{\circ}_f(M)$ has members with linear part

$$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and if $S^{r}_f(M)$ has an inversion symmetry T with linear part of T equaling $-I$, where I is the 3×3 identity matrix.

Theorem 6.2 *Let $f\colon M \to \mathbb{R}^3$ be a minimal surface, and let $\widetilde{f}_\theta\colon \widetilde{M} \to \mathbb{R}^3$ denote an associate surface on the universal covers. Suppose that*

1. *f lifts to a holomorphic immersion $f\colon M \to \mathbb{T}^3 = \mathbb{C}^3/L$;*
2. *f has the linear symmetries of a cube.*

Then for a dense set of angles θ in S^1, the $\widetilde{f}_\theta$ induce minimal immersions of M into flat tori. Furthermore, the triply-periodic surfaces corresponding to these $\widetilde{f}_\theta$ are invariant under generically distinct lattices of translation.

Proof. By Proposition 6.1, the rigid motions R_1, R_2, and T lift to symmetries of M in $\mathbb{T}^3 = \mathbb{C}^3/L$. Applying the lift of T to $\ell_1 \in L$, we get $T(\ell_1) = -\overline{\ell}_1$ which shows that complex conjugation leaves L invariant. Hence, there exists $\ell_2 = i(a,b,c)^t \in L$, where $a, b, c, \in \mathbb{R}$ and $a \neq 0$. Applying R_2 to ℓ_2 we find that $\ell_3 = R_2(a,b,c)^t i + (a,b,c)^t i = (a,-b,-c)^t i + (a,b,c)^t i = (2ai, 0, 0)^t \in L$. Now apply the lift of R_1 to get the three vectors: $(2ai,0,0)^t$, $R_1(2ai,0,0)^t = (0,2ai,0)^t$, $R_1(0,2ai,0)^t = (0,0,2ai)^t$. Similarly, we can find vectors $(d,0,0)^t, (0,d,0)^t, (0,0,d)^t \in L$ for some $d \in \mathbb{R}^+$. Hence, $L \otimes \mathbb{Q}$ has a $\mathbb{Q}$-simple basis, and by Theorem 5.4, $f\colon M \to \mathbb{R}^3$ satisfies Property **P**. This proves the first part of the theorem.

By the Theorem 6.1, only a finite number of these minimal surfaces can lie in any fixed ^{3}. Hence, the lattices for translation for the periodic f_θ are generically distinct. $\square$

Remark 6.2 *Several classical examples of periodic surfaces satisfy the conditions of the above theorem. These include the Schwarz diamond surface, the Novious model, and A. Schoen's* $O, C - TO$ *surface. Earlier, Schoen [53] proved that the Schwarz diamond surface satisfied Property* $\mathbf{P}$. *His result motivated our consideration of this problem.*

7 A family of conformally distinct periodic minimal surfaces

As we saw in Remark 6.1, there exist several geometrically distinct examples of isometric periodic minimal surfaces in ^{3}. However, the existence of these surfaces depends on the existence of a periodic surface with certain symmetry properties.

In this section, we again apply our theorems on isometry groups to get periodic minimal surfaces in ^{3}. We will show that Riemann surfaces satisfying certain conformal restrictions are periodic.

Lemma 7.1 *The group* $\mathcal{A}$ *of* 3×3 *complex matrices* A *satisfying* $AA^t = I$, *induces all the fractional linear transformations* $\mathcal{L}$ *on* $Q_3 \subset {}^2$.

Proof. If $AA^t = I$, then A induces a linear map, $\widetilde{A}\colon Q_3 \to Q_3$, in homogeneous coordinates. Note that $\widetilde{A} = id_{Q_3}$ if and only if A is plus or minus the identity matrix. Hence, we get a two-to-one homomorphism $\sim\colon \mathcal{A} \to \mathcal{L}$. Since $\mathcal{A}$ and $\mathcal{L}$ have the same dimension, $\sim$ is two-to-one, and $\mathcal{L}$ is connected, we have $\sim$ is onto. $\square$

Lemma 7.2 *Let* $f\colon M \to S^2 = Q_3 \subset {}^2$ *be a two-sheeted branched cover of* S^2 *by a surface of genus three. Then* f *is a canonical mapping.*

Proof. Let $c = [(\omega_1, \omega_2, \omega_3)^t]\colon M_3 \to \mathbb{CP}^2$ be a canonical mapping for the hyperelliptic surface M_3 of genus three. From the discussion in Section 10 of [16], we may assume, after a possible change of basis, that $c\colon M_3 \to Q_3 \subset \mathbb{CP}^2$, and c is a two-sheeted branched covering of Q_3.

Therefore, $c\colon M_3 \to Q_3$ is the hyperelliptic mapping on M_3. Since this mapping is unique up to a linear fractional transformation, by Lemma 7.1, there is an $A \in \mathcal{A}$ with $f = Ac = A[(\omega_1, \omega_2, \omega_3)^t]$. This implies that f is a canonical mapping. $\square$

Lemma 7.3 *Let $\omega_1, \ldots, \omega_n \in H^{1,0}(M)$, and let $[\omega_p] = [\omega_1(p), \ldots, \omega_n(p)] \in \mathbb{CP}^{n-1}$ in homogeneous coordinates. If $g\colon M \to M$ is conformal or anti-conformal, then $[(g^*\omega)_p] = [\omega_{g(p)}]$.*

Proof. Suppose $\omega_i(z) = f_i(z)dz$ in local coordinates. Then $(g^*\omega_i)(z) = f_i(g(z))g^*dz$. Therefore, $[(g^*\omega)_p] = [f_1(g(p)), \ldots, f_n(g(p))] = [\omega_{g(p)}]$. $\square$

Lemma 7.4 *Let $\omega_i, \alpha_i \in H^{1,0}(M)$, for $1 \le i \le n$. If $[\omega_p] = [\omega_1(p), \ldots, \omega_n(p)] = [\alpha_1(p), \ldots, \alpha_n(p)] = [\alpha_p]$ and $\omega_p \neq 0$ for all p, then $\omega_i = \lambda\alpha_i$ for some $\lambda \in \mathbb{C}$.*

Proof. $[\omega] = [\alpha]$ implies $\omega = f\alpha$, where f is meromorphic. If f is not constant, then $f(p) = 0$ for some p. But $\omega_p = f(p)\alpha_p = 0$, a contradiction to our hypothesis. Hence, f is a complex constant λ and $\omega = \lambda\,\alpha$. $\square$

Lemma 7.5 *Let $f\colon M \to S^2$ be a two-sheeted cover of S^2 branched over $P = \{p_1, \ldots, p_k\}$. Then any diffeomorphism $g\colon S^2 \to S^2$ with $g(P) = P$ is covered by a diffeomorphism $\widetilde{g}\colon M \to M$.*

Proof. If $f\colon M \to S^2$ is a two-sheeted cover of S^2 branched over P, then it can be constructed abstractly as follows: Take the representation

$$R\colon \pi_1(S^2 - P) \to \text{ Perm } \{a, b\}$$

that sends each of the standard generators to the nontrivial permutation. This representation gives rise to a two-sheeted cover $\widetilde{f}\colon \widetilde{M} \to S^2 - P$. To get $f\colon M \to S^2$, fill in the missing points above $P \subset S^2$.

Let $g\colon S^2 - P \to S^2 - P$ be a diffeomorphism. Then for the induced maps on fundamental groups, we have $(g_* \circ \widetilde{f}_*)(\pi_1(\widetilde{M})) = \widetilde{f}_*(\pi_1(\widetilde{M}))$. Elementary covering space theory implies that g lifts to $\widetilde{g}\colon \widetilde{M} \to \widetilde{M}$. When $g\colon S^2 \to S^2$ is a diffeomorphism with $g(P) = P$, then it is clear that the lift of $g|(S^2 - P)$ to $\widetilde{M}$ can be extended to a lift $\widetilde{g}\colon M \to M$. $\square$

The next proposition is useful for calculating the isometry group for a minimal surface. This is done in terms of the symmetries of the branch points of the Gauss map on the surface.

Proposition 7.1 *Let $f\colon M \to \mathbb{R}^3$ be a minimal isometric immersion. Suppose that an isometry $g\colon S^2 \to S^2$ with orthogonal matrix O lifts to a mapping $\widetilde{g}$ on M:*

$$\begin{tikzcd} M \arrow[r, "\widetilde{g}"] \arrow[d, "G"'] & M \arrow[d, "G"] \\ S^2 \arrow[r, "O"'] & S^2 \end{tikzcd},$$

$G = (\omega) =$ Gauss map where $\omega = (\omega_1, \omega_2, \omega_3)^t$. Then $\widetilde{g}\colon M \to M$ is an isometry, and $\widetilde{g}^*(\omega) = e^{i\theta}O(\omega)$ or $\widetilde{g}(\omega) = e^{i\theta}O(\overline{\omega})$, depending on whether g is orientation-preserving or orientation-reversing.

Proof. We will consider $S^2 = Q_3 \subset \mathbb{CP}^2$, and let $[\omega_p] = [\omega_1(p), \omega_2(p), \omega_3(p)]$. If $O\colon S^2 \to S^2$ is orientation-preserving, then by Lemma 7.3, $O[\omega_p] = [\widetilde{\omega}_{g(p)}] = [(g^*\omega)_p]$. However, in the case $O\colon S^2 \to S^2$ is orientation-reversing, then the mapping on Q_3 is expressed as $\sigma \circ O\colon Q_3 \to Q_3$, where σ is complex conjugation in homogeneous coordinates. For the orientation-reversing case, we have $\sigma \circ O[\omega_p] = [\omega_{g(p)}] = [(g^*\omega)_p]$. Hence, Lemma 7.4 implies $g^*\omega = e^{i\theta}O\omega$ if g is conformal, and $g^*\omega = e^{i\theta}O\overline{\omega}$ if g is anti-conformal. This calculation implies g is an isometry on M. $\square$

We now give an application of the last proposition. Let D denote the Riemann surface of genus 3 corresponding to the triply-periodic Schwarz diamond surface $\mathcal{D}$ in $\mathbb{R}^3$. Let $I(M)$ denote the isometry group of M, and let $|S|$ denote the size of a set S.

Corollary 7.1 *Let $f\colon D \to \mathbb{R}^3_1$ and $f_{\pi/2}\colon D \to \mathbb{R}^3_2$ be the Schwarz diamond and primitive surfaces, respectively. Then*

1. *If $g\colon M_3 \to \mathbb{R}^3$ is a minimal surface, then $|I(M_3)| \leq |I(D)| = 96$, with equality holding if and only if $g\colon M_3 \to \mathbb{R}^3$ is a conjugate surface to the Schwarz diamond surface.*

2. *$I(D) = S_{f_\theta}(D)$ if and only if $\theta = 0,\ \pi,\ \pm\pi/2$.*

Remark 7.1 *Corollary 7.1 shows that the Schwarz diamond and primitive surfaces display more symmetry than any other periodic surfaces of genus three.*

Proof. Let H_A denote the group of isometries of S^2 which leaves invariant a subset $A \subset S^2$. By Proposition 7.1 and Lemma 7.5, the isometry group $I(M_3)$ of a minimal surface $g\colon M_3 \to \mathbb{R}^3$ is isomorphic to $\mathbb{Z}_2 \times H_A$, where $A =$ the branch locus of the Gauss map. Since H_A is largest when A is the branch locus of the Gauss map for the Schwarz diamond surface, then $|I(M_3)| \leq |I(D)|$ with equality holding if and only if $g\colon M_3 \to \mathbb{R}^3$ is an associate to the Schwarz diamond surface. The Gauss map for a surface of genus three is the hyperelliptic quotient, and the branch points of this quotient mapping determine the surface uniquely. It is straightforward to compute that $|I(D)| = 96$ which proves Part 1.

Part 2 follows from Parts 1 and 2 of Theorem 5.5. $\square$

Let $I_p\colon \mathbb{R}^n \to \mathbb{R}^n$ denote inversion through $p \in \mathbb{R}^n$ and let $T_q\colon \mathbb{R}^n \to \mathbb{R}^n$ denote translation by $q \in \mathbb{R}^n$. If I_p or T_q leaves $M \subset \mathbb{R}^3$ invariant, then, when no confusion arises, we denote the restrictions $I_p|_M$ and $T_q|_M$ by I_p by T_q, respectively.

Definition 7.1 *If $L \subset \mathbb{R}^n$ is a subgroup of $\mathbb{R}^n$ and $v \subset \mathbb{R}^n$, then let $\langle L, v\rangle$ denote the subgroup of $\mathbb{R}^n$ generated by $L \cup \{v\}$.*

Proposition 7.2 *Suppose $f\colon M \to \mathbb{T}^n_1 = \mathbb{R}^n/L_1$ and $f_{\pi/2}\colon M \to \mathbb{T}^n_2 = \mathbb{R}^n/L_2$ are minimal surfaces. Then*

1. *$I_p \in S^r_f(M)$ for some $p \in \mathbb{T}^n_1$ if and only if $T_q \in S^r_{f_{\pi/2}}(M)$ for some $q \in \mathbb{T}^n$ of order two.*
2. *If $I_p \in S^r_f(M)$, then I_p has no fixed points, and $f_{\pi/2}$ induces a minimal immersion $f_{\pi/2}\colon M/I_p \to \mathbb{R}^n/\langle L_2, q_2\rangle$ of the non-orientable surface M/I_p, where q_2 is a lift of q (in Part 1) to $\mathbb{R}^n$.*
3. *Suppose M has genus 3, $n = 3$, and $I_p \in S^r_f(M)$. Then, after a translation, $f\colon M \to \mathbb{R}^3/L$ induces a minimal immersion $\widetilde{f}\colon M/(\Theta \circ I_p) \to \mathbb{R}^3/\langle L, I_p(h)\rangle$, where $h = 0 \in \mathbb{R}^3$ is a hyperelliptic point of M, and $\Theta\colon M \to M$ is the hyperelliptic automorphism.*

Proof. Part 1 follows from Corollary 5.2 and Proposition 6.1, 2.

Part 1 shows $f_{\pi/2}(M) \to \mathbb{R}^n$ is invariant under an additional translation T_q; this implies Part 2.

By Proposition 3.1, a minimal surface $f\colon M_3 \to \mathbb{R}^3$ is hyperelliptic and has an inversion symmetry through each hyperelliptic point. Suppose f is translated so that a hyperelliptic point h is at the origin in $\mathbb{R}^3$. Since $\Theta, I_p \in S_f(M_3), \Theta \circ I_p \neq id$, and the linear part of $\Theta \circ I_p = I_{3\times 3}$, we see that $\Theta \circ I_p$ extends to a symmetry in $\mathbb{R}^3$ given by translation by $I_p(h)$. Hence, f induces a minimal immersion $\widetilde{f}\colon M/(\Theta \circ I_p) \to \mathbb{R}^3/\langle L, I_p(h)\rangle$. This proves Part 3. □

Corollary 7.2 *The Schwarz $\mathcal{P}$ and $\mathcal{D}$ surfaces, and Schoen's crossed layers of parallels surfaces are all invariant under an additional orientation-reversing translation. Their quotients by these larger lattices are embedded, non-orientable, minimal surfaces of Euler characteristic $\chi = -2$ in flat three-tori.*

Many of the classical examples of periodic surfaces in $\mathbb{R}^3$ satisfy the hypothesis of Proposition 7.2, 2. Our next theorem proves the existence of a large family of such surfaces.

Theorem 7.1 *There is a real five-dimensional family V of periodic hyperelliptic Riemann surfaces of genus three. These are the surfaces which can be represented as two-sheeted covers of S^2 branched over four pairs of antipodal points. Furthermore,*

1. *There exist two distinct isometric minimal immersions for each $M_3 \in V$;*
2. *These immersions are embeddings;*
3. *V induces a five-dimensional family $\widetilde{V}$ of embedded non-orientable minimal surfaces of Euler characteristic $\chi = -2$.*

Proof. Suppose $\{p_1, p_2, \ldots, p_8\}$ consists of four pairs of antipodal points on $S^2 = Q_3 \subset \mathbb{CP}^2$ and let $f\colon M_3 \to S^2$ be the abstract two-sheeted cover of S^2 branched over these points. Note that the antipodal map on Q_3 is given in homogeneous coordinates by complex conjugation.

By Lemma 7.2, $f\colon M \to Q_3$ is canonical and represented in homogeneous coordinates by $f(p) = [\omega_1(p), \omega_2(p), \omega_3(p)]^t = [\omega(p)]$, where $\omega_i \in H^{1,0}(M)$ and $\sum_{i=1}^3 \omega_i^2 = 0$. By Proposition 7.1 and Lemma 7.5, the antipodal map on Q_3 lifts to an orientation-reversing isometry $\widetilde{a}\colon M_3 \to M_3$ such that $\widetilde{a}^*(\omega) = e^{i\theta}(-I)(\overline{\omega}) = e^{i\theta}(-1)(\overline{\omega})$. If $\beta = e^{-i\theta/2}\omega$, then $\widetilde{a}^*(\beta) = -\overline{\beta}$.

Let $g\colon M \to J(M) = \mathbb{C}^3/L$ be the Jacobi embedding defined by $\int_{p_0}^{p} \beta$. Suppose $\beta = (\beta_1, \beta_2, \beta_3)^t$, and $\ell = \int_\gamma \beta \in L$ for some $\gamma \in H_1(M, \mathbb{Z})$. Then $-\overline{\ell} = \int_\gamma -\overline{\beta} = \int_\gamma \widetilde{a}^*(\beta) = \int_{\widetilde{a}_*(\gamma)} \beta \in L$. This calculation shows that complex conjugation leaves L invariant. Thus, L contains two rank-three subgroups $L_1, L_2 \subset L$ with $L_1 \subset \mathbb{R}^3$ and $L_2 \subset i\mathbb{R}^3$. By projecting $J(M)$ orthogonally onto the real and imaginary subspaces, one obtains two minimal immersions of M into three-tori. More precisely, from the generalized Weierstrass representation, $f(p) = \mathrm{Re} \int_{p_0}^{p} \beta \colon M \to {}^3_1$ and $f_{\pi/2}(p) = \mathrm{Re}[e^{i\pi/2} \int_{p_0}^{p} \beta] \colon M \to {}^3_2$ are these minimal immersions.

We have shown that if $M_3 \in V$, then there exist two distinct isometric conformal minimal immersions of M_3 into three-tori. We now prove that these minimal immersions are embeddings. First note that the Schwarz diamond surface $\mathcal{D}$ and its conjugate surface $\mathcal{P}$ are embedded in their natural tori. Furthermore, by the construction of the minimal immersions in V, there exists a smooth family $f_t \colon M_3 \to {}^3_t$ joining any minimal immersion $f_1 \colon M_3 \to {}^3_1$ in V to either $f_0 \colon P \to {}^3_0$ or $f_2 \colon D \to {}^3_2$ where $f_0(P) = \mathcal{P}$ and $f_2(D) = \mathcal{D}$. For convenience, suppose we join f_0 to f_1. Let $r \in [0, 1]$ be the first time for which f_t is not one-one, and suppose $f_r(p) = f_r(q)$ for some $p \neq q$.

Since f_r is smooth and r is the first time f_r is not one-to-one, the maximum principle for minimal surfaces implies that $f_r(M) = \widehat{M}$ an embedded surface and that f_r, considered to be a map $f_r \colon M \to \widehat{M}$, is a covering space of $\widehat{M}$. Since the hyperelliptic points $\mathcal{H}$ of M are at the half lattice points of ^{3}, f_r is one-to-one on the set of hyperelliptic points. But these points are precisely the zeros of Gaussian curvature of $f_r(M)$. Then $f_r^{-1}(f_r(\mathcal{H}))$ are zeros of Gaussian curvature of M and so $\mathcal{H} = f_r^{-1}(f_r(\mathcal{H}))$, which implies f_r is not one-to-one on $\mathcal{H}$. This contradiction implies that each minimal immersion in Part 1 of the theorem is an embedding.

Part 3 of the theorem now follows directly from parts 2. and 3. of Proposition 7.2. $\square$

Remark 7.2 *All of the classical examples of periodic minimal surfaces of genus three, except Schoen's gyroid surface, are members of the family described in Theorem 7.1.*

We now give explicit analytic formulae for the periodic minimal surfaces in the family V. Suppose $M \in V$ with Gauss map $G \colon M \to S^2 = \ \cup$

$\{\infty\}$. We may assume, after a possible rigid motion of the lifted surface in $\mathbb{R}^3$, that G is a branched cover of $\mathbb{C} \cup \{\infty\}$ with branch points $P = \{a_1, \ldots, a_4, a_5 = -1/\overline{a}_1, \ldots, a_8 = -1/\overline{a}_4\}$ in the complex plane and where the product $a_1 a_2 a_3 a_4$ is a positive real number.

In this case the plane curve of M is $y^2 = (z - a_1) \ldots (z - a_8)$. In this representation G is the meromorphic function $z\colon M \to \mathbb{C} \cup \{\infty\}$. If $\eta = (1/y)dz$ and $\omega = [(1 - z^2), (1 + z^2)i,\ 2z)\eta]$, then $f(z) = \int^z \omega\colon M \to J(M)$ induces the Jacobi map of M. The projections $f_1 = \mathrm{Re}(f)$ and $f_2 = \mathrm{Im}(f)$ are the two minimal embeddings described in Theorem 7.1.

The only classical example of a periodic minimal surface with no straight lines or plane lines of curvature was Schoen's gyroid surface. From Weierstrass, a surface containing such lines always admits an orientation-reversing isometry with a non-empty fixed point set. By Proposition 7.1. Lemma 7.5, and Theorem 7.1, most of the minimal surfaces $f\colon M_3 \to \mathbb{R}^3$ in the family V have no orientation-reversing isometry with a non-empty fixed point set.

8 Specialized results

Theorem 8.1 *Let M_3 be the abstract hyperelliptic surface branched over the eight roots of unity in $\mathbb{C} \cup \{\infty\} = S^2$. Then:*

1. *The plane curve of M_3 is $y^2 = x^8 - 1$.*

2. *There exists a conformal minimal embedding $f\colon M_3 \to \mathbb{R}^3$ and $f(M_3)$ is one of Schoen's crossed layers of parallels surfaces (CLP).*

3. *In the induced metric, there is an isometry $\tau\colon M_3 \to M_3$ of order eight such that $f \circ \tau\colon M_3 \to \mathbb{R}^3$ is the conjugate surface to f.*

4. *$J(M_3)$ is isogeneous to a product of rectangular elliptic curves, two of which are isomorphic.*

Proof. Part 1 follows from the uniqueness of a two-sheeted cover of S^2 with given branch locus.

The existence of a minimal embedding of M_3 in some ^{3} is guaranteed by Theorem 7.1. To check that M_3 is some variant of the classical Schoen CLP surface, we note that all of the CLP surfaces are all branched over antipodal points on the equator of S^2. By making appropriate proportional changes in the lengths of the sides of the generating module for the CLP surface, we find a surface branched over the eight roots of unity. See [53] for a picture and further discussion of the CLP surface.

By Proposition 7.1, rotation $R_z: S^2 \to S^2$ around the z-axis by $45°$ lifts to an isometry $\tau: M_3 \to M_3$. If $\tau \in S_f(M)$, then ^{3} would have an order eight linear self-congruence, an impossibility. In fact, $\tau^2 \in S_f(M)$; thus, by Proposition 7.1, $\tau^*(\omega = (\omega_1, \omega_2, \omega_3)^t) = iR_z(\omega)$, where $f = \int \omega: M_3 \to {}^3$. Hence, $f \circ \tau: M_3 \to {}^3$ is the conjugate surface to f (up to congruence).

If $f: M_3 \to J(M) = {}^3/L$ is the holomorphic lift of the CLP surface, then a calculation similar to the one in the proof of Theorem 6.2, shows $\{(a,0,0),\ (b,0,0)i,\ (0,a,0),\ (0,b,0)i,\ (0,0,c),\ (0,0,d)i\} \subset L$. It follows that $J(M_3)$ is isogeneous to a product of rectangular elliptic curves, two of which are isomorphic. $\square$

By taking the lift of the CLP surface described in Theorem 8.1 to its universal cover $\widetilde{M}$, we get a minimal immersion $\widetilde{f}: \widetilde{M} \to {}^3$. If $\widetilde{\tau}: \widetilde{M} \to \widetilde{M}$ is the lift of the order eight isometry τ, $\tau(z) = e^{2\pi i/8} z$, then $\widetilde{f} \circ \widetilde{\tau}$ is congruent to the conjugate surface of $\widetilde{f}$. (Note τ does not lift to natural covering space $\widehat{M} \subset {}^3$ of M.) The next corollary is a simple consequence of this discussion.

Corollary 8.1 *There exists an orientation-preserving isometry $\widetilde{\tau}: \widetilde{M} \to \widetilde{M}$ of a nonproper, simply-connected, triply-periodic minimal surface $\widetilde{f}: \widetilde{M} \to {}^3$ such that $\widetilde{\tau}$ does not extend to a congruence for any associate surface $\widetilde{f}_\theta$. In fact, $\widetilde{\tau}$ takes straight lines on $\widetilde{f}(\widetilde{M})$ to plane lines of curvature.*

Remark 8.1 *The above corollary contrasts sharply with our earlier rigidity results:*

1. *Any* proper *triply-periodic surface in ^{3} is "rigid" (Theorem 5.3);*

2. *An orientation-reversing isometry of a simply-connected minimal surface in ^{3} always extends to a congruence for some associate surface (Theorem 5.5, 3.).*

Combining Theorems 8.1 and 8.2, we can prove the following rather surprising corollary.

Corollary 8.2 *The Schwarz diamond surface can be deformed through minimal surfaces of genus three in flat three-tori to its conjugate surface, the Schwarz primitive surface.*

Proof. In the family V of Theorem 7.1, we can continuously deform the Schwarz diamond surface $\mathcal{D}$ to the CLP surface and join the conjugate CLP surface to the conjugate surface of $\mathcal{D}$. By Theorem 8.2, 3. we may identify the CLP surface with the conjugate CLP surface as point sets. Consider this deformation, via the Gauss map, as a path α_t of 4 pairs of antipodal points on S^2 with $\alpha_0 = \mathcal{D}$ and $\alpha_1 =$ CLP. Let $\widetilde{\alpha}_t$ be the path of conjugate surfaces. Since $\widetilde{\alpha}_1 =$ CLP, deforming $\mathcal{D}$ along α_t and then backwards along $\widetilde{\alpha}_t$, describes a method for continuously deforming the point set of $\mathcal{D}$ through minimal surfaces of genus 3 to the point set of the conjugate surface of $\mathcal{D}$. $\square$

The next theorem gives easily verifiable conditions for a periodic minimal surface in ^{3} or its conjugate surface to be a boundary in ^{3}.

Theorem 8.2 (Boundary Theorem)

1. *If $f\colon M^n \to {}^{n+1}$ is an embedded, orientable, minimal submanifold and $M^n \subset {}^{n+1}$ is not totally geodesic, then M^n is homologous to zero in ${}^{n+1}$.*

2. *Suppose $f\colon M^k \to {}^n$ is a submanifold. If the translation $T_q \in S^r_f(M^k)$, then M^k is a boundary in ^{n}. If k is even and $I_q \in S^r_f(M^k)$, then*

M^k is a boundary in $\ ^n$. (Recall I_p is inversion through p and T_q is translation by q.)

3. *If $f\colon M \to\ ^n_1$ and $f_{\pi/2}\colon M \to\ ^n_2$ are minimal surfaces, then $f(M)$ is a boundary in $\ ^n_1$ if and only if $f_{\pi/2}(M)$ is a boundary in $\ ^n_2$.*

Proof. Since $f = \int[f^*(dx_1), \dots, f^*(dx_{n+1})]\colon M^n \to\ ^{n+1}$ is minimal, and $M^n \subset\ ^{n+1}$ is not contained in a subtorus, the induced map on the harmonic forms, $f^*\colon H^1(\ ^{n+1}) \to H^1(M^n)$, is injective. This in turn implies that $f_*\colon H_1(M^n,\ ) \to H_1(\ ^{n+1},\ )$ is surjective. Thus, we may assume there exist $\gamma_i\colon S^1 \to M^n \subset\ ^{n+1}$, $1 \le i \le n+1$, representatives for a basis of $H_1(\ ^{n+1},\ )$. Since M^n is orientable, the normal bundle of M^n is trivial, and so we can push the curves γ_i off M^n. Hence, $[\gamma_i] \cap [M^n] = 0$ where $\cap$ denotes the intersection pairing on homology. Since $H_n(\ ^{n+1},\ )$ is a free abelian group and $\cap$ is a nondegenerate pairing on homology, M must be a boundary in $\ ^{n+1}$. This proves Part 1.

Suppose $T_q \in S^r_f(M^k)$, and $I = (i_1, \dots, i_k)$. Let $dx_I = dx_{i_1} \dots dx_{i_k}$. Then

$$\int_{f_*(M^k)} dx_I = \int_{f_*(M^k)} T_q^*(dx_I) = \int_{T_q * f_*(M^k)} dx_I =$$

$$\int_{f_*(T_q * M^k)} dx_I = -\int_{f_*(M^k)} dx_I.$$

Hence, $\int_M \alpha = 0$ for all $\alpha \in H^k(\ ^n)$. Since $H_k(\ ^n,\ )$ is free and $\int_M \alpha = 0$ for all $\alpha \in H^k(\ ^n)$, M^k must be a boundary in $\ ^n$. A similar argument for the inversion I_p in place of T_q completes the proof of Part 2.

Suppose $f\colon M \to\ ^n_1$ is expressed as $f = \mathrm{Re} \int[\omega_1, \dots, \omega_n]$, where $\omega_j = \alpha_j + i * \alpha_j \in H^{1,0}(M)$, and $f_{\pi/2} = \mathrm{Im} \int[\omega_1 \dots, \omega_n]\colon M \to\ ^n_2$. Then for all i, j with $1 \le i < j \le n$, $\int_{f_*(M)} dx_i dx_j = \int_M \alpha_i \wedge \alpha_j = \int_M *\alpha_i \wedge *\alpha_j = \int_{f_{\pi/2}*(M)} dy_i dy_j$. This implies $f(M)$ is a boundary in $\ ^n_1$ if and only if $f_{\pi/2}(M)$ is a boundary in $\ ^n_2$. $\square$

Since the conjugate surface $M \subset\ ^3$ of the Novious model is not embedded, it is difficult to visualize whether or not it is a boundary in $\ ^3$. However, as the next corollary shows, M is a boundary in $\ ^3$.

Corollary 8.3 *If $f\colon M \to {}^3_1$ is an embedded minimal surface and the conjugate surface $f_{\pi/2}\colon M \to {}^3_2$ exists, then $f_{\pi/2}(M)$ is a boundary in ${}^3_2$.*

Proof. This corollary follows immediately from Theorem 8.2, 3. $\square$

Remark 8.2 *Part 1 of the Theorem 8.2 can be generalized to some other interesting cases. In [40], Nagano and Smyth show that if $f\colon M^n \to N^{n+1}$ is a harmonic map between compact oriented Riemannian manifolds and the Ricci curvature of N^{n+1} is non-negative, then $f^*\colon H^1(N^{n+1}) \to H^1(M^n)$ is injective when M^n does not lie in the orthogonal trajectory of a parallel vector field. Our proof of the Theorem 8.2 shows that if f is an embedding, then $f(M^n)$ is a boundary in N^{n+1} when $f(M^n)$ satisfies this nondegeneracy condition.*

If $f, g\colon M^n \to {}^k$ are homotopic and are defined by integrating harmonic one-forms, then f and g differ by a translation. This result follows directly from the fact that homotopic maps induce the same mapping on cohomology and from the Hodge Theorem: *Every real cohomology class on a closed Riemannian manifold contains a unique harmonic form.* We conclude this chapter with several simple but interesting applications of the above observation. The following proposition and its corollaries are well known but we include them here for the sake of completeness.

Proposition 8.1

1. *If $f, g\colon M^n \to {}^k$ are homotopic isometric minimal immersions, then they differ by a translation.*

2. *If $f, g\colon M^n \to {}^k$ are homotopic isometric minimal immersions of a complex manifold with respect to two Kahler metrics $<>_1, <>_2$, then f differs from g by a translation.*

Proof. We note that the proofs of 1. and 2. follow directly from the above observation and from the fact that the harmonic one-forms on a Kahler manifold only depend on the complex structure of M. $\square$

Corollary 8.4 *If $f, g\colon M^n \to {}^k$ are homotopic holomorphic immersions, then they differ by a translation.*

Corollary 8.5 *If $f, g\colon M^2 \to {}^k$ are homotopic conformal minimal immersions, then they differ by a translation.*

In particular, the last corollary shows that during a deformation through non-congruent minimal surfaces in ^{3}, the conformal structure of the surfaces must change.

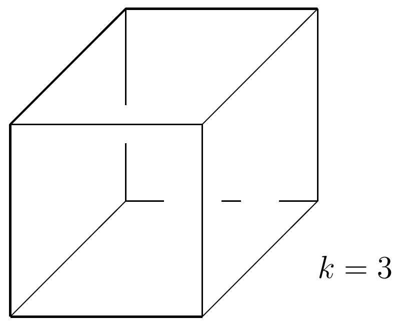
The following material deals with the next fundamental conjecture.

Conjecture 8.1 *Every flat ^{3} contains an embedded minimal surface of every genus except 0 and 2, an embedded nonorientable surface of $\chi = -2k$ for all $k \geq 1$ and an immersed nonorientable surface with $\chi = -(2k+1)$ for all $k \geq 1$.*

By our previous theorems, the immersion statement in the above conjecture is the best possible. The nonembedding theorem for nonorientable surfaces, due to Bredon and Wood [6] shows that the embedding part of the above conjecture is also the best possible. (Also recall Theorem 3.6 and Remark 3.2 for related statements.)

Example 8.1 *Construction of nonorientable periodic surfaces with $\chi = -(2k+1)$, $k > 1$.*

Let Γ be a polygonal curve inscribed on a cube with a stairway with k proportional steps.



Let $f\colon D \to \mathbb{R}^3$ be the unique solution to the classical Plateau problem for Γ and let $M \subset \mathbb{R}^3$ be the triply-periodic surface generated by continually reflecting the Plateau solution across the edges of Γ to get a triply-periodic minimal surface in $\mathbb{R}^3$. Then the quotient of M in $\mathbb{R}^3/L(M)$ induces a minimal immersion of a nonorientable surface with $\chi = -(2k+1)$ where k is the number of stairs. By carrying out the above construction on a rectangular figure rather than a cube one produces a nonorientable surface with $\chi = -(2k-1)$ in any $T^3 = \mathbb{R}^3/L$ where L is a lattice with an orthogonal basis. This construction together with Theorem 10.1 proves the next theorem.

Theorem 8.3 *If L is a lattice with an orthogonal basis, then $\mathbb{R}^3/L$ contains a minimally immersed surface of every topological type possible.*

Remark 8.3 *After making proportional changes in the lengths of the sides of the polygon described above and solving Plateau's problem for Γ, one might be able to show that some of the conjugate surfaces give rise to embedded orientable surfaces of every even genus $g > 2$ in all rectangular flat T^3.*

In Theorem 9.1, we will show that M_g will conformally minimally immerses fully in some flat T^{2g-1} when $g > 3$. Furthermore, the similar result holds for $g = 3$ if and only if M_g is hyperelliptic. These results lead one to ask the following:

Question 8.1 *What is the smallest dimension $n = n(g)$ such that every Riemann surface of genus g will conformally minimally immerse in in some flat n-torus?*

Another partial result in the direction of Conjecture 8.1 is the following.

Theorem 8.4 *Every flat $\ ^4$ contains an embedded minimal surface of every genus $g \geq 1$. Furthermore, these surfaces are holomorphic with respect to some orthogonal almost-complex structure J.*

Proof. The proof of this theorem is a straightforward examination of which flat $\ ^4$ are the Jacobians of some surface of genus 2 with respect to some orthogonal almost-complex structure J. The content of the theorem is that every flat $\ ^4$ is the Jacobian of some surface of genus 2. $\square$

Further analysis along the directions of the above theorem easily yields the following.

Theorem 8.5 *A generic flat $\ ^6$ contains a full, embedded, minimal surface of genus* 3.

9 The existence of minimal surfaces in T^k

In this section, we develop several techniques for proving the existence of compact minimal surfaces in higher-dimensional flat tori. In general, we will *not* assume that a given minimal surface $f\colon M \to \ ^k$ is full. We note that $f = \int(h_1, \ldots, h_k)\colon M \to \ ^k$ lies fully in a subtorus of dimension equal to the rank of $\{h_1, \ldots, h_k\} \subset H^1(M)$. For example, any minimal surface $f\colon M_g \to \ ^k$ is not full when $k > 2g$. It is important to keep this behavior in mind when considering results such as the following.

Proposition 9.1 *If $f\colon M \to \ ^k$ and $g\colon M \to \ ^r$ are conformal minimal immersions, then $(f, g)\colon M \to \ ^k \times \ ^r$ is a conformal minimal immersion.*

Proof. Since $f = \mathrm{Re}\int(\omega_1, \ldots, \omega_k)$ and $g = \mathrm{Re}\int(\alpha_1, \ldots, \alpha_r)$, where $\omega_i, \alpha_i \in H^{1,0}(M)$ and $\sum_{i=1}^k \omega_i^2 = 0 = \sum_{i=1}^r \alpha_i^2$, we have $\sum_{i=1}^k \omega_i^2 + \sum_{i=1}^r \alpha_i^2 = 0$. By the generalized Weierstrass representation,

$$(f, g) = \mathrm{Re}\int(\omega_1, \ldots, \omega_k, \alpha_1, \ldots, \alpha_r) \colon M \to {}^k \times {}^r$$

is a conformal minimal immersion. □

Note that if the f in Proposition 9.1 is an embedding, then so is (f, g) and (f, g) is an immersion if g is only a branched immersion.

Recalling some results from Section 5, we arrive at the following special result.

Corollary 9.1 *Let $f \colon M \to {}^3$ be an embedded minimal surface satisfying Property* **P**. *Then M fully embeds as a minimal surface in flat ^{k}, $3 \le k \le 6$.*

Proof. If $f = \mathrm{Re}(\omega_1, \omega_2, \omega_3) \colon M \to {}^3$ satisfies Property **P**, then by Theorem 5.4, 1., f lifts to a holomorphic immersion into ${}^3 = {}^3/L$ with an -simple lattice. Suppose after an orthogonal transformation that $\{v_1 = (a, 0, 0), v_2 = (b, d, 0), v_3 = (e, f, g), cv_1, cv_2, cv_3\}$ is a -simple basis for $L \otimes$. Then there exist holomorphic mappings $g = \int \omega_3 \colon M \to {}^1$ and $h = \int(\omega_2, \omega_3) \colon M \to {}^2$. By applying Proposition 9.1 and checking ranks, we get

1. $(f, g) \colon M \to {}^3 \times {}^1$ is a minimal embedding full in some ${}^4 \subset {}^3 \times {}^1$.

2. $(f, h) \colon M \to {}^3 \times {}^2$ is a minimal embedding full in some ${}^5 \subset {}^3 \times {}^2$.

□

The theorems in Section 6 show that many classical examples of surfaces satisfy the conditions of Corollary 9.1. These surfaces include the Schwarz $\mathcal{P}$ and $\mathcal{D}$ surfaces and the Novious model [46]. See [53] for a picture of the Novious model.

Lemma 9.1 *If* $v = (a_1, \ldots, a_n) \in {}^n$ *is non-zero, and if* $v \notin \widetilde{Q}_n = \{(x_1, \ldots, x_n) \in {}^n \mid \sum_{i=1}^n x_i^2 = 0\}$*, then there exists an* $n \times n$ *matrix* A *and a non-zero constant* $c \in$ *.*

1. $A^t A = cI$
2. $Av = e_1 = (1, 0, \ldots, 0)$.

Proof. We will prove the lemma by induction on the number z_v of non-zero a_i appearing in $v = (a_1, \ldots, a_n)$. If $z_v = 1$, then $v = (0, \ldots, 0,\ a_i,\ 0, \ldots, 0)$, and there exists a permutation matrix O with $Ov = (a_i,\ 0, \ldots, 0)$. $A = (1/a_i)\, O$ gives the required matrix in this case.

Assume now that the lemma holds for $z_v = k$. Since $v = (a_1, \ldots, a_n) \notin \widetilde{Q}_n$, then there is always a pair a_i, a_j, neither zero, with $a_i^2 + a_j^2 \neq 0$. Let O be a permutation matrix such that $Ov = (a_i, a_j, a_3, \ldots, a_n)$, and let $A_1 = (1/a_i)\, O$ so that $A_1 v = (1, b_1, \ldots, b_n)$. Let

$$A_2 = \begin{pmatrix} \frac{1}{1+b_1^2} & \frac{b_1}{1+b_1^2} & 0 \\ \frac{-b_1}{1+b_1^2} & \frac{1}{1+b_1^2} & 0 \\ 0 & 0 & I_{(n-2)\times(n-2)} \end{pmatrix},$$

where $I_{(n-2)\times(n-2)}$ is the $(n-2)\times(n-2)$ identity matrix. If $A_3 = A_2 A_1$, then $A_3 v$ has one more zero than v, and $A_3^t A_3 = c_1 I$. Hence, $A_3 v \notin \widetilde{Q}_n$. By induction, there exists an A_4 with $A_4 A_3 v = e_1 = (1, 0, \ldots, 0)$ such that $(A_4 A_3)^t\, (A_4 A_3) = c_2 I$. This concludes the proof of the lemma. $\square$

Lemma 9.2 *If the canonical curve of* M_g *is contained in some quadric in* ${}^{g-1}$*, then there exist a flat torus* ${}^{2g-1}$ *and a full conformal minimal immersion* $f\colon M_g \to {}^{2g-1}$*.*

Proof. Suppose that the canonical curve of M_g is contained in a quadric of rank n. After a change of basis, we may assume $c(M_g) \subset Q_n$ where Q_n is the standard quadric of rank n. Hence, there exist n linearly independent holomorphic one-forms $\omega_1, \ldots, \omega_n$ with $\sum_{i=1}^n \omega_i^2 = 0$. Pick $\gamma \in H_1(M_g, \ )$ so that $v = (a_1, \ldots, a_n) = \int_\gamma \omega = (\omega_1, \ldots, \omega_n)^t \notin Q_n$. This is always possible since the rational period vectors $P \otimes \ = \{\int_\gamma \omega \mid \gamma \in H_1(M_g, \ )\} \otimes \ $ are dense in $\ ^n$, and hence there is an $\ell \in P \otimes \ $ with $\ell \notin Q_n$. Since Q_n is defined by a homogeneous polynomial, no multiple of ℓ is a member of Q_n. Therefore, the required $\gamma \in H_1(M_g, \ )$ exists.

By the previous lemma, there is a matrix A with $A\omega = \alpha$, $\sum_{i=1}^n \alpha_i^2 = 0$ and $\int_\gamma \alpha = (1, 0, \ldots, 0)^t$. Now complete $\alpha_1, \ldots, \alpha_n$ to an ordered basis $\alpha_1, \ldots, \alpha_g$ of $H^{1,0}(M_g)$ in such a way that $\int_\gamma (\alpha_1, \alpha_2, \ldots, \alpha_g)^t = (1, 0, \ldots, 0)^t$.

Define $\beta_k \in H^{1,0}(M)$ by

$$\begin{array}{lcll} \beta_1 & = & \alpha_1 & \\ \beta_{2k-2} & = & \sqrt{2}\alpha_k & \\ \beta_{2k-1} & = & i\alpha_k & \text{for } 2 \leq k \leq n \\ \beta_{2k-2} & = & \alpha_k & \\ \beta_{2k-1} & = & i\alpha_k & \text{for } n+1 \leq k \leq g \quad . \end{array}$$

By the definition of β_k, we have $\sum_{i=1}^{2g-1} \beta_i^2 = 0$.

It is now straightforward to show that

$$f(p) = \mathrm{Re}[i \int_{p_0}^p \beta = (\beta_1, \ldots, \beta_{2g-1})] \colon M_g \to \ ^{2g-1}/L = \ ^{2g-1},$$

where $L = \mathrm{Re}\{i \int_\delta \beta \mid \delta \in H_1(M_g, \ )\}$, is a full conformal minimal immersion. We note that L is a lattice, since $i \int_\gamma \beta = (i, 0, \ldots, 0)$ is purely imaginary and because f is full. Note also that there are no branch points since whenever $\{h_1, \ldots, h_{2g-1}\} \subset H^1(M_g)$ are linearly independent, they never have a common zero. $\square$

Theorem 9.1

1. *A surface of genus 3 will minimally immerse fully in a flat* $\ ^5$ *if and only if it is hyperelliptic.*

2. *A surface of genus $g \geq 4$ will always minimally immerse fully in $\;^{2g-1}$. (Note: We already proved in Section 3 that a surface of genus 2 will never minimally immerse in $\;^{3}$.)*

Proof. By Theorem 4.2, 2., any minimal surface of genus 3 that minimally immerses in $\;^{5}$ is hyperelliptic. Since the canonical curve of a hyperelliptic surface of genus 3 is contained in a quadric, Lemma 9.2 proves Part 1. Since $c(M_g)$ is always contained in a quadric when $g \geq 4$, Lemma 9.2 also implies Part 2. $\square$

Definition 9.1 *We will say that a minimal immersion $f\colon M \to \;^{k}$* lifts *to $\;^{r}$ if there exist a commutative diagram:*

$$\begin{array}{ccc} \;^{r} & \overset{i}{\hookrightarrow} & \;^{k}/L \\ \widetilde{f}\ \uparrow & & \downarrow\ \mathrm{Re} \\ M & \overrightarrow{\ f\ } & \;^{k} \end{array},$$

where $L = \{\int_\gamma(\omega_1, \ldots, \omega_k) \mid \gamma \in H_1(M, \;)\}$, $\widetilde{f} = \int(\omega_1, \ldots, \omega_k)$, and $f =$ $\mathrm{Re}\,\widetilde{f}$. *Equivalently, L is a discrete subgroup of $\;^{k}$ of rank $= 2r$.*

Theorem 9.2 *If $f\colon M \to \;^{k}$ is a branched minimal surface that lifts to a holomorphic map into $\;^{r}$, then for each s, $k \leq s \leq 2r$, M conformally minimally immerses fully in some $\;^{s}$.*

Proof. Our proof is by induction on s, with $k \leq s \leq 2r$. If $s = k$, $f\colon M \to \;^{k}$ is the conformal minimal immersion which lifts to a complex torus $\;^{r}$.

Now assume that the theorem holds for $s = n$ with $k \leq n < 2r$, and we will show it true for $s = n + 1$. By our induction hypothesis, we have a conformal minimal immersion f that factors through the following commutative diagram:

$$\begin{array}{ccc} \;^{r} & \overset{i}{\hookrightarrow} & \;^{n}/L \\ \widetilde{f}\ \uparrow & & \downarrow\ \mathrm{Re} \\ M & \overrightarrow{\ f\ } & \;^{n} \end{array},$$

where L is a discrete subgroup of rank $= 2r$.

$$\text{Let } A = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & B \end{pmatrix},$$

where B is a 2×2 matrix such that $BB^t = I_{2\times 2}$, B is not real, and B is close to $I_{2\times 2}$. Note that $AA^t = I_{(n+1)\times(n+1)}$.

Now pick a basis $\{v_1, v_2, \ldots, v_{2r-n}\}$ for $\ker(\mathrm{Re}\,|L) \otimes \ \subset i\ ^n$. After an orthogonal transformation, we may assume that $v_j \in i\ ^j \subset i\ ^n$ for each j, $1 \le j \le 2r-n$. If the original v_i are appropriately chosen, the harmonic form components appearing in $\mathrm{Re}(A(\omega_1, \ldots, \omega_n, 0))$ will span an $(n+1)$-dimensional subspace of $H^1(M)$.

By the generalized Weierstrass representation, there is a *full* non-proper minimal immersion $g = \mathrm{Re}\int A(\omega_1, \ldots, \omega_n, 0) \colon \widetilde{M} \to\ ^{n+1}$ on the universal cover $\widetilde{M}$ of M. The map g factors as

$$\begin{array}{ccccc} & V^n & \overset{i}{\hookrightarrow} & \ ^{n+1} & \\ h & \uparrow & & \downarrow & \mathrm{Re} \\ & \widetilde{M} & \overrightarrow{\ g\ } & \ ^{n+1} & \end{array},$$

where V^n is the complex subspace spanned by $L' = \{A\int_\gamma(\omega_1, \ldots, \omega_n, 0) \mid \gamma \in H_1(M,\ )\}$.

Since we chose A so that $\{v_1, v_2, \ldots, v_{2r-n-1}\}$ spanned the kernel of $\mathrm{Re} \circ A \colon\ ^{n+1} \to\ ^{n+1}$, $A(L')$ is a lattice in $\ ^{n+1}$. Therefore, g induces a conformal minimal immersion $f' \colon M \to\ ^{n+1} =\ ^n/\,\mathrm{Re}(L')$. (It is an immersion for B sufficiently close to $I_{2\times 2}$.) It follows that f' lifts to $A(\ ^r) \subset\ ^{n+1}/A(L)$, and by induction, this finishes the proof of the theorem. $\square$

The hypothesis for the next corollary is satisfied for all known classical examples of periodic minimal surface in $\ ^3$.

Corollary 9.2 *Suppose an associate surface to $f = \mathrm{Re}\int(\omega_1, \ldots, \omega_k) \colon M \to\ ^k_1$ is periodic; i.e., there exists $f_\theta \colon M \to\ ^k_2$ for some θ. If $r =$ the number of -independent ω_i's defining f, then for each s, $k \le s \le 2r$, M conformally minimally immerses fully in a $\ ^s$.*

Proof. By Theorem 5.2, f lifts to a holomorphic map into $\ ^r$. Now apply Theorem 9.2. $\square$

Corollary 9.3 *If $f: M_3 \to \ ^3$ is a minimal surface of genus 3, then M_3 minimally immerses fully in a $\ ^4$, $\ ^5$, and $\ ^6$.*

Proof. In this case, f lifts to $J(M_3)$, which has complex dimension 3. Theorem 9.2 implies the corollary. $\square$

10 Proof of the main existence theorem

The main result of this section is that every flat three-torus contains an infinite number of examples in the family V of genus 3 periodic surfaces described in Theorem 7.1. What we will show is that every $\ ^3$ contains an infinite sequence $\overline{\Sigma}_1, \ldots, \overline{\Sigma}_k, \ldots$ of nonorientable embedded minimal surfaces with Euler characteristic $\chi = -2$ and such that $\lim_{i \to \infty} Area(\overline{\Sigma}_i) = \infty$. The existence of these new examples is based on an abstract mini-max type proof that is independent of the work in previous sections.

Let $T^3 = \ ^3 / \ ^3$ where $\ ^3$ is the integer lattice in $\ ^3$ and let $F \subset T^3$ be the quotient torus of the x_1x_2-plane in T^3. Let $\sigma: T^3 \to T^3$ be the diagonal translation of order 2. Consider the surface Σ in T^3 obtained from F and $F + (0, 0, \frac{1}{2})$ by taking their connected sum along vertical line segments ℓ and $\sigma(\ell)$. Do this so that $\sigma(\Sigma) = \Sigma$ and let $\overline{\Sigma} = \overline{\Sigma}/\sigma \subset \overline{T}^3 = T^3/\sigma$. See Figure 3 below for a picture of Σ. Let $\overline{F}$ denote the image of F in $\overline{T}^3$. It is straightforward to check that $\overline{\Sigma}$ is isotopic in $\overline{T}^3$ to $\mathcal{P}/\sigma$, where $\mathcal{P}$ is the Schwarz primitive surface. Let $R: T^3 \to T^3$ be the rotation around the diagonal vector $(1, 1, 1)$ by 120° and note that R commutes with σ. Let $\overline{R}: \overline{T}^3 \to \overline{T}^3$ denote the associated quotient linear isometry. Since $\mathcal{P}$ is invariant under R, $\mathcal{P}/\sigma$ is invariant under $\overline{R}$. Hence, $\overline{R}(\overline{\Sigma})$ is isotopic to $\overline{\Sigma}$.

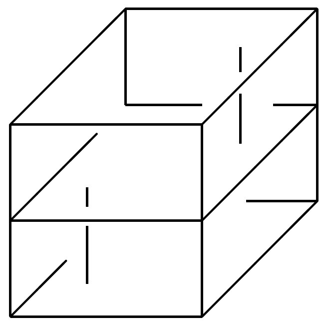


Figure 3:

If $\widetilde{F}$ is a subtorus of $\overline{T}^3$ that represents the same $_2$-homology class as $\overline{F}$, then there exists a linear automorphism $\overline{L}: \overline{T}^3 \to \overline{T}^3$ with $\overline{L}(\overline{F}) = \widetilde{F}$ and such that $\overline{L}$ lifts to a $L: T^3 \to T^3$. Note that the linear automorphisms of T^3 are generated by R and Δ, Δ defined by $e_1 \to e_1 + e_2$, $e_2 \to e_2$, $e_3 \to e_3$. Note also that $\overline{\Sigma}$ is isotopic to $\overline{\Delta}(\overline{\Sigma})$ where $\overline{\Delta}: \overline{T}^3 \to \overline{T}^3$ is the associated quotient map. Recall that the surface $\overline{\Sigma}$ is obtained from $\overline{F}$ by taking and adding a handle along a vertical line segment in $\overline{T}^3$. Since $\overline{\Delta}(\overline{F}) = \overline{F}$ and $\overline{\Delta}$ preserves the vertical, $\overline{\Delta}(\overline{\Sigma})$ is isotopic to $\overline{F}$ by adding a vertical handle. Hence $\overline{\Delta}(\overline{\Sigma})$ is isotopic to $\overline{\Sigma}$. Since $\overline{L}$ is a composition of products of $\overline{\Delta}$ and R, both of which preserve the isotopy class of $\overline{\Sigma}$, the isotopy class of $\widetilde{F}$ can be obtained by a single surgery on $\overline{\Sigma}$. This proves the following lemma.

Lemma 10.1 *Suppose $\widetilde{F} \subset \overline{T}^3$ is a subtorus that represents the $_2$-homology class of $\overline{\Sigma}$. Then $\widetilde{F}$ is isotopic to a surface obtained by doing surgery on $\overline{\Sigma}$.*

This lemma will be used in the proof of the following theorem.

Theorem 10.1 *Let ^{3} be an arbitrary flat three-torus. Then there exists an infinite sequence of embedded minimal surfaces $\Sigma_1, \ldots, \Sigma_k, \ldots$ in the family V given in Theorem 7.1. Furthermore, the Σ_k can be choosen to have area greater than k.*

Before proving Theorem 10.1, we briefly outline the main idea. After lifting to two-sheeted covers of flat three-tori, it is sufficient to prove that our original ^{3} contains an infinite sequence $\overline{\Sigma}_1, \ldots, \overline{\Sigma}_k, \ldots$ of nonorientable minimal surfaces the family $\widetilde{V}$ described in Theorem 7.1 and such that $Area(\overline{\Sigma}_k) > k$.

After composing with a linear isomorphism of ^{3} with T^3, consider $\overline{\Sigma}$ to be contained in ^{3}. By Lemma 10.1, the surface $\overline{\Sigma}$ is "isotopic by surgery" to any fixed flat two-torus in the $_2$-homology class of $\overline{\Sigma}$. Suppose $_1$ and $_2$ are two flat tori in ^{3} which represent the $_2$-homology class of $\overline{\Sigma}$ but represent different -homology classes. Furthermore, choose $_1$ and $_2$ so that $Area(\ _1) > Area(\ _2) > n$. Recall that flat two-tori in ^{3} are strong local minima to the area functional on the space of $_2$-currents representing a $_2$-homology class and the local minima $_1$, $_2$ can be joined by a path, Σ_t, $1 > t > 2$, such that the Σ_t limit as varifolds to $_1$, $_2$, respectively, as $t \to 1$ or 2 and for any such path $Area(\Sigma_t) > Area(\ _2)$. The general of mini-max principle for minimal surfaces, first developed by Morse-Tompkins [39] and Shiffman [56], states that in the space of paths Σ_t joining the local area minima $_1$, $_2$, there should exist an a path whose maximum area surface $\widetilde{\Sigma}$ has area which is minimal over all such paths. The surface $\widetilde{\Sigma}$ is then called a *mini-max* surface and it is an unstable minimal surface. Since $\widetilde{\Sigma}$ is a mini-max, its area is at least as big as $Area(\ _2) > n$.

While the above guiding mini-max principle is easy to state, in principle it is usually difficult to apply because the spaces involved are infinite dimensional. By working with paths of harmonic maps, we shall reduce the question of finding the minimal $\widetilde{\Sigma}$ to finding the required mini-max on a finite dimensional space which is the Teichmoeller space of $\overline{\Sigma}$. The following proof of the existence of $\widetilde{\Sigma}$ given below was found around 1980. It seemed clear to the author at that time that the results of Meeks, Simon and Yau [34] and Simon [57] could be used to prove the existence of the minimax $\widetilde{\Sigma}$. This second approach has been made rigorous by Hass, Pitts and Rubenstein ([17] and [49]) and yields a variation of our proof on the existence of

$\widetilde{\Sigma}$. Their approach is much more general than ours and works in the case of general Riemannian three-manifolds. This completes our outline of the proof of Theorem 10.1.

Proof of Theorem 10.1. Let $\mathcal{T}$ denote the Teichmoeller space associated to the nonorientable surface M of Euler characteristic $\chi = -2$. Let $L: T^3 = {}^3/{}^3 \to {}^3$ be a linear isomorphism and let $f: M \to {}^3$ be the map induced by $L \circ i$ where $i: M \to \overline{\Sigma} \subset T^3$ is a parametrization of $\overline{\Sigma}$. For every $M_\tau \in \mathcal{T}$, there is a corresponding harmonic map $f_\tau: M_\tau \to {}^3$ in the homotopy class obtained by integrating the harmonic forms that represent the three cohomology classes represented by the closed forms $f^*(dx_1)$, $f^*(dx_2)$, and $f^*(dx_3)$. Consider the energy functional $E: \mathcal{T} \to {}^+$ where $E(M_\tau)$ is the energy of f_τ. The critical points of E correspond to the harmonic, branched, conformal maps. A straightforward calculation shows that $f: M_\tau \to {}^3$ induces an isomorphism of $H_1(M_\tau, \ )/(\text{Torsion})$ to $H_1({}^3, \ )$. Corollary 1 in [31] states that the f_τ that are critical points of E, i.e. the minimal f_τ, are smooth embeddings. (In our case this regularity property also follows from the fact that the E-critical f_τ are smooth embeddings via Theorem 7.1.)

Let $\overline{\mathcal{T}}$ denote the compactification of $\mathcal{T}$ and recall that $\overline{\mathcal{T}}$ is a compact, complex analytic variety. An estimate of Schoen and Yau (see, for example, Lemma 3.1 in [54]) (also see Sachs and Uhlembeck [52]) implies: If $M_\tau \in \mathcal{T}$ and M_τ contains a simple closed geodesic α in the hyperbolic metric on M_τ and $f_\tau(\alpha)$ is homotopically nontrivial in ^{3}, then the energy of f_τ is very large when the length of α is very small. Suppose $\gamma \subset M$ is a simple closed curve and $f(\gamma)$ is homotopically nontrivial in ^{3}. It follows that if M_{τ_i} converges to a "surface" $N \in \partial\overline{\mathcal{T}}$ so that γ becomes homotopically trivial in the limit surface, then the energy $E(M_{\tau_i}) \to \infty$ as $i \to \infty$. On the other hand, since ^{3} has nonpositive curvature, if $\{M_{\tau_j}\} \subset \mathcal{T}$ is an infinite sequence diverging to $\partial\mathcal{T}$ and $E(M_{\tau_j})$ is uniformly bounded, then a subsequence of the $f(M_{\tau_i})$ converges to a harmonic map of a "surface" $\widetilde{f}: N \to {}^3$ where $N \in \partial\mathcal{T}$. (This is clear since the total energies of the maps f_{τ_i} are uniformly bounded and there can be no "bubbling" since ^{3} cannot contain a minimal 2-sphere. Also see the proof of the solution of the free boundary value problem in [36] where a similar situation occurs.) From the simple topology of M, we conclude that N is a torus obtained by a surgery on a simple closed curve on M and $\widetilde{f}$ is a linear harmonic map on a flat two-torus $N = {}^2$. Note that $\widetilde{f}$ is well defined up to translation as a limit of some subsequence of the f_{τ_j}.

Let $\partial\widetilde{\mathcal{T}}$ denote the tori in $\partial\mathcal{T}$ which are obtained by surgery on a simple closed curve on M that is homotopically trivial under f in ^{3}. Let $\mathcal{T} = \mathcal{T} \cup \partial\widetilde{\mathcal{T}}$. By the above discussion, we can extend E to a continuous proper map $\widetilde{E}\colon \widetilde{\mathcal{T}} \to {}^+$.

Note that each path component C of $\partial\widetilde{\mathcal{T}}$ naturally corresponds to a Teichmoeller space of conformal structures on a flat two-torus. Since the associated harmonic maps of ^{2} into ^{3} are linear, it is well known that $\widetilde{E}|C$ is proper and has a unique critical point (the global minima for $\widetilde{E}|C$) corresponding to the conformal structure of the image flat torus in ^{3}. By Lemma 10.1 and the earlier discussion, $\partial\widetilde{\mathcal{T}}$ contains an infinite number of path components and $\widetilde{E}$ restricted to each path component has a unique critical point. Each of these boundary critical points is a minima for $\widetilde{E}$ on its path component. Note also that this minima of $\widetilde{E}|C$ is also a local minima for $\widetilde{E}$, since a flat torus is area minimizing. Also by Lemma 10.1, the values of $\widetilde{E}$ at these local minima for $\widetilde{E}$ on $\partial\widetilde{\mathcal{T}}$ are arbitrarily large since the flat two-tori representing the same $_2$-homology class as $f(M) = \overline{\Sigma}$, have arbitrarily large area in different -homology classes.

We will prove that $\widetilde{\mathcal{T}}$ contains a sequence of critical values $M_{\tau_i} \in \mathcal{T}$ of $\widetilde{E}$ and $E(M_{\tau_i}) \to \infty$ as $i \to \infty$. This existence proof will also show that the harmonic map $f_{\tau_i}\colon M_{\tau_i} \to {}^3$ is harmonic, branched, conformal, minimal surface.

First we construct a map $F\colon \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ with the property that $\widetilde{E}(M_\tau) \leq \widetilde{E}(F(M_\tau))$ with equality if and only if f_τ is conformal. Suppose $f_\tau\colon M_\tau \to {}^3$ is an immersion, then $F(M_\tau)$ is the surface $f(M_\tau)$ with the pulled back conformal structure. Since the energy of a map is greater than or equal to twice its area with equality if and only if the map is conformal and since harmonic maps into ^{3} minimize energy for a fixed conformal structure, $\widetilde{E}(M_\tau) \leq \widetilde{E}(F(M_\tau))$ with equality if and only if f_τ is conformal. Suppose $M_\tau \in \mathcal{T}$. The main theorem in [31] concerning the regularity of the Albanese map of a nonorientable surface into its Albanese variety states, in our case, that: A given harmonic map $f_\tau \colon M_\tau \rightarrow {}^3$ is either a smooth embedding or else is a rank 1 map which is also a smooth embedding outside of some simple closed curve and this simple closed curve collapses to a point in ^{3} (since $f_\tau \colon M \rightarrow {}^3$ is in this case the Albanese map of M_τ). In particular, it still makes sense to pull back the "conformal" structure on M_τ to get a new conformal structure $F(M_\tau)$ and $E(F(M_\tau) \leq E(M_\tau)$. It follows that the fixed points of F are the branched conformal minimal immersions related to $\widetilde{E}$.

We now prove the existence of the required critical points M_{τ_i}. Choose a sequence of local minimal $N_i \in \partial\mathcal{T}$ of $\widetilde{E}$ such that $\widetilde{E}(N_i) \rightarrow \infty$ as $i \rightarrow \infty$. Since N_i is a strong local minimal of $\widetilde{E}$, for ε sufficiently small the compact component $C(i,T)$ of $\widetilde{E}^{-1}([0, T = \widetilde{E}(N_i) + \varepsilon])$ containing N_i contains no other fixed point of F. There is a smallest $t_i > \widetilde{E}(N_i)$, such that N_i is in the component $C(1,t_i)$ of $\widetilde{E}^{-1}([0,t_i])$ that contains N_1. Let $M_{\tau_i} \in \partial C(i,t_i) \cap \partial C(1,t_i)$. Note that $F(C(i,t)) \subset C(i,t)$. Also, note that $F(M_{\tau_i}) \in \mathrm{Int}(C(i,t))$, unless $M_{\tau_i} \in \mathcal{T}$ and f_{τ_i} is conformal. Similarily, $F(M_{\tau_i}) \in \mathrm{Int}(C(1,t_i))$ unless f_{τ_i} is conformal. Since $\mathrm{Int}(C(i,t_i)) \cap \mathrm{Int}(C(1,t_i)) = \emptyset$, f_{τ_i} must be conformal. Since the f_{τ_i} is conformal, $E(f_{\tau_i}) = 2 \cdot Area(f_{\tau_i})$, and $E(f_{\tau_i}) > \widetilde{E}(N_i) > i$, we conclude that $Area(f_{\tau_i}) > \frac{i}{2}$. As remarked at the beginning of the proof, the critical points of E are smooth embeddings. The existence of these f_{τ_i} completes the proof of Theorem 10.1. $\square$

Corollary 10.1 *Let M be either a closed orientable surface of odd genus or a closed nonorientable surface of odd Euler characteristic. Then every flat three-torus ^{3} contains an infinite sequence $\{M_1, M_2, \ldots\}$ of embedded* *minimal surfaces, each diffeomorphic to M. Furthermore, this sequence can be choosen so that $Area(M_n) > n$ for all n.*

Proof. Let M and ^{3} be given and fixed. Elementary covering space theory and the proof of Theorem 10.1 imply that ^{3} is a finite cover of another flat three-torus $\widehat{\ }^3$ and such that ^{3} contains a surface $\Sigma \in V$ of Euler characteristic $\chi = -2$ and such that the lift $\widehat{\Sigma}$ of Σ to ^{3} is diffeomorphic to M. (See Figure 2 at the end of Section 3 for a representative possible picture of Σ.) Recall from the proof of Theorem 10.1 that $\widehat{\ }^3$ contains an infinite sequence, $\Sigma_1, \Sigma_2, \dots$ of embedded minimal surfaces that are homotopic to Σ and such that $Area(\Sigma_n) > n$. Hence, the lifts $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2, \dots$ to ^{3} yield a sequence of embedded minimal surfaces that are diffeomorphic to M and such that $Area(\widetilde{\Sigma}_n) > n$. This completes the proof of the corollary. □

11 Recent developments and further credits

The most important applications of periodic minimal surfaces arise from problems in classical function theory. More precisely, periodic holomorphic minimal surfaces in ^{n} play a fundamental role in the classical theory of Riemann surface theory via Abel's theorem. Namely, a necessary and sufficient condition for a divisor $\Sigma(p_i - z_i)$ on a closed Riemann surface M to be a divisor of a meromorphic function is for this sum of points to equal the identity element in the Jacobian of M when we consider M to be a subset of this torus. In fact, the Jacobian $J(M)$ is naturally isomorphic to the abelian group $Div_0/Div(\mathcal{M}^*)$ where Div_0 are the divisors of degree zero on M and $Div(\mathcal{M}^*)$ are the divisors of the not identically zero meromorphic functions on M.

A nonorientable "Riemann surface" M with a conformal-anticonformal structure no longer has holomorphic forms but it still has harmonic one-forms. By integration of a basis of $H^1(M)$, one obtains a harmonic map $f\colon M \to A(M)$ where $f(p) = \int^p (h_1, \dots, h_n)$ and $A(M) = {}^n/P$ where $P =$

$\{\int_\gamma (h_1, \ldots, h_n) \mid \gamma \in H_1(M)\}$. As shown in [31], f is not always one-to-one, however, it is a smooth one-to-one immersion when f is a branched minimal immersion. We used this regularity theorem in the proof of Theorem 10.1. The minimal surfaces in $\mathbb{R}^n$ that arise from compact nonorientable minimal surfaces in their Albanese varieties yield a rich collection of nonholomorphic examples of n-periodic minimal surfaces in $\mathbb{R}^n$.

Micallef has proven some interesting results on stable n-periodic minimal surfaces. First he has shown that an orientable stable minimal surface in a four-dimensional flat torus is always holomorphic with respect to some orthogonal almost-complex structure on the torus [37]. He has also shown that a full, stable, hyperelliptic minimal surface in a flat torus is holomorphic with respect to some orthogonal almost-complex structure [38]. On the other hand, a simple dimension calculation shows that there are stable minimal surfaces of genus four in most flat eight dimensional tori that are not holomorphic with respect to any orthogonal almost-complex structure.

The classical theory of triply-periodic minimal surfaces in $\mathbb{R}^3$, rather than being motivated by problems in classical function theory, is inspired by the intricacy and geometric beauty of the classical examples. In [53] Schoen published a number of images of models of the classical examples as well as many new examples that he found. Based on Schoen's manuscript, Karcher has recently written a short treatise on some of the examples of Schoen. Karcher's manuscript includes computer graphics images of some of these examples as well as some related, interesting, new examples [23]. Also, see the publications [21] and [22] of Karcher for a variety of new examples of periodic minimal surfaces.

More recently, interest in triply-periodic minimal surfaces has arisen from their appearance as dividing surfaces or as surface interfaces in physical problems. For example, in materials such as crystals, that arrange themselves in a triply-periodic structure, triply-periodic minimal surfaces appear in surprising contexts. For instance, a solid state physicist might model a Fermi surface (or equipotential surface) in a crystal of salt by the Schwarz primitive surface $\mathcal{P}$ that closely approximates the actual Fermi surface geometrically.

Triply-periodic minimal surfaces have been used to model liquid crystals that arise in oil-water-surfactant microemulsions and to model dividing surfaces in certain microemulsions of block copolymers. Since one might expect the area or energy of these surfaces to be stable with respect to smooth variations, it is natural to ask why minimal surfaces arise in such problems when it is well known that there are no stable orientable minimal surfaces (other than the plane) in ^{3}. (See [11] or [13].) A possible reason for their occurrence is that there do exist orientable minimal surfaces in flat three-tori that are stable with respect to volume preserving variations. Since every physically allowable variations for a surface interface preserves volume of the complements, it is not so surprising that a least energy interface might be representable by a minimal surface. The Schwarz $\mathcal{P}$ and $\mathcal{D}$ surfaces, as well as A. Schoen's gyroid surface, have all been shown to be stable in their natural flat three-tori with respect to volume preserving variations (see the work of M. Ross [51]). This stability property for these three surfaces helps to explain why all three apparently occur as surface interfaces in liquid crystal experiments. The recent appearance of the Schwarz $\mathcal{D}$, as well as other periodic minimal surfaces, as dividing surfaces in block copolymer microemulsions has been hailed as major breakthrough in the field of polymer science and has led to the hope of the creation of materials with new physical properties. In general, these dividing surfaces are assumed to be close to constant mean curvature periodic surfaces, not just zero mean curvature ones. For a discussion of this recent work see the work of Anderson, Henke, Hoffman, Martin and Thomas [1][2]. (Also see [3].)

In the remainder of this section we will bring some of the material that appears in the earlier sections up to date as well as to give credits to others for overlaps with their work. Starting at the beginning, one must of course give credit to Riemann [50] and Schwarz [55] who gave the first explicit examples of triply-periodic minimal surfaces and who contributed the basic theoretical results on which the subject began. Some of the elementary results of Section 3, such as Theorem 3.1 and Corollaries 3.1, 3.2 and 3.3 were almost certainly known to these two fathers of the subject. In any case, these results, as well as a majority of the remaining results of Section 3, were found independently by Nagano and Smyth at about the same time as this author. They also obtained a few of the results in some latter sections and we refer the interested reader to their papers listed in our bibliography for further results in the subject.

Recently, some of the results in Section 5 on the rigidity of periodic minimal surfaces has been adapted to deal with the case of complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$. These adaptations by Hoffman and Meeks [19] have proven to be important in the construction of new examples of complete embedded minimal surfaces of finite topology in $\mathbb{R}^3$.

Choi, Meeks and White [10] were able to generalize our rigidity theorem for triply-periodic minimal surfaces in $\mathbb{R}^3$ (Theorem 5.3). They prove that any properly embedded minimal surface $f: M \to \mathbb{R}^3$ with more than one end is, up to congruence, the unique isometric minimal immersion of M into $\mathbb{R}^3$. In particular, the symmetry group of such an M is equal to its isometry group. Also, Meeks and Rosenberg [33] have been able to show that any properly embedded, doubly-periodic minimal surface is, in the same way, rigid. These results give rise to a conjecture that is closely related to our earlier Conjecture 5.1 on the rigidity of nonsimply connected, properly embedded, minimal surfaces.

Conjecture 11.1 *The symmetry group of a properly embedded, minimal surface in* $\mathbb{R}^3$ *is equal to its isometry group.*

The above conjecture represents an apparently deep property of properly embedded minimal surfaces. All known properly embedded minimal surfaces M in $\mathbb{R}^3$ either have more than one end, and so are rigid by the earlier stated result of Choi-Meeks-White, or $M/\operatorname{Sym}(M)$ has finite topology and $\operatorname{Sym}(M)$ is infinite. The previous statement and the next theorem show that Conjecture 11.1 holds for all known examples of minimal surfaces.

Theorem 11.1 *Suppose M is a properly embedded, minimal surface in $\mathbb{R}^3$ with infinite symmetry group and such that the quotient of M by its symmetry group has finite topology. Then, the symmetry group of M is equal to the isometry group of M.*

The proof of the above theorem, which we postpone for the present moment, is based on a recent result of Meeks and Rosenberg [32]. Roughly stated, their theorem says that if M is a properly embedded minimal surface in $\mathbb{R}^3$ and if a fundamental region of a discrete subgroup of $\mathrm{Sym}(M)$ has finite topology, then this fundamental region also has finite total curvature. If M is a properly embedded, connected surface in $\mathbb{R}^3$ with infinite symmetry group and M is not a surface of revolution, then $\mathrm{Sym}(M)$ contains a discrete subgroup G that acts freely on $\mathbb{R}^3$ as a group of translations or G is generated by a screw motion symmetry. Thus, M is either triply-periodic, doubly-periodic or one-periodic (generated by a translation or screw motion symmetry). One can then study M by studing the quotient surface $M/G \subset \mathbb{R}^3/G$. The rough statement of the Meeks-Rosenberg theorem given earlier then translates into the following:

Theorem 11.2 (Meeks-Rosenberg) *A properly embedded minimal surface in a complete nonsimply connected flat three manifold has finite total curvature if and only if it has finite topology.*

Meeks and Rosenberg also show that complete minimal surface of finite total curvature in a flat three-manifold can be expressed in terms of analytic data on the conformal compactification of the surface which, by a theorem of Huber [20], is a compact Riemann surface. We shall call this data the Weierstrass data for the surface. It follows from Theorem 11.2 and the theory of finite total curvature minimal surfaces in nonsimply connected, flat, three-manifolds $\mathbb{R}^3/G$, as developed in [32], that if $M \subset \mathbb{R}^3/G$ has finite topology, then the asymptotic behavior of M is very restricted. In fact, if G is generated by a screw motion that is a rotation around the z-axis by an angle θ, $0 \leq$

$\theta \leq \pi$, then each end of $M \subset \mathbb{R}^3/G$ is asymptotic to a helicoid end, a flat end or to a vertical half annulus in $\mathbb{R}^3/G$. When an end of M is asymptotic to a vertical flat end, as is Scherk's singly-periodic surface, the end is called a *Scherk type end* of M. With this background discussion completed, we prove the following restatement of Theorem 11.2.

Theorem 11.3 *Let M be a connected, properly embedded, minimal surface in $\mathbb{R}^3$, invariant under an infinite discrete group G of isometries of $\mathbb{R}^3$. If M/G has finite topology, then every isometry of M extends to an isometry of $\mathbb{R}^3$.*

Proof. Suppose M is not a plane. We can assume (by taking a finite index subgroup) that G acts freely on M in an orientation preserving manner and $N = \mathbb{R}^3/G$ is isometric to $\mathbb{T}^3$, $\mathbb{T}^2 \times \mathbb{R}$ or $\mathbb{R}^3/S_\theta$ where S_θ is a right hand screw motion with axis being the z-axis and with rotation angle θ. We will also assume that G is a maximal subgroup satisfying these properties.

If N is isometric to $\mathbb{T}^3$ or $\mathbb{T}^2 \times \mathbb{R}$, then M is doubly-periodic and this case of Theorem 11.3 was proved in [33]. Hence, we can assume $N = \mathbb{R}^3/S_\theta$.

We know all the ends of M/G are of the same type: planar, Scherk or helicoidal type ends. We consider each possibility separately. We use the following statement proved in [10]: *M is rigid (i.e. any two isometric minimal immersions of M in $\mathbb{R}^3$ are congruent) provided there is a plane in $\mathbb{R}^3$ whose intersection with M contains a compact cycle along which M is transverse to the plane.*

First suppose all the ends of M/G are planar. Then these planar ends lift to parallel planar ends in $\mathbb{R}^3$, and one can choose a plane between two consecutive planar ends of M that meets M transversally in a compact set. Hence Theorem 11.3 is true in this case.

Now assume the ends of M/G are Scherk type ends. Choose a vertical plane P in $\mathbb{R}^3$, not parallel to any end, and such that P is tangent to M at exactly one point p. P/G is an annulus and intersects M/G is an analytic compact cycle with a singularity at p. It is compact because P is not parallel to any Scherk end. Now it's not hard to see that a small parallel translation of P yields a transverse intersection of P/G with M/G, that contains a simple closed curve null homotopic on P/G. This curve lifts to ^{3} and Theorem 11.3 follows here as well.

Assume now that M/G has helicoidal type ends. By Theorem 5.5, 6., $S_f^\circ(M)$ is a normal subgroup of $Iso(M)$ where $f\colon M \to {}^3/G$ denotes the inclusion map. Suppose $h\colon M \to M$ is an isometry. Since $S_f^\circ(M)$ is normal, h is the lift of a map $\widehat{h}\colon M/S_f^\circ(M) \to M/S_f^\circ(M)$. Since the ends of M/G are helicoidal, all elements of $S_f^\circ(M)$ are screw motions or rotations with a common axis and we may assume that this axis is the z-axis. In particular, $S_f^\circ(M)$ is abelian and G is a normal subgroup with $S_f^\circ(M)/G$ a cyclic group. Hence, the quotient map $Q\colon M/G \to M/S_f^\circ(M)$ is a finite cyclic branched covering space. Let E be some collection of annular ends representatives of $M/S_f^\circ(M)$ that are disjoint from the branch locus of Q and such that E is invariant under $\widehat{h}$. Let $\widetilde{Q}\colon \widetilde{E} \to E$ denote the cyclic covering space where $\widetilde{E} = Q^{-1}(E)$ and $\widetilde{Q} = Q|\widetilde{E}$. Note $\widetilde{E}$ contains annular end representatives of M/G. Since M/G is embedded in ${}^3/G$, $S_f^\circ(M)/G$ acts on $\widetilde{E}$ so that for every component C of $\widetilde{E}$, $\widetilde{Q}|C\colon C \to \widetilde{Q}(C)$ has constant degree. Elementary covering space theory (for cyclic covering spaces) implies that $\widehat{h}|E$ lifts to a map $\widetilde{h}\colon \widetilde{E} \to \widetilde{E}$.

Suppose $(\omega_1, \omega_2, \omega_3)$ are the Weierstrass forms for $f\colon M \to {}^3$. Since M has "horizontal helicoidal type ends", $h^*(\omega_3) = e^{i\theta}\omega_3$ for some θ. Let $\widetilde{\omega}_3$ denote the quotient form of ω_3 on M/G. Since M/G has helicoidal type ends and M/G is embedded in ${}^3/G$, $\widetilde{\omega}_3$ has a local expression on any component C of $\widetilde{E}$, conformally parametrized by the punctured unit disk, of the form $(a_{-1}/z + a_0 + a_1 z + \ldots)dz$ where a_{-1} is a fixed nonzero imaginary number whose absolute value is independent of C. (See [32] for this representation.) Note that $\{a_{-1}, -a_{-1}\}$ is an invariant of $\widetilde{\omega}_3$, i.e., is not dependent on the parametrization of C by the unit disk. By considering $\widetilde{h}|C\colon C \to \widetilde{h}(C)$ to be a conformal parametrization of $\widetilde{h}(C)$ by C, the invariance of a_{-1} up to sign shows that $(\widetilde{h}|\widetilde{E})^*(\widetilde{\omega}_3|\widetilde{E}) = \pm\widetilde{\omega}_3|\widetilde{E}$. This implies that $h^*(\omega_3) = \pm\omega_3$ and hence $e^{i\theta} = \pm 1$. This means that h extends to an isometry of ^{3}. The proof of the nonorientation preserving case, $h^*(\omega_3) = e^{i\theta}\overline{\omega}_3$, is similar to the orientation preserving case and will be left to the reader. This completes the proof of the theorem in the case that M/G has helicoidal type ends. Since every end of M/G is planar, Scherk or helicoidal, Theorem 11.3 is proved. $\Box$

Remark 11.1 *The embeddedness assumption in Theorems 11.2 and 11.3 is clearly needed as is demonstrated by the following example. Let H_θ denote the associate surface of the helicoid H for any $\theta \neq \pm\frac{\pi}{2}, 0, \pi$. Then the orientation reversing symmetries of H_θ do not extend to ^{3} even though H_θ is a properly immersed minimal surface invariant under a group of translations and the quotient surface by these translations has finite topology.*

In [30] Meeks proved that a nonflat embedded minimal surface $M \subset {}^3$ of genus g separates ^{3} into two 1-handlebodies. Such a decomposition of a closed three-manifold is called a Heegaard splittings of genus g. In general, a flat three-manifold can have topologically distinct Heegaard splittings of the same genus. However, Meeks conjectured that two Heegaard splittings of ^{3} of the same genus are isotopic (see Conjecture 1 in ([28])). It would follow from this result that if M_1, M_2 are two triply-periodic minimal surfaces in ^{3}, then there exists a homeomorphism of ^{3} taking M_1 to M_2. Meeks conjectured this result as well. Recently, C. Frohman [14] proved a deep and general topological result that implies Meeks' conjecture for triply-periodic minimal surfaces in ^{3}. Shortly after Frohman proved his theorem, M. Boileau and J. P. Otal [5] proved the stronger result that Heegaard splittings of ^{3} are unique up to isotopy. Finally, based on Frohman's study of one-ended Heegaard splittings of ^{3}, Frohman and Meeks proved that a one-ended minimal surface in ^{3} is topologically unknotted [15]. In otherwords, two such homeomorphic minimal surfaces differ by a homeomorphism of ^{3}. A topological result of Callahan, Hoffman and Meeks [9] states that a doubly or triply-periodic minimal surface must have one end and, as a consequence of the Frohman-Meeks theorem, any two such surfaces differ by a homeomorphism of $\mathbb{R}^3$. Another related topological uniqueness theorem was proved by Meeks and Yau [35]. Their theorem states that a properly embedded, minimal surface in $\mathbb{R}^3$ of finite topology is unknotted.

References

[1] D. Anderson. *Studies in the Microstructure of Microemulsions*. PhD thesis, University of Minnesota, June 1986.

[2] D. Anderson, C. Henke, D. Hoffman, and E. L. Thomas. Periodic area-minimizing surfaces in block copolymers. *Nature*, 334(6184):598–601, 1988. August 18 issue.

[3] D. M. Anderson, H. T. Davis, J. C. C. Nitsche, and L. E. Scriven. Periodic surfaces of prescribed mean curvature. to appear in Advances in Chem. Phys.

[4] S. Andersson, S. T. Hyde, K. Larsson, and S. Lidin. Minimal surfaces and structures: From inorganic and metal crystals to cell membranes and biopolymers. *Chemical Reviews*, 88:221–242, 1988.

[5] M. Boileau and J. P. Otal. Scindements de Heegaard du tore ^{3}. preprint.

[6] G. E. Bredon and J. W. Wood. Non-orientable surfaces in orientable 3-manifolds. *Inventiones Math.*, 7:83–110, 1969.

[7] E. Calabi. Quelque applications de l'Analyse complex aux surfaces d'Aire minima. In H. Rossi, editor, *Topics in Complex Manifolds*. Les Presses de l'Université de Montréal, 1968.

[8] M. Callahan, D. Hoffman, and W. H. Meeks III. Embedded minimal surfaces with an infinite number of ends. *Inventiones Math.*, 96:459–505, 1989.

[9] M. Callahan, D. Hoffman, and W. H. Meeks III. The structure of singly-periodic minimal surfaces. *Inventiones Math.*, 99:455–481, 1990.

[10] T. Choi, W. H. Meeks III, and B. White. A rigidity theorem for properly embedded minimal surfaces in ^{3}. *Journal of Differential Geometry*. March, 1990.

[11] M. do Carmo and C. K. Peng. Stable minimal surfaces in ^{3} are planes. *Bulletin of the AMS*, 1:903–906, 1979.

[12] W. Fischer and E. Koch. On 3-periodic minimal surfaces. *Zeitschrift für Kristallographie*, 179:31–52, 1987.

[13] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33:199–211, 1980.

[14] C. Frohman. The topological uniqueness of triply-periodic minimal surface in ^{3}. *Journal of Differential Geometry*, 31:277–283, 1990.

[15] C. Frohman and W. H. Meeks III. The topological uniqueness of complete one-ended minimal surfaces and Heegaard surfaces in ^{3}. preprint.

[16] R. C. Gunning. *Lectures on Riemann Surfaces*. Princeton University Press, Princeton, N.J., 1966.

[17] J. Hass, J. Pitts, and H. Rubenstein. personal communication.

[18] D. Hoffman and W. H. Meeks III. Minimal surfaces based on the catenoid. to appear in the Mathematical Monthly.

[19] D. Hoffman and W. H. Meeks III. Embedded minimal surfaces of finite topology. *Annals of Mathematics*, 131:1–34, 1990.

[20] A. Huber. On subharmonic functions and differential geometry in the large. *Commentari Mathematici Helvetici*, 32:181–206, 1957.

[21] H. Karcher. Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.*, 62:83–114, 1988.

[22] H. Karcher. Construction of minimal surfaces. *Surveys in Geometry*, pages 1–96, 1989.

[23] H. Karcher. The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions. *Manuscripta Math.*, 64:291–357, 1989.

[24] S. Lang. *Abelian Varieties*. Springer-Verlag, New York, 1983.

[25] H. B. Lawson, Jr. *Lectures on Minimal Submanifolds*. Publish or Perish Press, Berkeley, 1971.

[26] W. H. Meeks III. *The Geometry and the Conformal Structure of Triply-Periodic Minimal Surfaces in* ^{3}. PhD thesis, University of California, Berkeley, 1975.

[27] W. H. Meeks III. The conformal structure and geometry of triply-periodic minimal surfaces in ^{3}. *Bulletin of the AMS*, 83:134–136, 1977.

[28] W. H. Meeks III. *Lectures on Plateau's Problem*. Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, 1978.

[29] W. H. Meeks III. A survey of the geometric results in the classical theory of minimal surfaces. *Bol. Soc. Brasil Mat.*, 12:29–86, 1981.

[30] W. H. Meeks III. The topological uniqueness of minimal surfaces in three-dimensional Euclidean space. *Topology*, 20:389–410, 1981.

[31] W. H. Meeks III. Regularity of the Albanese map for nonorientable surfaces. *Journal of Differential Geometry*, 29(1):345–352, 1989.

[32] W. H. Meeks III and H. Rosenberg. The geometry of periodic minimal surfaces. preprint.

[33] W. H. Meeks III and H. Rosenberg. The global theory of doubly periodic minimal surfaces. *Inventiones Math.*, 97:351–379, 1989.

[34] W. H. Meeks III, L. Simon, and S. T. Yau. The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature. *Annals of Math.*, 116:221–259, 1982.

[35] W. H. Meeks III and S. T. Yau. The topological uniqueness theorem of complete minimal surfaces of finite topological type. preprint.

[36] W. H. Meeks III and S. T. Yau. Topology of three-manifolds and the embedding problems in minimal surface theory. *Annals of Math.*, 112:441–484, 1980.

[37] M. Micallef. Stable minimal surfaces in Euclidean space. *Journal of Differential Geometry*, 19(1):57–84, 1984.

[38] M. Micallef. Stable minimal surfaces in flat tori. In *Complex differential geometry and nonlinear differential equations*, volume 49. American Math. Soc., 1984.

[39] M. Morse and C. Tompkins. Existence of minimal surfaces of general critical type. *Annals of Math.*, 40:443–472, 1939.

[40] T. Nagano and B. Smyth. Minimal varieties and harmonic maps in tori. *Commentari Mathematici Helvetici*, 5:249–265, 1975.

[41] T. Nagano and B. Smyth. Sur les surfaces minimales hyperelliptiques dans un tore. *C. R. Acad. Sci. Paris*, 280:1527–1529, 1975.

[42] T. Nagano and B. Smyth. Minimal surfaces in tori by Weyl groups. I. *Proceedings of the AMS*, 61(1):102–104, 1976.

[43] T. Nagano and B. Smyth. Minimal surfaces in tori by Weyl groups. II. In *Proceedings of the U.S.-Japan Seminar on Minimal Submanifolds*, Tokyo, 1977. Kaigashi Publ. Ltd. Tokyo.

[44] T. Nagano and B. Smyth. Periodic minimal surfaces. *Commentari Mathematici Helvetici*, 35:29–55, 1978.

[45] T. Nagano and B. Smyth. Periodic minimal surfaces and Weyl groups. *Acta Math.*, 145(1–2):1–27, 1980.

[46] E. R. Neovius. *Bestimmung zweier spezieller periodischer Minimalflächen*. Akad. Abhandlungen, Helsingfors, 1883.

[47] J. C. C. Nitsche. *Lectures on Minimal Surfaces*, volume 1. Cambridge University Press, 1988.

[48] R. Osserman. *A Survey of Minimal Surfaces*. Dover Publications, New York, 2nd edition, 1986.

[49] J. Pitts and H. Rubenstein. Applications of minimax to minimal surfaces and the topology of three-manifolds. In *Proceedings for the Center for Mathematical Analysis*, pages 137–170, Canberra, Australia, 1987. Australian National University.

[50] B. Riemann. *Ouevres Mathématiques de Riemann*. Gauthiers-Villars, Paris, 1898.

[51] M. Ross. *Stability properties of complete two-dimensional minimal surfaces in Euclidean space*. PhD thesis, Stanford University, 1989.

[52] J. Sachs and K. Uhlembeck. Minimal immersions of closed Riemannian surfaces. *Transactions of the AMS*, 271:639–652, 1982.

[53] A. Schoen. Infinite periodic minimal surfaces without self-intersections. Technical Note D-5541, NASA, Cambridge, Mass., May 1970.

[54] R. Schoen and S. T. Yau. Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature. *Annals of Math.*, 110:127–142, 1979.

[55] H. A. Schwarz. *Gesammelte Mathematische Abhandlungen*, volume 1. Springer, Berlin, 1890.

[56] M. Shiffman. The Plateau problem for non-relataive minima. *Annals of Math.*, 40(4):834–854, 1939.

[57] L. Simon. On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary metric. to appear.