Motion of Level Sets by Mean Curvature I

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Abstract

to be added later¸

1 Introduction

We set forth in this paper rigorous justification of a new approach for defining and then investigating the evolution of a hypersurface in . ^{n} moving according to its mean curvature. This problem has been long studied using parametric methods of differential geometry: see, for instance, Gage []-[], Gage-Hamilton [], Grayson [], Huisken []-[], etc., etc. In this classical setup, we are given at time 0 a smooth hypersurface Γ_0 which is, say, the connected boundary of a bounded open subset of . ^{n}. As time progresses we allow the

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surface to evolve, by moving each point in the opposite direction to the mean curvature vector, at a velocity equal to $(n-1)$ times the absolute value of the mean curvature at that point. Assuming this evolution is smooth, we define thereby for each $t > 0$ a new hypersurface Γ_t. The primary problem is then to study geometric properties of $\{\Gamma_t\}_{t>0}$ in terms of Γ_0.

For the case $n = 2$ this program has been successfully carried out in great detail: see [], [], etc. For $n \geq 3$, however, it is fairly clear that even if Γ_0 is smooth, a smooth evolution as envisioned above cannot exist beyond some initial time interval. Imagine for instance Γ_0 to be the boundary of a "dumbbell" shaped region in ^{3}, as illustrated.

In view of [] and numerical calculations of Sethian [], we expect that as time evolves, the surface will smoothly evolve (and shrink) up until a critical time $t_* > 0$ when the two ends pinch off, as drawn.

After this time, the classical motion via mean curvature is undefined. In addition, if it were possible to define the subsequent motion in some reasonable way, we expect Γ_t for $t > t_*$ to comprise two pieces which pull apart at time t_*. If this were so, then Γ_t would have changed topological type.

This possibility suggests inherent problems in the classical differential geometric approach of regarding Γ_0 as a parameterized surface: the parametrization will in general develop singularities .

What is needed is an alternative description of the evolution for all times $t > 0$, sufficiently general so as to allow for the possible onset of singularities and attendant topological complications. To our knowledge there have been two different such undertakings, by Brakke [] and by Osher-Sethian []. Brakke's dissertation [] recasts the mean curvature motion problem (even in arbitrary codimension) into the setting of varifold theory from geometric measure theory (cf. Allard []). Brakke defines and then constructs an appropriate generalized varifold solution , which is defined for all time (although it may vanish after a finite time.) He then deduces many geometric properties and under an additional density assumption establishes partial regularity. The principal drawback seems to be the lack of any uniqueness assertion.

A completely different viewpoint is to be found in the paper [] by Osher and Sethian. Their approach, recast slightly, is this. Given the initial hypersurface Γ_0 as above, select some function $g\colon \mathbb{R}^n \to \mathbb{R}$ so that

$$\Gamma_0 = \{x \in \mathbb{R}^n | g(x) = 0\}. \tag{1.1}$$

Consider then the parabolic PDE

$$u_t = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}) u_{x_i x_j} \qquad \text{in } \mathbb{R}^n \times [0, \infty) \tag{1.2}$$

$$u = g \qquad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{1.3}$$

for the unknown $u = u(x,t) (x \in \mathbb{R}^n, t \geq 0)$. Now the PDE (1.2) says that each *level set of* u *evolves according to its mean curvature*, at least in regions where u is smooth and its spacial gradient Du does not vanish. Consequently, focusing our attention on the set $\{u = 0\}$, it seems reasonable in view of (1.1), (1.2) to *define*

$$\Gamma_t \equiv \{x \in \mathbb{R}^n \mid u(x,t) = 0\} \tag{1.4}$$

for each time $t > 0$. Osher and Sethian [] introduce various techniques to study (1.2) and related PDE's numerically, thereby to track computationally the evolution of Γ_0 into $\Gamma_t (t \geq 0)$. (Notice by the way that our utilizing (1.1)–(1.3) amounts, in the language of fluid mechanics, to adopting an Eulerian viewpoint, as opposed to the Lagrangian, parametric viewpoint of classical differential geometry ; see also equation (2.2) below).

Our purpose here is to provide theoretical justification for this approach. The undertaking is analytically subtle principally because the mean curvature evolution equation (1.2) is nonlinear, degenerate, and indeed even undefined at points where $Du = 0$. In addition, it is not so clear that our definition (1.3) is independent of the choice of initial function g verifying (1.1). We will resolve these problems by introducing an appropriate definition of a weak solution for (1.2), inspired by the notion of so-called "viscosity solutions" of nonlinear PDE as in Crandall-Lions [], Crandall-Evans-Lions [], Lions [], Jensen [], etc. We then prove that there exists a unique weak solution of (1.2) and further that definition (1.3) is then independent of the choice of initial function g satisfying (1.1). We additionally check that $\{\Gamma_t\}_{t\geq 0}$ so defined agrees with the classical notion of motion via mean curvature, over any time interval for which the latter exists. Finally we employ the PDE (1.2) to deduce assorted geometric properties of $\{\Gamma_t\}_{t\geq 0}$.

The main theoretical advantage of (1.1)–(1.3) as compared with Brakke's varifold methods seems to us to be the uniqueness assertion: the set Γ_t is unambiguously defined by (1.3) once we have a uniqueness assertion for the PDE (1.2). The primary disadvantage is that our techniques work only in codimension one.

We hope to establish in a forthcoming companion paper a partial regularity theorem for $\{\Gamma_t\}_{t\geq 0}$.

Our paper is organized as follows. In Section 2 we motivate and introduce our definition of weak solution for (1.2) and in Section 3 prove the uniqueness of a weak solution. Section 4 establishes the existence of a weak solution to (1.2). In Section 5 we verify the independence of the definition (1.3) on the choice of g. Section 6 contains a consistency check that the definition (1.3) agrees with the classical motion by mean curvature, if and so long as the latter exists. Sections 7 and 8 contain various geometric assertions, examples of pathologies and conjectures.

2 Definition and Elementary Properties of Weak Solution

2.1 Heuristics

We start with a formal derivation of the mean curvature evolution PDE (1.2). For this, suppose temporarily $u = u(x,t)$ is a smooth function whose spatial gradient $Du = (u_{x_1}, \ldots, u_{x_n})$ does not vanish in some open region O of $\mathbb{R}^n \times (0, \infty)$. Assume further that each level set of u smoothly evolves according to its mean curvature, as described in §1. We focus our attention onto any one such level set, and for definiteness consider the zero sets

$$\Gamma_t \equiv \{x \in \mathbb{R}^n \mid u(x,t) = 0\} \quad (t \geq 0). \tag{2.1}$$

Let $\nu = \nu(x,t)$ be a smooth unit normal vector field to $\{\Gamma_t\}_{t \geq 0}$ in O. Then

$$\frac{1}{n-1} \, div(\nu)\nu$$

is the mean curvature vector field. Thus if we fix $t \geq 0$, $x \in \Gamma_t \cap O$, the point x evolves according to the nonautonomous ODE

$$\begin{cases} \dot{x}(s) = -[div(\nu)\nu]\,(x(s),s) \quad (s > t) \\ x(t) = x \end{cases} \tag{2.2}$$

As $x(s) \in \Gamma_s \, (s \geq t)$, we have

$$u(x(s),s) = 0 \quad (s > t),$$

and so

$$0 = \frac{d}{ds}\, u(x(s),s) = -[(Du \cdot \nu)\, div(\nu)]\,(x(s),s) + u_t(x(s),s).$$

Setting $s = t$, we discover

$$u_t = (Du \cdot \nu)\, div(\nu) \ \text{ at } (x,t).$$

Choosing then

$$\nu \equiv \frac{Du}{|Du|}, \tag{2.3}$$

it follows that

$$u_t = |Du| \, div\left(\frac{Du}{|Du|}\right) = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} \ \text{ at } (x,t). \tag{2.4}$$

Similar reasoning demonstrates this PDE to hold throughout the region O.

Now, conversely, assume in some region O u is a smooth solution of (2.4) with Du nonvanishing. Fix $t > 0$, $x \in \Gamma_t \cap O$ and solve then the ODE (2.2), (2.3). Since u solves (2.4), we deduce as above

$$u(x(s), s) = 0 \quad (s > t)$$

Consequently the zero sets, and similarly all the level sets, of u evolve in O according to their mean curvatures.

Since the motion of any level set thus depends only upon its own geometry, and not that of any other level set, our PDE (2.4) should be invariant under an arbitrary relabelling of these sets. Thus if Ψ: $\to$ is smooth with

$$\Psi' \neq 0$$

we expect that

$$v = \Psi(u)$$

will also be a solution of (2.4) in the region O. A direct calculation verifies this. Hence we see that an *arbitrary monotonic function of a solution is still a solution*, this in strong contrast to the situation for uniformly parabolic PDE's. Indeed, we may informally interpret (2.4) as being somehow "uniformly parabolic along each level set", but being also totally degenerate across different level sets.

2.2 Weak solutions

The foregoing heuristics done with, we turn now to the full mean curvature evolution equation:

$$u_t = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2})\, u_{x_i x_j} \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{2.5}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}, \tag{2.6}$$

the function $g: \mathbb{R}^n \to \mathbb{R}$ being given. We want to define a notion of weak solution to (2.5). Since however, the right hand side of the PDE cannot be put into divergence form, we are not able to define a weak solution by means of formal integration by parts of derivatives onto a smooth test function (as for instance in Bombieri, De Giorgi, Giusti [, Section 1]). We will instead follow Evans[], Lions [], Jensen [], etc. and define our weak solution in terms of pointwise behavior with respect to a smooth test function. The primary difficulty will be to modify extant theory to cover the possibility that Du may vanish.

Definition 2.1 *A function $u \in C(\mathbb{R}^n \times [0,\infty)) \cap L^\infty(\mathbb{R}^n \times [0,\infty))$ is a* weak subsolution *of (2.5) provided for each $\phi \in C^\infty(\mathbb{R}^{n+1})$ if*

(2.7) $\quad u - \phi$ *has a local maximum at a point* $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$

then

$$\begin{cases} \phi_t \le (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\, \phi_{x_i x_j} \quad \textit{at } (x_0,t_0) \\ \\ \textit{if } D\phi(x_0,t_0) \ne 0, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \phi_t \le (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j} \quad \textit{at } (x_0,t_0) \\ \\ \textit{for some } \eta \in \mathbb{R}^n \textit{ with } |\eta| \le 1, \quad \textit{if } D\phi(x_0,t_0) = 0. \end{cases} \tag{2.9}$$

Definition 2.2 *A function* $u \in C(\mathbb{R}^n \times [0,\infty)) \cap L^\infty(\mathbb{R}^n \times [0,\infty))$ *is a* weak supersolution *of (2.5) provided for each* $\phi \in C^\infty(\mathbb{R}^{n+1})$, *if*

(2.10) $u - \phi$ *has a local minimum at a point* $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$

then

$$\text{(2.11)} \qquad \begin{cases} \phi_t \geq (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\, \phi_{x_i x_j} \quad \text{at } (x_0,t_0) \\ \\ \text{if } D\phi(x_0,t_0) \neq 0, \end{cases}$$

and

$$\text{(2.12)} \qquad \begin{cases} \phi_t \geq (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \quad \text{at } (x_0,t_0) \\ \\ \text{for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \neq 1, \quad \text{if } D\phi(x_0,t_0) = 0. \end{cases}$$

Definition 2.3 *A function* $u \in C(\mathbb{R}^n \times [0,\infty)) \cap L^\infty(\mathbb{R}^n \times [0,\infty))$ *is a* weak solution *of (2.5) provided u is both a weak subsolution and a weak supersolution.*

As preliminary motivation for these definitions, suppose u is a smooth function on $\mathbb{R}^n \times (0,\infty)$ satisfying

$$u_t \leq (\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2})\, u_{x_i x_j}$$

whenever $Du \neq 0$. Our function u is thus a classical subsolution of (2.5) on $\{Du \neq 0\}$. Suppose now $Du(x_0,t_0) = 0$. Assume additionally there are points $(x_k,t_k) \to (x_0,t_0)$ for which $Du(x_k,t_k) \neq 0\, (k = 1,\ldots)$. Then

$$u_t \leq (\delta_{ij} - \eta^k_j\eta^k_j)u_{x_i x_j} \quad \text{at } (x_k,t_k)$$

for $\eta^k \equiv \frac{Du(x_k,t_k)}{|Du(x_k,t_k)|}$. Since $|\eta^k| = 1\, (k = 1,\ldots)$ we may as necessary pass to a subsequence so that $\eta^k \to \eta$ in $\mathbb{R}^n$, $|\eta| = 1$. Passing to limits above, we find

$$u_t \leq (\delta_{ij} - \eta_i\eta_j)u_{x_i x_j} \ \text{ at } (x_0,t_0).$$

If, on the other hand, there do not exist such points $\{(x_k, t_k)\}_{k=1}^{\infty}$, then $Du = 0$, and so $D^2u = 0$ and u is a function of t only, near (x_0, t_0). Moving to the edge of the set $\{Du = 0\}$, we see that u is a nonincreasing function of t. Thus

$$u_t \leq (\delta_{ij} - \eta_i \eta_j) u_{x_i x_j} \quad \text{at} \;\; (x_0, t_0)$$

for any $\eta \in \mathbb{R}^n$.

Further motivation for our definition of weak solution, and in particular, explanation as to why we assume only $|\eta| \leq 1$ in (2.9), (2.12), will be found in subsection 2.4.

2.3 An equivalent definition

We write $z = (x, t)$, $z_0 = (y_0, t_0)$.

Definition 2.4 *A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a weak subsolution of (2.5) if whenever $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and*

$$\text{(2.13)} \qquad \begin{cases} u(x,t) \leq u(x_0, t_0) + p \cdot (x - y_0) + q(t - t_0) \\ +\frac{1}{2}(z - z_o)^T R(z - z_0) + o(|z - z_0|^2) \quad \text{as } z \to z_0. \end{cases}$$

for some $p \in \mathbb{R}^n$, $q \in \mathbb{R}$, $R = (r_{ij}) \in S^{n+1 \times n+1}$, then

$$\text{(2.14)} \qquad q \leq (\delta_{ij} - \frac{p_i p_j}{|p|^2})\, r_{ij} \quad \text{if } p \neq 0$$

and

$$\text{(2.15)} \qquad q \leq (\delta_{ij} - \eta_i \eta_j)\, r_{ij} \quad \text{for some } |\eta| \leq 1, \;\; \text{if } p = 0.$$

Definition 2.5 *A function $u \in C(\mathbb{R}^n \times [0, \infty)) \cap L^\infty(\mathbb{R}^n \times [0, \infty))$ is a weak supersolution of (2.5) if whenever $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and*

$$\text{(2.16)} \qquad \begin{cases} u(x,t) \geq u(x_0, t_0) + p \cdot (x - x_0) + q(t - t_0) \\ +\frac{1}{2}(z - z_0)^T R(z - z_0) + o(|z - z_0|^2) \quad \text{as } z \to z_0 \end{cases}$$

for some $p \in \mathbb{R}^n$, $q \in \mathbb{R}$, $R = (r_{ij}) \in S^{n+1 \times n+1}$, *then*

$$q \geq (\delta_{ij} - \frac{p_i p_j}{|p|^2})\, r_{ij} \qquad \text{if } \; p \neq 0 \tag{2.17}$$

and

$$q \geq (\delta_{ij} - \eta_i \eta_j)\, r_{ij} \qquad \text{for some } \; |\eta| \leq 1, \quad \text{if } \; p = 0. \tag{2.18}$$

Theorem 2.6 *Definitions 2.1 and 2.4 are equivalent, and Definitions 2.2 and 2.3 are equivalent.*

2.4 Properties of weak solutions

Theorem 2.7

(i) *Assume* u_k *is a weak solution of (2.5) for* $k = 1, 2, \ldots$ *and (2.13)* $u_k \to u$ *boundedly and locally uniformly on* $\mathbb{R}^n \times [0, \infty)$. *Then* u *is a weak solution.*

(ii) *An analogous assertion hold for weak subsolutions and supersolutions.*

Proof. 1. Choose $\phi \in C^\infty(\mathbb{R}^{n+1})$ and suppose first $u - \phi$ has a *strict* local maximum at some point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. As $u^k \to u$ uniformly near (x_0, t_0),

(2.19) $u_k - \phi$ has a local maximum at a point $(x_k, t_k)\, (k = 1, 2, \ldots)$

with

$$(x_k, t_k) \to (x_0, t_0) \qquad \text{as } \; k \to \infty. \tag{2.20}$$

Since u_k is a weak solution, we have *either*

$$\phi_t \leq (\delta_{ij} - \frac{\phi_{x_i} \phi_{x_i}}{|D\phi|^2})\, \phi_{x_i x_j} \qquad \text{at } \; (x_k, t_k) \tag{2.21}$$

if $D\phi(x_k,t_k)\neq 0$, *or*

$$\phi_t \leq (\delta_{ij} - \eta_i^k \eta_j^k)\,\phi_{x_i x_j} \quad \text{at} \;\; (x_k,t_k) \tag{2.22}$$

for some $\eta^k \in \mathbb{R}^n$ with $|\eta^k| \leq 1$, if $D\phi(x_k,t_k) = 0$.

2. Assume first $D\phi(x_0,t_0)\neq 0$. Then $D\phi(x_k,t_k)\neq 0$ for all large enough k. Hence we may pass to limits in the inequalities (2.21) to discover

$$\phi_t \leq (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}), \quad \text{at} \;\; (x_0,t_0). \tag{2.23}$$

3. Next, suppose $D\phi(x_0,t_0) = 0$. We set

$$\xi^k = \begin{cases} \frac{D\phi}{|D\phi|}(x_k,t_k) & \text{if} \;\; D\phi(x_k t_k) \neq 0 \\ \eta^k & \text{if} \;\; D\phi(x_k,t_k) = 0 \end{cases} \tag{2.24}$$

Passing if necessary to a subsequence we may assume

$$\xi^k \to \eta.$$

Then $|\eta| \leq 1$. Utilizing now (2.22), we deduce as well

$$\phi_t \leq (\delta_{ij} - \eta_i \eta_j)\,\phi_{x_i x_j} \quad \text{at} \;\; (x_0,t_0). \tag{2.25}$$

4. If $u-\phi$ has only a local maximum at (x_0,t_0) we apply the above argument to

$$\psi(x,t) \equiv \phi(x,t) + |x-x_0|^4 + (t-t_0)^4,$$

so that $u-\psi$ has a strict local maximum at (x_0,t_0). Hence u is a weak subsolution. Similar reasoning verifies that u is a weak supersolution as well. □

Theorem 2.8 *Assume u is a weak solution of (2.5) and $\Psi\colon \mathbb{R} \to \mathbb{R}$ is continuous. Then*

$$v \equiv \Psi(u)$$

is a weak solution.

Proof. 1. Assume first Ψ is smooth, with

$$\Psi' > 0 \qquad \text{on} \ \ . \tag{2.26}$$

Let $\phi \in C^\infty(\ ^{n+1})$ and suppose $v - \phi$ has a local maximum at (x_0, t_0). Adding as necessary a constant to ϕ, we may assume

$$\begin{cases} v(x_0, t_0) = \phi(x_0, t_0) \\ v(x, t) \leq \phi(x, t) & \text{for all } \ (x, t) \ \text{ near } \ (x_0, t_0). \end{cases} \tag{2.27}$$

In view of (2.26)

$$\Phi \equiv \Psi^{-1}$$

is defined and smooth near $u(x_0, t_0)$, with

$$\Phi' > 0. \tag{2.28}$$

From (2.27) therefore, we see

$$\begin{cases} u(x_0, t_0) = \psi(x_0, t_0) \\ u(x, t) \leq \psi(x, t) & \text{for all } \ (x, t) \ \text{ near } \ (x_0, t_0) \end{cases} \tag{2.29}$$

where

$$\psi \equiv \Phi(\phi). \tag{2.30}$$

2. Since u is a weak solution we see

$$\psi_t \leq (\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2})\, \psi_{x_i x_j} \qquad \text{at } \ (x_0, t_0) \tag{2.31}$$

if $D\psi(x_0, t_0) \neq 0$ and

$$\psi_t \leq (\delta_{ij} - \eta_i \eta_j)\, \psi_{x_i x_j} \qquad \text{at } \ x_0, t_0) \tag{2.32}$$

for some $|\eta| \leq 1$, if $D\psi(x_0, t_0) = 0$. Now $D\phi(x_0, t_0) = 0$ if and only if $D\psi(x_0, t_0) = 0$. Consequently (2.31) obtains if $D\phi(x_0, t_0) \neq 0$; in which case we substitute (2.30) to compute

$$\Phi' \phi_t \leq (\delta_{ij} - \frac{(\Phi')^2 \phi_{x_i} \phi_{x_j}}{(\Phi')^2\, |D\phi|^2})\, (\Phi' \phi_{x_i x_j} + \Phi'' \phi_{x_i} \phi_{x_j}) \quad \text{at } \ (x_0, t_0)$$

Since $\Phi' > 0$ we simplify and obtain

$$\phi_t \leq (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\, \phi_{x_i x_j} \quad \text{at } (x_0, t_0). \tag{2.33}$$

Suppose on the other hand $D\phi(x_0, t_0) = 0$. Then (2.32) is valid for some $|\eta| \leq 1$. We substitute (2.30) and compute

$$\Phi'\phi_t \leq (\delta_{ij} - \eta_i\eta_j)\,(\Phi'\phi_{x_i x_j} + \Phi''\phi_{x_i}\phi_{x_j}) \quad \text{at } (x_0, t_0).$$

Since $D\phi = 0$, the term involving Φ'' is zero. Thus

$$\phi_t \leq (\delta_{ij} - \eta_i\eta_j)\, \phi_{x_i x_j} \quad \text{at } (x_0, t_0). \tag{2.34}$$

We similarly have the opposite inequalities to (2.33), (2.34) should $v - \phi$ have a local minimum at (x_0, t_0).

3. Now assume instead of (2.20) that

$$\Psi' < 0 \quad \text{on} \quad . \tag{2.35}$$

Then

$$\Phi' < 0 \quad \text{on}$$

as well. Thus (2.21) now implies

$$\begin{cases} u(x_0, t_0) = \psi(x_0, t_0) \\ u(x, t) \geq \psi(x, t) & \text{for all } (x, t) \text{ near } (x_0, t_0) \end{cases}$$

Since u is a weak solution, either

$$\psi_t \geq (\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2})\, \psi_{x_i x_j} \quad \text{at } (x_0, t_0)$$

if $D\psi(x_0, t_0) \neq 0$ or

$$\psi_t \geq (\delta_{ij} - \eta_i\eta_j)\, \psi_{x_i x_j} \quad \text{at } (x_0, t_0)$$

for some $|\eta| \leq 1$, if $D\psi(x_0, t_0) = 0$. Since now $\Phi' < 0$, we as above deduce either (2.33) or (2.34).

4. We have so far shown that $v = \Psi(u)$ is a weak solution provided Ψ is smooth, with $\Psi' \neq 0$. Approximating and using Theorem 2.2 we draw the same conclusion if

$$\Psi' \geq 0 \quad \text{or} \quad \Psi' \leq 0 \quad \text{on} \quad .$$

5. Next assume Ψ is smooth and there exist finitely many points $-\infty = a_0 < a_1, < a_2 < \ldots < a_m < a_{m+1} = +\infty$ such that

$$(2.36) \qquad \Psi \;\text{ is monotone on the intervals }\; (a_j, a_{j+1}) \; (j = 0, \ldots, m)$$

and

$$(2.37) \quad \Psi \;\text{ is constant on the intervals }\; (a_j - \sigma,\, a_j + \sigma) \, (j = 1, \ldots, m)$$

for some $\sigma > 0$.

Suppose $v - \phi$ has a maximum at (x_0, t_0). Then

$$u(x_0, t_0) \in (a_j - \frac{\sigma}{2},\, a_{j+1} + \frac{\sigma}{2}) \quad \text{for some } \; j \in \{0, \ldots, m\}.$$

As Ψ is monotone on $(a_j - \sigma, a_{j+1} + \sigma)$ and u is continuous, we can apply steps 1-4 is some neighborhood of (x_0, t_0) to deduce (2.33) or (2.34). The reverse inequalities similarly obtain if $v - \phi$ has a minimum.

6. Finally suppose only that Ψ is continuous. We construct a sequence of smooth functions $\{\Psi^k\}_{k=1}^{\infty}$ each verifying the structural assumptions (2.36), (2.37) so that

$$\Psi^k \to \Psi \qquad \text{uniformly on } \; [-||u||_{L^\infty},\, ||u||_{L^\infty}].$$

Hence

$$v^k = \Psi^k(u) \to v \equiv \Psi(u)$$

boundedly and uniformly. Then Theorem 2.2 asserts v to be a weak solution. $\square$

3 Uniqueness and Comparison of Weak Solutions

3.1 Preliminaries

Our plan, as in Jensen [], Jensen-Lions-Souganidis [], is to regularize using sup *and* inf *convolutions*, defined as follows. Assume $w: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is continuous and bounded. Then if $\epsilon > 0$, we write

$$(3.1) \qquad w^\epsilon(x,t) \equiv \sup_{\substack{y \in \mathbb{R}^n \\ s \in [0,\infty)}} \{w(y,s) - \tfrac{1}{\epsilon}(|x-y|^2 + (t-s)^2)\},$$

$$(3.2) \qquad w_\epsilon(x,t) \equiv \inf_{\substack{y \in \mathbb{R}^n \\ s \in [0,\infty)}} \{w(y,s) + \tfrac{1}{\epsilon}(|x-y|^2 + (t-s)^2)\}.$$

for $x \in \mathbb{R}^n$, $t \in [0,\infty)$. Note that since w is continuous and bounded, the "sup" and "inf" above can be replaced by "max" and "min".

Lemma 3.1 (Properties of sup **and** inf **convolutions)**

There exist constants A,B,C depending only on $||w||_{L^\infty(R^n \times [0,\infty))}$, *such that:*

(i) $w_\epsilon \leq w \leq w^\epsilon$ *on* $\mathbb{R}^n \times [0,\infty)$

(ii) $||w^\epsilon, w_\epsilon||_{L^\infty(R^n \times [0,\infty))} \leq A$

(iii) *If* $y \in \mathbb{R}^n$, $s \in [0,\infty)$ *and* $w^\epsilon(x,t) = w(y,s) - \frac{l}{\epsilon}(|x-y|^2 + (t-s)^2)$,

then

$$(3.3) \qquad |x-y|,\ |t-s| \leq C\epsilon^{\frac{1}{2}} \equiv \sigma(\epsilon)$$

A similar assertion holds for w_ϵ.

(iv) $w^\epsilon, w_\epsilon \to w$ *as* $\epsilon \to 0^+$, *uniformly on compact subsets of* $\mathbb{R}^n \times [0,\infty)$.

(v) *Lip* $(w^\epsilon,\, w_\epsilon) \leq \frac{B}{\epsilon}$.

(vi) *The mapping*

$$(x,t) \longmapsto w^\epsilon(x,t) + \frac{1}{\epsilon}\,(|x|^2 + t^2)$$

is convex, and the mapping

$$(x,t) \longmapsto w_\epsilon(x,t) - \frac{1}{\epsilon}\,(|x|^2 + t^2)$$

is concave.

(vii) *Assume w is a weak subsolution of (2.5) in $\mathbb{R}^n \times (0,\infty)$. Then w^ϵ is a weak subsolution on $\mathbb{R}^n \times (\sigma(\epsilon),\infty)$. Similarly, if w is a weak supersolution of (2.5), w_ϵ is a weak supersolution.*

(viii) *The function w^ϵ is twice differentiable a.e. and satisfies*

$$w^\epsilon_t \leq (\delta_{ij} - \frac{w^\epsilon_{x_i} w^\epsilon_{x_j}}{|Dw^\epsilon|^2})\, w^\epsilon_{x_i x_j} \tag{3.4}$$

at each point of twice differentiability in $\mathbb{R}^n \times (\sigma(\epsilon),\infty)$ where $Dw^\epsilon \neq 0$. Similarly, w_ϵ is twice differentiable a.e. and satisfies

$$w_{\epsilon t} \geq (\delta_{ij} - \frac{w_{\epsilon\, x_i} w_{\epsilon\, x_j}}{|Dw^\epsilon|^2})\, w_{\epsilon\, x_i x_j} \tag{3.5}$$

at each point of twice differentiability in $\mathbb{R}^n \times (\sigma(\epsilon),\infty)$ where $Dw_\epsilon \neq 0$.

Proof. 1. Assertions (i) and (ii) are clear from the definitions, for

$$A = ||w||_{L^\infty(R^n \times [0,\infty))}\,.$$

Statement (iii) follows from (ii), and then (iv) is a consequence of the uniform continuity of w on compact sets. In light of estimate (3.3) we have (v) as well.

2. For each $y \in \mathbb{R}^n$, $s \in [0, \infty)$, the mapping

$$(x,t) \longmapsto w(y,s) - \frac{1}{\epsilon}\left(|x-y|^2 + (t-s)^2\right) + \frac{1}{\epsilon}\left(|x|^2 + t^2\right)$$

is affine. Consequently

$$\begin{aligned}(x,t) \longmapsto \quad & \sup_{\substack{y \in \mathbb{R}^n \\ s \in [0,\infty)}} \left[w(y,s) - \tfrac{1}{\epsilon}\left(|x-y|^2 + (t-s)^2\right) + \tfrac{1}{\epsilon}\left(|x|^2+t^2\right)\right] \\ & = w^\epsilon(x,t) + \tfrac{1}{\epsilon}\left(|x|^2 + t^2\right)\end{aligned}$$

is convex.

3. Assume $\phi \in C^\infty(\mathbb{R}^{n+1})$ and $w^\epsilon - \phi$ has a local maximum at a point (x_0, t_0), with $t_0 > \sigma(\epsilon)$.

We then employ (3.3) to choose $(y_0, s_0) \in \mathbb{R}^n \times (0, \infty)$ so that

$$w^\epsilon(x_0, t_0) = w(y_0, s_0) - \frac{1}{\epsilon}\left(|x_0 - y_0|^2 + (t_0 - s_0)^2\right).$$

Set

$$\psi(x,t) \equiv \phi(x + x_0 - y_0,\ t + t_0 - s_0). \tag{3.6}$$

Since $w^\epsilon - \phi$ has a local maximum at (x_0, t_0) we compute

$$\begin{aligned} w(y_0, s_0) \quad & -\tfrac{1}{\epsilon}(|x_0 - y_0|^2 + (t_0 - s_0)^2) - \phi(x_0, t_0) \\ & = w^\epsilon(x_0, t_0) - \phi(x_0, t_0) \\ & \geq w^\epsilon(x,t) - \phi(x,t) \\ & \geq w(y,s) - \tfrac{1}{\epsilon}(|x-y|^2 + (t-s)^2) - \phi(x,t) \end{aligned}$$

for all (x,t) near (x_0, t_0) and all $(y,s) \in \mathbb{R}^n \times [0, \infty)$. Fix (y,s) close to (y_0, s_0) and set

$$x = y + x_0 - y_0,\ t = s + t_0 - s_0$$

above, to discover

$$w(y_0, s_0) - \phi(x_0, t_0) \geq w(y,s) - \phi(y + x_0 - y_0,\ s + t_0 - s_0).$$

Using (3.6) we rewrite this to read

$$w(y_0, s_0) - \phi(y_0, s_0) \geq w(y, s) - \psi(y, s)$$

for all (y, s) near (y_0, s_0). Hence $w - \psi$ has a local maximum at (y_0, s_0) and thus

$$\psi_t \leq (\delta_{ij} - \frac{\psi_{x_i}\psi_{x_i}}{|D\psi|^2})\,\psi_{x_i x_j} \quad \text{ at } (y_0, s_0)$$

if $D\psi(y_0, s_0) \neq 0$, and

$$\psi_t \leq (\delta_{ij} - \eta_i \eta_j)\,\psi_{x_i x_j} \quad \text{ at } (y_0, s_0)$$

for some $|\eta| \leq 1$, if $D\psi(y_0, s_0) = 0$. Since

$$\begin{cases} D\psi(y_0, s_0) = D\phi(x_0, t_0),\ \psi_t(y_0, s_0) = \phi_t(x_0, t_0) \\ \\ D^2\phi(y_0, s_0) = D^2\phi(x_0, t_0), \end{cases}$$

we immediately obtain

$$\phi_t \leq (\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\,\phi_{x_i x_j} \quad \text{ at } (x_0, t_0)$$

or

$$\phi_t \leq (\delta_{ij} - \eta_i \eta_j)\,\psi_{x_i x_j} \quad \text{ at } (x_0, t_0)$$

according to whether $D\phi(x_0, t_0) = 0$ or not.

4. Owing to (vi), $w^\epsilon(x, t) + \frac{1}{\epsilon}(|x|^2 + t^2)$ is convex in (x, t) or so is twice differentiable a.e. according to a theorem of Alexandroff (see, e.g., Resetnjak [], Krylov [, Appendix 2], etc.) Thus w^ϵ is twice differentiable a.e. $\square$

3.2 Comparison principle, uniqueness

Theorem 3.2 *Assume that u is a weak subsolution and v is a weak supersolution of (2.5). Suppose further*

$$(3.7) \qquad u \leq v \quad \textit{on } {}^n \times \{t = 0\}.$$

Finally assume that

$$\text{(3.8)} \qquad \begin{cases} u \ \textit{ and } \ v \ \textit{ are constant, with } \ u \le v, \\ \textit{on } \ .\ ^n \times [0,\infty) \cap \{|x| + t \ge R\} \end{cases}$$

for some $R \ge 1$. *Then*

$$\text{(3.9)} \qquad u \le v \ \ \textit{on} \ .\ ^n \times [0,\infty).$$

In particular, weak solutions of (2.5) are unique.

Proof. 1. Should (3.9) fail, then

$$\max_{(x,t)\in R^n\times[0,\infty)} (u - v) \equiv a > 0;$$

and so for $\alpha > 0$ small enough,

$$\text{(3.10)} \qquad \max_{(x,t)\in R^n\times[0,\infty)} (u - v - \alpha\, t) \ge \frac{a}{2} > 0.$$

According to (3.8) we have

$$\text{(3.11)} \qquad u^\epsilon = u, \ v_\epsilon = v \qquad \text{on } \ \{|x| + t \ge 2R\}$$

for all small $\epsilon > 0$. Note further $u^\epsilon \to u$ and $v_\epsilon \to v$ uniformly. Consequently if we fix $\epsilon > 0$ small enough,

$$\text{(3.12)} \qquad \max_{(x,t)\in R^n\times[0,\infty)} (u^\epsilon - v_\epsilon - \alpha\, t) \ge \frac{a}{4} > 0.$$

2. For each $\delta > 0$ define for $x, y \in \ .\ ^n$, $t, t+s \in [0,\infty)$

$$\text{(3.13)} \qquad \Phi(x,y,t,s) \equiv u^\epsilon(x+y, t+s) - v_\epsilon(x,t) - \alpha\, t - \frac{1}{\delta}(|y|^4 + s^4).$$

Owing to (3.12) we see

$$\text{(3.14)} \qquad \max_{(x,t),(x+y,t+s)\in R^n\times[0,\infty)} \Phi \ge \frac{a}{4} > 0.$$

Choose now $(x_1, t_1), (x_1 + y_1, t_1 + s_1) \in \mathbb{R}^n \times [0, \infty)$ so that

$$\Phi(x_1, y_1, t_1, s_1) = \max \Phi(x, t),\ (x + y, t + s) \in \mathbb{R}^n \times [0, \infty). \tag{3.15}$$

Note in view of (3.11), (3.13) and Lemma 3.1 (ii) that such points exist. Since $\Phi(x_1, y_1, t_1, s_1) > 0$, (3.13) implies

$$|y_1|, |s_1| \leq C\delta^{1/4} \tag{3.16}$$

3. We *claim* next that if $\epsilon, \delta > 0$ are fixed small enough,

$$t_1, t_1 + s, > \sigma(\epsilon), \tag{3.17}$$

with $\sigma(\epsilon)$ defined in (3.3). Indeed if $t_1 \leq \sigma(\epsilon)$, then

$$\begin{aligned} \tfrac{a}{4} &\leq \Phi(x_1, y_1, t_1, s_1) && \\ &\leq u^\epsilon(x_1 + y_1, t_1 + s_1) - v_\epsilon(x_1, t_1) && \\ &= u(x_1 + y_1, t_1 + s_1) - v(x_1, t_1) \quad &+o(1) \quad &\text{as } \epsilon \to 0 \\ &= u(x_1 + y_1, s_1) - v(x_1, 0) &+o(1) \quad &\text{as } \epsilon \to 0 \\ &= u(x_1, 0) - v(x_1, 0) &+o(1) \quad &\text{as } \epsilon, \delta \to 0 \\ &\leq o(1) && \text{as } \epsilon, \delta \to 0, \end{aligned}$$

where we employed Lemma 3.1 (ii), (3.16), (3.7) and the continuity of u, v. This is a contradition for $\epsilon, \delta > 0$ small enough, whence $t_1 > \sigma(\epsilon)$. Owing to (3.16) we may as necessary adjust δ smaller to ensure (3.17). *Hereafter in the proof* $\alpha, \epsilon, \delta > 0$ *are fixed.*

According to Lemma 3.1, (vii),

$$u^\epsilon \text{ is a weak subsolution of (2.5) near } (x_1 + y_1, t_1 + s_1) \tag{3.18}$$

and

$$v_\epsilon \text{ is a weak supersolution of (2.5) near } (x_1, t_1) \tag{3.19}$$

4. We now demonstrate

$$y_1 \neq 0. \tag{3.20}$$

Assume for contradiction that in fact $y_1 = 0$. Then (3.13), (3.15) imply

$$
(3.21) \qquad \begin{cases} u^\epsilon(x_1, t_1, +s_1) - v_\epsilon(x_1, t_1) - \alpha\, t_1 - \frac{1}{\delta} s_1^4 \\ \geq u^\epsilon(x+y, t+s) - v_\epsilon(x, t) - \alpha\, t - \frac{1}{\delta}(|y|^4 + s^4) \end{cases}
$$

for all $(x,t), (x+y, t+s) \in \mathbb{R}^n \times [0, \infty)$. Put $x = x_1, t = t_1$, above and simplify to obtain the inequality

$$
u^\epsilon(x_1 + y, t_1 + s) \leq u^\epsilon(x_1, t_1 + s_1) + \frac{1}{\delta}|y|^4 + \frac{1}{\delta}(s^4 - s_1^4)
$$

for $(x_1 + y, t_1 + s) \in \mathbb{R}^n \times [0, \infty)$. Set $r = s - s_1$ and rewrite to find

$$
\begin{aligned} u^\epsilon(x_1 + y, t_1 + s_1 + r) \ &\leq u^\epsilon(x_1, t_1 + s_1) + \tfrac{4}{\delta} s_1^3 r + \tfrac{6}{\delta} s_1^2 r^2 \\ &+O(|r|^3 + |y|^4) \ \text{ as } \ (y, r) \to (0, 0). \end{aligned}
$$

Since u^ϵ is a weak subsolution near $(x_1 + y_1, t_1 + s_1) = (x_1, t_1 + s_1)$ we may invoke $(2.7)'$, $(2.9)'$ with $x_0 = x_1, t_0 = t_1 + s_1, p = 0, q = \frac{4}{\delta} s_1^3$, $r_{n+1,n+1} = \frac{6}{\delta} s_1^2$, $r_{ij} = 0$ otherwise; giving

$$
(3.22) \qquad \frac{4}{\delta} s_1^3 \leq 0.
$$

Now go back and insert $y = x_1 - x, s = t_1 + s_1 - t$ into (3.21). This yields after simplifications:

$$
\begin{aligned} v_\epsilon(x, t) \ &\geq v_\epsilon(x_1, t_1) + (\tfrac{4 s_1^3}{\delta} - \alpha)(t - t_1) - \tfrac{6}{\delta} s_1^2 (t - t_1)^2 \\ &+O(|x - x_1|^4 + |t - t_1|^3) \ \text{ as } \ (x, t) \to (x_1, t_1). \end{aligned}
$$

Now v_ϵ is a weak supersolution near (x_1, t_1). Thus $(2.10)'$, $(2.12)'$, with $x_0 = x_1$, $t_0 = t_1$, $p = 0$, $q = \frac{4 s_1^3}{\delta} - \alpha$, $r_{n+1,n+1} = -\frac{6 s_1^2}{\delta}$, $r_{ij} = 0$ otherwise, imply

$$
(3.23) \qquad \frac{4 s_1^3}{\delta} - \alpha \geq 0.
$$

But now we have a contradiction with (3.22), since $\alpha > 0$. This establishes (3.20).

5. Note next that, in general, if $f\colon \mathbb{R}^m \to \mathbb{R}$ is convex, then so is the mapping $(w, z) \longmapsto f(w + z)$ on $\mathbb{R}^{2m}$. Consequently Lemma 3.1 (vi) asserts

$$
(x, y, t, s) \longmapsto u^\epsilon(x + y, t + s) + \frac{1}{\epsilon}((x + y)^2 + (t + s)^2)
$$

to be convex. As

$$(x,t) \longmapsto -v_\epsilon(x,t) + \frac{1}{\epsilon}\,(|x|^2 + t^2)$$

is convex as well, we see that

$$(x,y,t,s) \longmapsto \Phi(x,y,t,s) + C(|x|^2 + |y|^2 + t^2 + s^2)$$

is convex near (x_1, y_1, t_1, s_1), for some sufficiently large constant $C = C(\epsilon, \delta)$. Since Φ additionally attains its maximum at (x_1, y_1, t_1, s_1) we may invoke Jensen []: there exist points $\{(x^k, y^k, t^k, s^k)\}_{k=1}^{\infty}$ such that

$$(x^k, y^k, t^k, s^k) \to (x_1, y_1, t_1, s_1), \tag{3.24}$$

$$\begin{array}{l} \Phi,\ u^\epsilon \ \text{ and } \ v_\epsilon \ \text{ are each twice differentiable} \\ \text{at } \ (x^k, y^k, t^k, s^k) \quad (k = 1, \dots), \end{array} \tag{3.25}$$

$$D_{x,y,t,s}\Phi(x^k, y^k, t^k, s^k) \to 0, \tag{3.26}$$

and

$$D^2_{x,y,t,s}\Phi(x^k, y^k, t^k, s^k) \le o(1) I_{2n+2} \ \text{ as } \ k \to \infty. \tag{3.27}$$

6. Using (3.13), (3.25), we see

$$\left\{ \begin{array}{ll} D_x\Phi(x^k, y^k, t^k, s^k) & = Du^\epsilon(x^k + y^k, t^k + s^k) - Dv_\epsilon(x^k, t^k) \\ & \equiv p^k - \overline{p}^k. \end{array} \right. \tag{3.28}$$

and

$$\left\{ \begin{array}{ll} D_y\Phi(x^k, y^k, t^k, s^k) & = Du^\epsilon(x^k + y^k, t^k + s^k) - \frac{4}{\delta}|y^k|^2 y^k \\ & = p^k - \frac{4}{\delta}|y^k|^2 y^k \end{array} \right. \tag{3.29}$$

Since $y^k \to y_1$, we deduce from (3.26) that

$$p^k, \overline{p}^k \to \frac{4}{\delta}|y_1|^2 y_1 \equiv p \qquad \text{in } \ {}^n. \tag{3.30}$$

Assertion (3.20) tells us $p \neq 0$ and so $p^k, \overline{p}^k \neq 0$ for large enough k.

Again employing (3.13), (3.25) we note

$$
\text{(3.31)} \qquad \begin{cases} \Phi_t(x^k, y^k, t^k, s^k) & = u^\epsilon_t(x^k + y^k, t^k + s^k) - v_{\epsilon t}(x^k, t^k) - \alpha \\ & \equiv q^k - \overline{q}^k - \alpha. \end{cases}
$$

As u^ϵ and v_ϵ are Lipschitz, we may assume, upon passing to a subsequence and reindexing if necessary, that

$$
\text{(3.32)} \qquad q^k \to q,\ \overline{q}^k \to \overline{q} \quad \text{in} \quad .
$$

Then (3.26) and (3.31) ensure

$$
\text{(3.33)} \qquad q - \overline{q} = \alpha.
$$

7. Next, (3.13) and (3.25) imply

$$
\text{(3.34)} \qquad \begin{aligned} D^2_x\phi(x^k, y^k, t^k, s^k) & = D^2u^\epsilon(x^k + y^k, t^k + s^k) - D^2 v_\epsilon(x^k, t^k) \\ & \equiv R^k - \overline{R}^k. \end{aligned}
$$

Now (3.27) forces

$$
\text{(3.35)} \qquad R^k - \overline{R}^k \le \epsilon_k I_n
$$

where $\epsilon_k \to 0$. Furthermore, Lemma 3.1 (vi) shows

$$
R^k \ge -CI_n,\ \overline{R}^k \le CI_n,
$$

for $C = C(\epsilon)$. Thus

$$
-CI_n \le R^k \le \overline{R}^k + \epsilon_k I_n \le CI_n.
$$

We may consequently suppose, passing as necessary to subsequences, that

$$
\text{(3.36)} \qquad R^k \to R,\ \overline{R}^k \to \overline{R} \quad \text{in } S^{n\times n},
$$

with

$$
\text{(3.37)} \qquad R \le \overline{R}.
$$

8. Now recall (3.25) holds and $p_0^k \equiv Du^\epsilon(x^k+y^k,t^k+s^k)$, $\overline{p}^k \equiv Dv_\epsilon(x^k,t^k)$ are non-zero for large k. Since u^ϵ is a weak subsolution near (x_1+y_1,t_1+s_1) and v_ϵ is a weak supersolution near (x_1,t_1), we thus have

$$q^k \leq (\delta_{ij} - \frac{p_i^k p_j^k}{|p^k|^2})\, r_{ij}^k$$

and

$$\overline{q}^k \geq (\delta_{ij} - \frac{\overline{p}_i^k \overline{p}_j^k}{|\overline{p}^k|^2})\, \overline{r}_{ij}^k$$

for all large k. We send k to infinity, recalling (3.30), (3.32) and (3.36) to obtain

$$q \leq (\delta_{ij} - \frac{p_i p_j}{|p|^2})\, r_{ij}$$

and

$$\overline{q} \geq (\delta_{ij} - \frac{p_i p_j}{|p|^2})\, \overline{r}_{ij}.$$

Subtract:

$$q - \overline{q} \leq (\delta_{ij} - \frac{p_i p_j}{|p|^2})\,(r_{ij} - \overline{r}_{ij}).$$

Now the matrix $((\delta_{ij} - \frac{p_i p_j}{|p|^2}))$ is nonnegative and $R - \overline{R}$ is nonpositive, by (3.37). Consequently

$$q - \overline{q} \leq 0,$$

a contradiction to (3.33). □

3.3 Contraction property

Theorem 3.3 *Assume that u and v are weak solutions of (2.5), such that*

$$\text{(3.38)} \qquad \begin{cases} u \text{ and } v \text{ are constant, with } u = v, \\ \text{on } \mathbb{R}^n \times [0,\infty) \cap \{|x| + t \geq R\} \end{cases}$$

for some $R > 1$. Then

$$\text{(3.39)} \qquad \max_{0 \leq t < \infty} ||u(\cdot,t) - v(\cdot,t)\,||_{L^\infty(R^n)} = ||u(\cdot,0) - v(\cdot,0)\,||_{L^\infty(R^n)}.$$

Proof. Should (3.39) fail, we may assume

$$\max_{(x,t)\in R^n\times[0,\infty)} (u-v) \equiv a > ||u(\cdot,0) - v(\cdot,0)\,||_{L^\infty(R^n)} \equiv b.$$

Then as in the proof of Theorem 3.2 as above, there exist $\alpha, \epsilon, \delta > 0$ such that

$$\max_{(x,t),(x+y,t+s)\in R^n\times[0,\infty)} \Phi > b, \quad \text{where}$$

Φ is defined by (3.13). We find a point (x_1, y_1, t_1, s_1) verifying (3.15) and check (3.17) is valid provided $\epsilon, \delta > 0$ are small enough. The rest of the proof follows that for Theorem 3.2. $\square$

4 Existence of Weak Solutions

4.1 Approximation; geometric interpretation

We turn our attention now to constructing a weak solution of the initial value problem (2.5), (2.6). We will assume that

$$g \text{ is constant on } \mathbb{R}^n \cap \{|x| \geq S\} \tag{4.1}$$

for some $S > 0$, and additionally, for the moment at least, g is smooth.

Our intention is to approximate (2.5), (2.6) by the PDE

$$u^\epsilon_t = (\delta_{ij} - \frac{u^\epsilon_{x_i} u^\epsilon_{x_j}}{|Du^\epsilon|^2 + \epsilon^2})\, u^\epsilon_{x_i x_j} \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{4.2}$$

$$u^\epsilon = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{4.3}$$

for $0 < \epsilon < 1$. (The superscript ϵ here and hereafter is only a label and does not mean the $\sup -$ convolution (3.1).)

We interpret (4.2), (4.3) as follows. Assuming for the moment $u^\epsilon = u^\epsilon(x,t)$ to be a smooth solution of (4.2), (4.3), write $y = (x, x_{n+1}) \in \mathbb{R}^{n+1}$ and define

$$v^\epsilon(y,t) \equiv u^\epsilon(x,t) - \epsilon x_{n+1} \tag{4.4}$$

Then $|D_y v^\epsilon|^2 = |Du^\epsilon|^2 + \epsilon^2$. Thus our PDE becomes

$$v^\epsilon_t = (\delta_{ij} - \frac{v^\epsilon_{y_i} v^\epsilon_{y_j}}{|Dv^\epsilon|^2})\, v^\epsilon_{y_i y_j} \qquad \text{in } \mathbb{R}^{n+1} \times [0,\infty) \tag{4.5}$$

$$v^\epsilon = g^\epsilon \qquad \text{on } \mathbb{R}^{n+1} \times \{t = 0\}, \tag{4.6}$$

for

$$g^\epsilon(y) \equiv g(x) - \epsilon x_{n+1}.$$

As noted in §2, the PDE (4.5) says that each level set of v^ϵ evolves according to its mean curvature. This is in particular the case for the zero level sets

$$\Gamma^\epsilon_t \equiv \{y \in \mathbb{R}^{n+1} \mid v^\epsilon(y,t) = 0\}.$$

But according to (4.4) each Γ^ϵ_t is a graph:

$$\Gamma^\epsilon_t = \{y = (x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = \frac{1}{\epsilon}\, u^\epsilon(x,t)\},$$

and Ecker and Huisken [] have shown the evolution of an entire graph by mean curvature remains a smooth entire graph for all time.

Geometrically, if as in §1 we are given Γ_0 as the boundary of a smooth, bounded, simply connected open set U in $\mathbb{R}^n$, we select a smooth function g with $g = 0$ on Γ_0, $g < 0$ in U, $g > 0$ in $\mathbb{R}^n - U$. Then $\Gamma^\epsilon_0 \subset \mathbb{R}^{n+1}$ is the graph $\{x_{n+1} = \frac{1}{\epsilon}\, g(x)\}$ as drawn.

For small $\epsilon, \Gamma^\epsilon_0$ roughly approximates the cylinder $\Gamma_0 \times \mathbb{R}$. We may thus hope that for moderate $t > 0$ and small $\epsilon > 0$, the smooth graph Γ^ϵ_t will be close to the cylinder $\Gamma_t \times \mathbb{R}$, Γ_t denoting the evolution of Γ_0 via its mean curvature in $\mathbb{R}^n$.

The idea then is that the complicated, possibly singular behavior of $\{\Gamma_t\}_{t\geq 0}$ in $\mathbb{R}^n$ will be approximated by the smooth evolution $\{\Gamma_t^\epsilon\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$, in the sense that for a given $t > 0$,

$$\Gamma_t^\epsilon \approx \Gamma_t \times \mathbb{R}$$

if $\epsilon > 0$ is very small. The illustrations provided make this expectation appear plausible, although there are a number of subtleties.

4.2 Solution of the approximate equations

We now investigate the approximations (4.2), (4.3) analytically.

Theorem 4.1

(i) *For each* $0 < \epsilon < 1$ *there exists a unique smooth, bounded solution* u^ϵ *of (4.2), (4.3).*

(ii) *Additionally,*

$$\sup_{0<\epsilon<1} ||u^\epsilon,\, Du^\epsilon,\, u_t^\epsilon||_{\,L^\infty(R^n)\times[0,\infty))} \leq C||g||_{C^{1,1}(R^n)} \tag{4.7}$$

Proof. 1. For each $0 < \sigma < 1$, consider the PDE

$$u_t^{\epsilon,\sigma} = a_{ij}^{\epsilon,\sigma}(Du^{\epsilon,\sigma})\, u_{x_i x_j}^{\epsilon,\sigma} \quad \text{in } \mathbb{R}^n \times [0,\infty) \tag{4.8}$$

$$u^{\epsilon,\sigma} = g \quad \text{on} \ \ ^n \times \{t = 0\}, \tag{4.9}$$

for

$$a_{ij}^{\epsilon,\sigma}(p) \equiv (1+\sigma)\delta_{ij} - \frac{p_i p_j}{|p|^2 + \epsilon^2} \ \ (p \in \ ^n, \ 1 \le i, \ j \le n)$$

The smooth bounded coefficients $\{a_{ij}\}$ satisfy also the uniform ellipticity condition

$$\sigma |\xi|^2 \le a_{ij}^{\epsilon,\sigma}(p)\xi_i \xi_j, \quad (\xi \in \ ^n)$$

for each $p \in \ ^n$, and consequently classical PDE theory gives the existence of a unique smooth bounded solution $u^{\epsilon,\sigma}$: see, e.g., Ladyzhenskaja, Solonnikov, Uralcéva []. By the maximum principle,

$$||u^{\epsilon,\sigma}||_{L^\infty(R^n \times [0,\infty))} = ||g||_{L^\infty(R^n)} \tag{4.10}$$

2. Now differentiate (4.8) with respect to x_ℓ:

$$u_{x_\ell t}^{\epsilon,\sigma} = a_{ij}^{\epsilon,\sigma}(Du^{\epsilon,\sigma})\, u_{x_\ell x_i x_j}^{\epsilon,\sigma} + a_{ij\,p_k}^{\epsilon,\sigma}(Du^{\epsilon,\sigma}) u_{x_\ell x_k}^{\epsilon,\sigma} u_{x_i x_j}^{\epsilon,\sigma}$$

The maximum principle then implies

$$||Du^{\epsilon,\sigma}||_{L^\infty R^n \times [0,\infty))} = ||Dg||_{L^\infty(R^n)} \tag{4.11}$$

Similarly

$$||u_t^{\epsilon,\sigma}||_{L^\infty(R^n \times [0,\infty))} = ||u_t(\cdot, 0)||_{L^\infty(R^n)} \le C||D^2 g||_{L^\infty(R^n)}. \tag{4.12}$$

3. Since

$$(1 - \frac{L^2}{L^2 + \epsilon^2})|\xi|^2 \le a_{ij}^{\epsilon,\sigma}(p)\xi_i \xi_j \qquad (\xi \in \ ^n)$$

provided $|p| \le L$, we deduce from (4.10)-(4.12) and classical estimates that we have bounds, uniform in $0 < \sigma < 1$, on the derivatives of all orders of $\{u^{\epsilon,\sigma}\}_{0<\sigma<1}$. Consequently, uniqueness of the limit implies for each multiindex α,

$$D^\alpha u^{\epsilon,\sigma} \to D^\alpha u^\epsilon \ \text{ locally uniformly as } \ \sigma \to 0,$$

for a smooth function u^ϵ solving (4.2), (4.3). Estimate (4.7) follows from (4.10), (4.11), (4.12). □

4.3 Passage to limits

Theorem 4.2 *Assume* $g : \mathbb{R}^n \to \mathbb{R}$ *is continuous and verifies (4.1). Then there exists a weak solution* u *of (2.5), (2.6), such that*

$$(4.13) \qquad u \ \text{ is constant on } \ \mathbb{R}^n \times [0,\infty) \cap \{|x| + t \geq R\}.$$

for some $R > 0$*, depending only on the* S *from (4.1).*

Proof. 1. Suppose temporarily g is smooth. Employing estimate (4.7) we can extract a subsequence $\{u^{\epsilon_k}\}_{k=1}^{\infty}, \subset \{u^{\epsilon}\}$, $0 < \epsilon \leq 1$ so that $\epsilon_k \to 0$ and $u^{\epsilon_k} \to u$ locally uniformly in $\mathbb{R}^n \times [0,\infty)$ for some bounded, Lipschitz function u.

2. We assert now that u is a weak solution of (2.5), (2.6). For this, let $\phi \in C^{\infty}(\mathbb{R}^{n+1})$ and suppose $u - \phi$ has a *strict* local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$. As $u^{\epsilon_k} \to u$ uniformly near (x_0, t_0), $u^{\epsilon_k} - \phi$ has a local maximum at a point (x_k, t_k), with

$$(4.14) \qquad (x_k, t_k) \to (x_0, t_0) \qquad \text{as } \ k \to \infty.$$

Since u^{ϵ_k} and ϕ are smooth, we have

$$Du^{\epsilon_k} = D\phi,\ u_t^{\epsilon_k} = \phi_t,\ D^2 u^{\epsilon_k} \leq D^2\phi \qquad \text{at } \ (x_k, t_k)$$

Thus (4.2) implies

$$(4.15) \qquad \phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2 + \epsilon_k^2}\right)\phi_{x_i x_j} \leq 0 \qquad \text{at } \ (x_k, t_k).$$

Suppose first $D\phi(x_0, t_0) \neq 0$. Then $D\phi(x_k, t_k) \neq 0$ for large k. We consequently may pass to limits in (4.15), recalling (4.14) to deduce,

$$(4.16) \qquad \phi_t \leq \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} \quad \text{at } \ (x_0, t_0).$$

Next, assume instead $D\phi(x_0, t_0) = 0$. Set

$$\eta^k \equiv \frac{D\phi(x_k, t_k)}{(|D\phi|(x_k, t_k)|^2 + \epsilon_k^2)^{\frac{1}{2}}},$$

so that (4.15) becomes

$$\phi_t \leq (\delta_{ij} - \eta_i^k \eta_j^k)\, \phi_{x_i x_j} \qquad \text{at} \ \ (x_k, t_k). \tag{4.17}$$

Since $|\eta^k| \leq 1$, we may assume, upon passing to a subsequence and reindexing if necessary, that

$$\eta^k \to \eta \qquad \text{in} \ . \ ^n$$

for some $|\eta| \leq 1$. Sending k to infinity in (4.17), we discover

$$\phi_t \leq (\delta_{ij} - \eta_i \eta_j)\, \phi_{x_i x_j} \qquad \text{at} \ \ (x_0, t_0). \tag{4.18}$$

If $u - \phi$ has a local maximum, but not necessarily a strict local maximum at (x_0, t_0), we repeat the argument above with $\phi(x, t)$ replaced by

$$\widetilde{\phi}(x, t) = \phi(x, t) + |x - x_0|^4 + (t - t_0)^4,$$

again to obtain (4.16) or (4.18).

Consequently, u is a weak subsolution. That u is a weak supersolution follows analogously.

3. Finally we verify u satisfies (4.13). Upon rescaling as necessary, we may as well assume

$$|g| \leq 1 \ \ \text{on} \ . \ ^n, \ g = 0 \ \ \text{on} \ . \ ^n \cap \{|x| \geq 1\}. \tag{4.19}$$

Consider now the auxilary function (cf. Brakke [, p.25])

$$v(x, t) \equiv \Psi(\frac{|x|^2}{2} + (n-1)\, t) \qquad (x \in . \ ^n, \ t > 0), \tag{4.20}$$

for

$$\Psi(s) = \begin{cases} 0 & (s \geq 2) \\ (s-2)^3 & (0 \leq s \leq 2). \end{cases}$$

Then $\Psi \in C^2([0,\infty))$,

$$\Psi'(s) = \begin{cases} 0 & (s \geq 2) \\ 3(s-2)^2 & (0 \leq s \leq 2), \end{cases}$$

$$\Psi''(s) = \begin{cases} 0 & (s \geq 2) \\ 6(s-2) & (0 \leq s \leq 2) \end{cases}$$

In particular,

(4.21) $$|\Psi''(s)| \leq C(\Psi'(s))^{1/2} \qquad (s \geq 0)$$

Now

$$\begin{aligned} v_t - (\delta_{ij} - \tfrac{v_{x_i} v_{x_j}}{|Dv|^2+\epsilon^2})\, v_{x_i x_j} &= (n-1)\Psi' - (\delta_{ij} - \tfrac{(\Psi')^2 x_i x_j}{(\Psi')^2|x|^2+\epsilon^2})\,(\Psi'\delta_{ij} + \Psi'' x_i x_j) \\ &= \Psi'[(n-1) - (\delta_{ij} - \tfrac{(\Psi')^2 x_i x_j}{(\Psi'^2|x|^2+\epsilon^2})\,\delta_{ij}] \\ &\quad -\Psi''[(\delta_{ij} - \tfrac{(\Psi')^2 x_i x_j}{(\Psi')^2|x|^2+\epsilon^2})\, x_i x_j] \\ &\equiv A + B \end{aligned}$$

(4.22)

We further compute

(4.23) $$A = -\Psi' \frac{\epsilon^2}{(\Psi')^2|x|^2+\epsilon^2} \leq 0,$$

since $\Psi' \geq 0$. Furthermore,

$$|B| = |\Psi''| \frac{\epsilon^2 |x|^2}{(\Psi')^2|x|^2+\epsilon^2}\,.$$

Now if $|\Psi'| \leq \epsilon$, then

(4.24) $$\begin{aligned} |B| \leq |\Psi''||x|^2 &\leq C|\Psi''| && (\text{since } \ \Psi''=0 \ \text{ if } \ |x| \geq 2) \\ &\leq C(\Psi')^{\frac{1}{2}} && (\text{by } \ (4.21)) \\ &\leq C\epsilon^{\frac{1}{2}}\,. \end{aligned}$$

On the other hand if $|\Psi'| \geq \epsilon$, we have

$$|B| \leq |\Psi''| \, \frac{\epsilon^2}{(\Psi')^2} \leq \frac{C\epsilon^2}{|\Psi'|^{3/2}} \leq C\epsilon^{1/2} \tag{4.25}$$

Combining (4.22) - (4.25) we find

$$v_t - (\delta_{ij} - \frac{v_{x_i} v_{x_j}}{|Dx|^2 + \epsilon^2}) \, v_{x_i x_j} \leq C\epsilon^{\frac{1}{2}}$$

and so

$$w_t^\epsilon \leq (\delta_{ij} - \frac{w_{x_i}^\epsilon w_{x_j}^\epsilon}{|Dw^\epsilon|^2 + \epsilon^2}) \, w_{x_i x_j}^\epsilon \qquad \text{in } \ ^n \times (0, \infty) \tag{4.26}$$

for

$$w^\epsilon(x,t) = v(x,t) - Ct\epsilon^{\frac{1}{2}}. \tag{4.27}$$

Now

$$w^\epsilon(x,0) = \Psi\,(\frac{|x|^2}{2}) = 0 \qquad \text{if } |x| \geq 2.$$

and

$$w^\epsilon(x,0) = \Psi\,(\frac{|x|^2}{2}) \leq -1 \qquad \text{if } |x| \leq 2.$$

Consequently, we see from (4.19) that

$$w^\epsilon \leq g \qquad \text{on } \ ^n \times \{t = 0\}. \tag{4.28}$$

Applying the maximum principle to (4.2), (4.3), (4.26) and (4.27), we deduce

$$w^\epsilon \leq u^\epsilon \qquad \text{in } \ ^n \times [0, \infty)$$

for each $0 < \epsilon < 1$. Sending $\epsilon = \epsilon_k$ to zero, we then have

$$\Psi\,(\frac{|x|^2}{2} + (n-1)\,t) = v(x,t) \leq u(x,t)$$

for all $x \in \ ^n$, $t \geq 0$. Thus

$$u \geq 0 \qquad \text{if} \quad \frac{|x|^2}{2} + (n+1)\,t \geq 2.$$

Similarly,

$$\tilde{w}^\epsilon_t \geq (\delta_{ij} - \frac{\tilde{w}^\epsilon_{x_i}\tilde{w}^\epsilon_{x_j}}{|D\tilde{w}^\epsilon|^2+\epsilon^2})\, \tilde{w}^\epsilon_{x_i x_j} \qquad \text{in } \mathbb{R}^n \times (0,\infty) \tag{4.29}$$

$$\tilde{w}^\epsilon \geq g \qquad \text{on } \mathbb{R}^n \times (0,\infty), \tag{4.30}$$

for $\tilde{w}^\epsilon \equiv -w^\epsilon$. As above we consequently deduce

$$u \leq 0 \quad \text{ if } \quad \frac{|x|^2}{2} + (n+1)\,t \geq 2.$$

Assertion (4.13) is proved.

4. According to the uniqueness assertion Theorem 3.2, in fact the full limit

$$\lim_{\epsilon\to 0} u^\epsilon = u$$

exists. Note also from Theorem 3.3 that

$$||u - \tilde{u}||_{L^\infty(R^n\times[0,\infty))} = ||g-\tilde{g}||_{L^\infty(R^n)}\,, \tag{4.31}$$

if $\tilde{u}$ is the solution built as above for a smooth initial function $\tilde{g}$ verifying (4.1).

Suppose at last g satisfies (4.1), but is only continuous. We select smooth $\{g^k\}_{k=1}^\infty$, satisfying (4.1) (for the same S) so that $g^k \to g$ uniformly on $\mathbb{R}^n$. Denote by u^k the solution of (2.5), (2.6) constructed above with initial function g^k. Utilizing (4.30) we see that the limit

$$\lim_{k\to\infty} u^k = u$$

exists uniformly on $\mathbb{R}^n \times [0,\infty)$. According to Theorem 2.2 u is a weak solution of (2.5), (2.6). $\square$

5 Definition of the Generalized Evolution by Mean Curvature

We now make precise the definition of the motion $\{\Gamma_t\}_{t>0}$ for a given initial hypersurface Γ_0. In fact, let us assume now only that

$$\Gamma_0 \text{ is a compact subset of } \mathbb{R}^n. \tag{5.1}$$

Choose then any continuous function $g\colon \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\Gamma_0 = \{x \in \mathbb{R}^n \mid g(x) = 0\} \tag{5.2}$$

and

$$g \text{ constant on } \mathbb{R}^n \cap \{|x| \geq S\} \tag{5.3}$$

for some $S > 0$. Utilizing Theorem 3.2, 4.1, we see that there is a unique weak solution of the mean curvature evolution equation

$$u_t = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2})\, u_{x_i x_j} \qquad \text{in } \mathbb{R}^n \times (0, \infty) \tag{5.4}$$

$$u = g \text{ on } \mathbb{R}^n \times \{t = 0\}, \tag{5.5}$$

with

$$u \text{ constant on } \mathbb{R}^n \times [0, \infty) \cap \{|x| + t \geq R\} \tag{5.6}$$

for some $R > 0$.

Define then the compact set

$$\Gamma_t \equiv \{x \in \mathbb{R}^n \mid u(x, t) = 0\} \tag{5.7}$$

for each $t > 0$. We call $\{\Gamma_t\}_{t>0}$ the *generalized evolution by mean curvature* of the original compact set Γ_0.

We must first verify that $\{\Gamma_t\}_{t>0}$ is well-defined.

Theorem 5.1 *Assume* $\widehat{g}: \mathbb{R}^n \to \mathbb{R}$ *is continuous, with*

$$\Gamma_0 = \{x \in \mathbb{R}^n \mid \widehat{g}(x) = 0\} \tag{5.8}$$

and

$$\widehat{g} \text{ constant} \quad \text{on } \mathbb{R}^n \cap \{|x| \geq S\}. \tag{5.9}$$

Suppose $\widehat{u}$ *is the unique weak solution of (5.4)-(5.6), with* $\widehat{g}$ *replacing* g. *Then*

$$\Gamma_t = \{x \in \mathbb{R}^n \mid \widehat{u}(x,t) = 0\} \tag{5.10}$$

for each $t > 0$.

Consequently our definition (5.7) does not depend upon the particular choice of initial function g *verifying (5.2), (5.3).*

Proof. 1. First, we may as well assume $g \geq 0$ on $\mathbb{R}^n$ and thus $u \geq 0$, in $\mathbb{R}^n \times (0, \infty)$. Indeed, if g is negative somewhere, we can consider the PDE (5.4)–(5.6) with $|g|$ replacing g, the unique solution of which, owing to Theorems 2.3, 3.2 is $|u|$. Our definition (5.7) is unchanged if we replace u by $|u|$. Similarly we may suppose $\widehat{g}$, $\widehat{u} \geq 0$. Set

$$\widehat{\Gamma}_t \equiv \{x \in \mathbb{R}^n \mid \widehat{u}(x,t) = 0\} \qquad (t \geq 0).$$

2. For $k = 1, 2, \ldots$ write $E_0 = \emptyset$ and

$$E_k \equiv \{x \in \mathbb{R}^n \mid g(x) \geq \frac{1}{k}\}\,;$$

so that

$$E_1 \subset \ldots E_k \subset E_{k+1} \subset \ldots, \; \mathbb{R}^n - \Gamma_0 = \bigcup_{k=1}^{\infty} E_k \tag{5.11}$$

Define

$$a_k \equiv \max_{R^n - E_{k-1}} \widehat{g} > 0 \qquad (k = 1, \ldots) \tag{5.12}$$

Then $a_1 \geq a_2 \geq \ldots$ and $lim_{k\to\infty} a_k = 0$, according to (5.8) and (5.11). Next define the continuous function

$$\Psi\colon [0,\infty) \to [0,\infty)$$

satisfying

$$\begin{cases} \Psi(0) = 0, & \\ \Psi(\frac{1}{k}) = a_k & (k = 1, \ldots) \\ \Psi \ \text{linear on} \ [\frac{1}{k+1}, \frac{1}{k}] & (k = 1, \ldots) \\ \Psi \ \text{constant on} \ [1,\infty) & \\ \Psi \ \text{non-decreasing} & \end{cases}$$

3. Write

$$\widetilde{g} = \Psi(g),\ \widetilde{u} = \Psi(u).$$

Then $\widetilde{u}$ solves (5.4)–(5.6), with $\widetilde{g}$ replacing g. Now

$$\widetilde{g} = \widehat{g} \qquad \text{on } \Gamma_0.$$

Furthermore, if $x \in E_k - E_{k-1}$, then

$$\begin{aligned} \widetilde{g}(x) &= \Psi(g(x)) \\ &\geq \Psi(\tfrac{1}{k}) \\ &= a_k \\ &\geq \widehat{g}(x) \qquad \text{by (5.12)} \end{aligned}$$

Thus $\widetilde{g} \geq \widehat{g}$ on $\mathbb{R}^n$. Consequently, Theorem 3.2 asserts

$$\widetilde{u} = \Psi(u) \geq \widehat{u} \geq 0 \quad \text{on } \mathbb{R}^n \times [0,\infty).$$

Thus if $x \in \Gamma_t$, $\widehat{u}(x,t) = 0$ and so $x \in \widehat{\Gamma}_t$. Hence

$$\Gamma_t \subseteq \widehat{\Gamma}_t.$$

The opposite inclusion is similarly proved and so

$$\Gamma_t = \widehat{\Gamma}_t \qquad \text{for each } t > 0.$$

$\square$

In light of this theorem, we can regard the mappings

$$\Gamma_0 \longmapsto \Gamma_t \qquad (t \geq 0)$$

as defining a time-dependent evolution on the collection $\mathcal{K}$ of compact subsets of $\mathbb{R}^n$. Let us write

$$\mathcal{M}(t)\Gamma_0 \equiv \Gamma_t \qquad (t \geq 0) \tag{5.13}$$

explicitly to display the dependence of Γ_t on t and Γ_0. Then

$$\mathcal{M}(t) : \mathcal{K} \to \mathcal{K}$$

for each $t \geq 0$, and $\mathcal{M}(0)$ is the identity operator. We will call $\{\mathcal{M}(t)\}_{t \geq 0}$ the *mean-curvature semigroup on* $\mathcal{K}$.

To justify this terminology, let us verify the semigroup property

$$\mathcal{M}(t+s) = \mathcal{M}(t)\mathcal{M}(s) \qquad (t, s \geq 0). \tag{5.14}$$

Indeed if $t, s > 0$ and $\Gamma_0 \in \mathcal{K}$, choose any continuous function g satisfying (5.2), (5.3). Let u be the corresponding unique weak solution of (5.4)–(5.6). Then

$$\mathcal{M}(t+s)\Gamma_0 = \Gamma_{t+s} = \{x \in \mathbb{R}^n \mid u(x, t+s) = 0\}. \tag{5.15}$$

$$\mathcal{M}(s)\Gamma_0 = \Gamma_s = \{x \in \mathbb{R}^n \mid u(x, s) = 0\}. \tag{5.16}$$

To compute $\mathcal{M}(t)\Gamma_s$ we select any continuous function $\widehat{g}$ so that

$$\Gamma_s = \{x \in \mathbb{R}^n \mid \widehat{g}(x) = 0\} \tag{5.17}$$

and $\widehat{g}$ is constant outside some large ball. We then find the unique weak solution $\widehat{u}$ of (5.4)–(5.6) (with $\widehat{g}$ replacing g) and set

$$\mathcal{M}(t)\Gamma_s = \widehat{\Gamma}_t = \{x \in \mathbb{R}^n \mid \widehat{u}(x, t) = 0\}. \tag{5.18}$$

According to Theorem 5.1, this construction is independent of the particular choice of $\widehat{g}$ satisfying (5.17). In particular, we may as well take

$$\widehat{g}(x) = u(x,s) \qquad (x \in \mathbb{R}^n).$$

Owing then to the uniqueness of a weak solution to (5.4)-(5.6) we have

$$\widehat{u}(x,t) = u(x,t+s) \qquad (x \in \mathbb{R}^n,\ t > 0)$$

Consequently (5.15), (5.18) imply

$$\mathcal{M}(t+s)\Gamma_0 = \mathcal{M}(t)\mathcal{M}(s)\Gamma_0,$$

as required. This establishes (5.14).

Note that we make no assertions concerning continuity of the mapping

$$(t, \Gamma_0) \longmapsto \mathcal{M}(t)\Gamma_0.$$

6 Consistency with Classical Motion by Mean Curvature

We must now check that our generalized evolution by mean curvature agrees with the classical motion, if and so long as the latter exists. Let us therefore suppose for this section that Γ_0 is a smooth hypersurface, the connected boundary of a bounded open set $U_0 \subset {}^n$. According to Hamilton [], Gage-Hamilton [], there exists a time $t_* > 0$ and a family $\{\Sigma_t\}_{0 \le t < t_*}$ of smooth hypersurfaces evolving from $\Sigma_0 = \Gamma_0$ according to classical motion by mean curvature. In particular for each $0 \le t < t_*, \Sigma_t$ is diffeomorphic to Γ_0, and is the boundary of an open set U_t diffeomorphic to U_0.

Theorem 6.1 *We have*

$$\Sigma_t = \Gamma_t \qquad (0 \le t < t_*)$$

where $\{\Gamma_t\}$ is the generalized evolution by mean curvature defined in §5.

Proof. 1. Fix $0 < t_0 < t_*$, and define then for $0 \le t \le s$ the (signed) distance function

$$d(x,t) = \begin{cases} -dist(x, \Sigma_t) & \text{if } \ x \in U_t \\ dist(x, \Sigma_t) & \text{if } \ x \in {}^n \setminus U_t \end{cases}$$

As $\Sigma = \bigcup_{0 \le t \le t_0} \Sigma_t$ is smooth, d is smooth in the regions

$$Q^+ \equiv \{(x,t) \mid 0 \le d(x,t) \le \delta_0, \, 0 \le t \le t_0\}$$

and

$$Q^- \equiv \{(x,t) \mid -\delta_0 \le d(x,t) \le 0, \, 0 \le t \le t_0\}$$

for $\delta_0 > 0$ sufficiently small.

2. Now for each point $(x,t) \in Q^+$ there exists a unique point $y \in \Sigma_t$ verifying $d(x,t) = |x-y|$. Consider now near (y,t) the smooth unit vector field $\nu = Dd$

pointing from Σ into Q^+. Then

$$d_t(x,t) = (\operatorname{div}\nu)\,(y,t) \tag{6.1}$$

since $\{\Sigma_t\}_{0\le t\le s}$ is a classical evolution by mean curvature. Additionally, the eigenvalues of $D^2 d(x,t)$ are (see e.g. [[?], p.355])

$$\left\{\frac{-k_1}{1-k_1 d}, \dots, \frac{-k_{n-1}}{1-k_{n-1}d},\ 0\right\}, \tag{6.2}$$

$k_1, \dots, k_{n-1}$ denoting the principal curvatures of Σ_t at the point y, calculated with respect to the unit normal field ν. Thus

$$\Delta d(x,t) = -\sum_{i=1}^{n-1} \frac{k_i}{1-k_i d} \tag{6.3}$$

However $(\operatorname{div}\,\nu)$ $(y,t) = -(k_1 + \cdots + k_{n-1})$; and so (6.1) (6.3) imply

$$d_t - \Delta d = \left(\sum_{i=1}^{n-1} \frac{k_i^2}{1-k_i d}\right) d \qquad \text{at } (x,t). \tag{6.4}$$

Since the quantity

$$\sum \frac{k_i^2}{1-k_i d}$$

is uniformly bounded and $d \ge 0$ in Q^+, we deduce from (6.4) that

$$\overline{d} \equiv \alpha e^{-\lambda t} d \tag{6.5}$$

satisfies

$$\overline{d}_t - \Delta \overline{d} \le 0 \qquad \text{in } Q^+ \tag{6.6}$$

if $\lambda > 0$ is fixed large enough and $\alpha > 0$ (to be selected later.) Furthermore, $|Dd|^2 = |\nu|^2 = 1$ and so

$$d_{x_i} d_{x_i x_j} = 0 \ \text{ in } Q^+, \qquad 1 \le j \le n.$$

The function $\overline{d}$ satisfies the same identity, whence (6.6) implies for each $\epsilon \geq 0$ that

$$\overline{d}_t - \left(\delta_{ij} - \frac{\overline{d}_{x_i} \overline{d}_{x_j}}{|D\overline{d}|^2 + \epsilon^2} \right) \overline{d}_{x_i x_j} \leq 0 \qquad \text{in } Q^+. \tag{6.7}$$

We see therefore that $\overline{d}$ is a smooth subsolution of the approximate mean curvature evolution PDE (4.2) in Q^+.

3. Choose any Lipschitz function $g: \mathbb{R}^n \to \mathbb{R}$ so that $g(x) = dist(x, \Sigma_0)$ near Σ_0, $\{g = 0\} = \Sigma_0$ and $g(x)$ is a positive constant for large $|x|$. For $0 < \epsilon < 1$ the approximating PDE (4.2) (4.3) then has a continuous solution u^ϵ which is smooth in $\mathbb{R}^n \times (0, \infty)$. Additionally we have

$$\Gamma_t = \{x \in \mathbb{R}^n \mid u(x,t) = 0\} \qquad t \geq 0. \tag{6.8}$$

Now $u = g = \delta_0 > 0$ on $\{(x,0) \mid dist(x, \Sigma_0) = dist(x, \Gamma_0) = \delta_0\}$ and as u is continuous, we thus have

$$u \geq \frac{d}{2} > 0 \text{ on } \{(x,t) \mid d(x,t) = \delta_0\} \tag{6.9}$$

for $0 \leq t \leq t_0$, provided $t_0 > 0$ is small enough. Hence (6.9) implies

$$u^\epsilon \geq \frac{d}{4} \text{ on } \{(x,t) \mid d(x,t) = \delta_0\}$$

for $0 \leq t \leq t_0$, $0 < \epsilon \leq \epsilon_0$ if $\epsilon_0 > 0$ is sufficiently small. Consequently there exists $0 < \alpha < 1$ so that

$$u^\epsilon \geq \overline{d} \qquad \text{on } \{(x,t) \mid d(x,t) = \delta_0\} \tag{6.10}$$

for $0 \leq t \leq t_0$, $0 < \epsilon < \epsilon_0$, $\overline{d}$ defined by (6.5). Since $0 < \alpha < 1$, we have

$$u^\epsilon \geq \overline{d} \text{ on } \{(x,0) \mid 0 \leq d(x,t) \leq \delta_0\} \tag{6.11}$$

Furthermore, $g \geq 0$ implies $u^\epsilon \geq 0$ and so

$$u^\epsilon \geq \overline{d} \text{ on } \{(x,t) \mid d(x,0) = 0\} \tag{6.12}$$

4. Combining (6.10)-(6.12) we see that $u^\epsilon \geq \overline{d}$ on the parabolic boundary of Q^+. Since $\overline{d}$ solves (6.7) and u^ϵ solves (4.2), the maximum principle implies

$$u^\epsilon \geq \overline{d} \text{ in } Q^+$$

Let $\epsilon \to 0$ to deduce

$$u > 0 \text{ in the interior of } Q^+ \tag{6.13}$$

A similar argument using instead

$$\overline{d} = -\alpha\epsilon^{-\lambda t} d$$

shows

$$u > 0 \text{ in the interior of } Q^-. \tag{6.14}$$

Since $u > 0$ in $(R^n \setminus \{dist(x, \Sigma_0) \leq \delta_0\}) \times [0, t_0]$, we deduce from (6.13) (6.14) and (6.8) that

$$\Gamma_t \subseteq \Sigma_t = \{x \mid d(x,t) = 0\} \qquad 0 \leq t \leq t_0. \tag{6.15}$$

5. Now define a new function $\widehat{g}: \mathbb{R}^n \to \mathbb{R}$ so that $\widehat{g}(x) = d(x, 0)$ (the signed distance function to $\Sigma_0 = \Gamma_0$) near $\Sigma_0 = \Gamma_0$, $\{\widehat{g} = 0\} = \Sigma_0$, and $\widehat{g}(x)$ is a positive constant for large $|x|$. Let $\widehat{u}$ denote the unique weak solution of (2.5) (2.6) (4.13) for this new initial function $\widehat{g}$. According to Theorem 5.1

$$\Gamma_t = \{x \in \mathbb{R}^n \mid \widehat{u}(x,t) = 0\} \qquad (t \geq 0) \tag{6.16}$$

Since $\widehat{g} < 0$ in U_0 we know by continuity that $\widehat{u} < 0$ somewhere in U_t, provided $0 \leq t \leq t_0$ and t_0 is small. Similarly $\widehat{u} > 0$ somewhere in $\mathbb{R}^n - \overline{U}_t$ for each $0 \leq t \leq t_0$. Fix any point $x_0 \in \Sigma_t$ and draw a smooth curve C in $\mathbb{R}^n$, intersecting Σ_t precisely at x_0 and connecting a point $x_1 \in U_t$ where $\widehat{u}(x_1, t) < 0$ to a point $x_2 \in \mathbb{R}^n - \overline{U}_t$ where $\widehat{u}(x_2, t) > 0$. As $\widehat{u}$ is continuous, we must have $\widehat{u}(x,t) = 0$ for some point x on the curve C. However (6.15) and (6.16) say that the set $\{\widehat{u}(\cdot,t) = 0\}$ lies in Σ_t. Thus $\widehat{u}(x_0,t) = 0$. Since x_0 denoted any point on Σ_t we deduce from (6.15) (6.16) that

$$\Gamma_t = \Sigma_t \qquad \text{if } \ 0 \leq t \leq t_0. \tag{6.17}$$

We have consequently demonstrated that the classical motion $\{\Sigma_t\}_{0\leq t<t}$ and the generalized motion $\{\Gamma_t\}_{t\geq 0}$ agree at least on some short time interval $[0, t_0]$.

6. Write

$$s \equiv \sup_{0\leq t<t^*} \{t \mid \Gamma_t = \Sigma_t\},$$

and suppose $s < t^*$. Then

$$\Gamma_t = \Sigma_t \ \text{ for all } \ 0 \leq t < s$$

and so, using the continuity of u solving (2.5) and (2.6) for g as above, we have

$$\Gamma_s \supseteq \Sigma_s$$

On the other hand if $x \in \;^n - \Sigma_s$, there exists $r > 0$ so that $B(x,r) \subset \;^n - \Sigma_t$ for all $s - \epsilon \leq t \leq s$, $\epsilon > 0$ small enough. Using this we easily deduce $x \notin \Gamma_s$. Hence

$$\Gamma_s = \Sigma_s.$$

But then applying steps 1–5 we deduce

$$\Gamma_t = \Sigma_t$$

for all $s \leq t \leq s + s_0 < t^*$, if $s_0 > 0$ is small enough. This contradicts the definition of s, and so in fact

$$s = t^*.$$

$\square$

7 Geometric Properties of Generalized Evolution by Mean Curvature

We devote this section to establishing some elementary properties of the generalized evolution by mean curvature

$$\Gamma_0 \longmapsto \mathcal{M}(t)\Gamma_0 \equiv \Gamma_t \qquad (t \geq 0). \tag{7.1}$$

for Γ_0 a compact subset of ^{n}.

7.1 Localization and extinction

First of all, it is known that if Γ_0 is the sphere $\partial B(0, R)$, then

$$\Gamma_t = \begin{cases} \partial B(0,\, R(t)) & \text{if } 0 \leq t < t^* \\ \{0\} & \text{if } t = t^* \\ \emptyset & \text{if } t > t^* \end{cases} \tag{7.2}$$

where

$$R(t) \equiv (R^2 - 2(n-1)\,t)^{\frac{1}{2}} \qquad \text{for } 0 \leq t \leq t^* \equiv \frac{R^2}{2(n-1)}\,. \tag{7.3}$$

This assertion follows in our approach by noting

$$u(x,t) = \Psi\left(|x|^2 + 2(n-1)\,t\right)$$

is a weak solution of (5.4), where $\Psi: \ \to \ $ is smooth with

$$\begin{cases} \Psi' \geq 0,\ \Psi < 0 \quad \text{on } [0, R), \\ \Psi > 0 \text{ on } (R, 3R), \quad \Psi \equiv 1 \text{ on } [3R, \infty). \end{cases} \tag{7.4}$$

By making comparisons with the shrinking sphere (7.2) we derive now some elementary properties of the general motion (7.1). (Cf. Brakke [, p. 29-30]).

Theorem 7.1 *(a) If* $\Gamma_0 \subset B\,(0, R)$, *then*

$$\Gamma_t = \emptyset \qquad \text{for } t > \frac{R^2}{2(n-1)} \tag{7.5}$$

(b) We have

$$\Gamma_t \subseteq co\,(\Gamma_0) \qquad (t \geq 0). \tag{7.6}$$

with $co\,(\Gamma_0)$ *denoting the convex hull of* Γ_0.

Proof. 1. Assume first $\Gamma_0 \subset B(0, R-\epsilon)$ for some $\epsilon > 0$. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be continuous, with

$$\Gamma_0 = \{g = 0\}, \qquad g = 1 \text{ on } \mathbb{R}^n \cap \{|x| \geq 2R\}.$$

Set

$$\widehat{g}(x) = \Psi\,(|x|^2),$$

with Ψ satisfying (7.4) selected so that

$$\widehat{g} \leq g \qquad \text{on } \mathbb{R}^n.$$

Then

$$\widehat{u} \leq u \qquad \text{on } \mathbb{R}^n \times [0, \infty),$$

for $\widehat{u}(x,t) = \Psi\,(|x|^2 + 2(n-1)\,t)$ and u the weak solution of (5.4)-(5.6). Thus $u > 0$, and so $\Gamma_t = \emptyset$, if $t > \frac{R^2}{2(n-1)}$.

In the general case, replace R by $R + \epsilon$ in this argument and send $\epsilon \to 0$.

2. Suppose $\Gamma_0 \subset \mathbb{R}^n_+ = \{x_n > 0\}$. Choose $R >> 1$ so large that

$$\Gamma_0 \subset B\,(Re_n, R),$$

for $e_n = (0, 0, \ldots, 0, 1)$. By the argument in step l., we deduce

$$\Gamma_t \subset B\,(Re_n,\, R(t))$$

for $0 \leq t \leq \frac{R^2}{2(n-1)}$, $R(t)$ defined as above. In particular,

$$\Gamma_t \subseteq \mathbb{R}^n_+ \qquad \text{for all } \; t \geq 0.$$

Replacing $\mathbb{R}^n_+$ in this argument by any open half space containing Γ_0, we deduce (7.6). □

7.2 Comparison of different sets moving by mean curvature

Theorem 7.2 *Let Γ_0, $\widehat{\Gamma}_0$ be compact subsets of $\mathbb{R}^n$ and denote by $\{\Gamma_t\}_{t \geq 0}$, $\{\widehat{\Gamma}_t\}_{t \geq 0}$ the corresponding generalized motions by mean curvature. Suppose also*

$$\Gamma_0 \subseteq \widehat{\Gamma}_0. \tag{7.7}$$

Then

$$\Gamma_t \subseteq \widehat{\Gamma}_t \qquad \textit{for each} \; t > 0. \tag{7.8}$$

We see therefore that if a compact set Γ_0 lies within another $\widehat{\Gamma}_0$ at time zero, then the subsequent evolution Γ_t of Γ_0 lies within the subsequent evolution $\widehat{\Gamma}_t$ of $\widehat{\Gamma}_0$. We will see in §8 that this assertion provides us with a powerful tool for studying specific examples.

Proof. Choose continuous functions $g, \widehat{g}: \mathbb{R}^n \to [0, \infty)$ so that

$$\Gamma_0 = \{g = 0\}, \; \widehat{\Gamma}_0 = \{\widehat{g} = 0\},$$

and $g, \widehat{g}$ constant on $\mathbb{R}^n \cap \{|x| \geq S\}$ for some $S > 0$. Replacing g by $g + \widehat{g}$ if necessary, we may assume

$$\widehat{g} \leq g \qquad \text{on} \; \mathbb{R}^n. \tag{7.9}$$

Now let $\widehat{u}, u$ denote the corresponding weak solutions of (5.4)-(5.6). Then (7.9) implies

$$0 \leq \widehat{u} \leq u \qquad \text{on} \; \mathbb{R}^n \times (0, \infty)$$

Thus $x \in \Gamma_t$ implies $x \in \widehat{\Gamma}_t$, and so (7.8) is valid. $\square$

Theorem 7.3 *Assume* $\Gamma_0, \widehat{\Gamma}_0$ *are nonempty compact sets and* $\{\Gamma_t\}_{t\geq 0}, \{\widehat{\Gamma}_t\}_{t\geq 0}$ *are the subsequent generalized motions by mean curvature. Then*

$$dist(\Gamma_0, \widehat{\Gamma}_0) \leq dist(\Gamma_t, \widehat{\Gamma}_t) \qquad (t \geq 0). \tag{7.10}$$

By definition $dist(\Gamma_t, \widehat{\Gamma}_t) = +\infty$ *if* $\Gamma_t = \emptyset, \widehat{\Gamma}_t = \emptyset$, *or both.*

Proof. 1. We may assume $dist(\Gamma_0, \widehat{\Gamma}_0) > 0$. Choose $g: {}^n \rightarrow$ so that

$$\begin{cases} \Gamma_0 = \{g = 0\},\ \widehat{\Gamma}_0 = \{g = 1\}, \\ g = 2 \ \text{ on } \ {}^n \cap \{|x| \geq s\} \ \text{ for some S}, \\ Lip(g) = dist(\Gamma_0, \widehat{\Gamma}_0)^{-1}. \end{cases} \tag{7.11}$$

Then

$$\Gamma_t = \{u = 0\},\ \widehat{\Gamma}_t = \{u = 1\}, \tag{7.12}$$

with u denoting the corresponding weak solution of (5.4)–(5.6).

2. From the contraction property, Theorem 3.3, we see that

$$Lip(u(\cdot, t)) \leq Lip(g) \qquad (t \geq 0). \tag{7.13}$$

If $\Gamma_t \neq \emptyset, \widehat{\Gamma}_t \neq \emptyset$, choose points $x \in \Gamma_t$, $\widehat{x} \in \widehat{\Gamma}_t$ so that

$$|x - \widehat{x}| = dist(\Gamma_t, \widehat{\Gamma}_t).$$

Then using (7.11)–(7.13) we compute

$$1 = u(\widehat{x}, t) - u(x, t) \leq Lip(u)|x - \widehat{x}| \leq dist(\Gamma_0, \widehat{\Gamma}_0)^{-1}\, dist(\Gamma_t, \widehat{\Gamma}_t).$$

This proves (7.10). $\square$

Inequality (7.10) says in particular that two hypersurfaces evolving under generalized motion by mean curvature do not ever move closer to each other than they were initially. In particular, $\Gamma_t \cap \widehat{\Gamma}_t = \emptyset$ for all $t > 0$ provided $\Gamma_0 \cap \widehat{\Gamma}_0 = \emptyset$. Notice that this property is essential for our approach of representing the evolving surfaces as the level sets of a continuous function.

7.3 Positive mean curvature

Now let us assume that Γ_0 is a smooth connected hypersurface, the boundary of a bounded open set $U \subset \mathbb{R}^n$. We will suppose additionally that

$$div(\nu) > 0 \qquad \text{on } \Gamma_0, \tag{7.14}$$

ν denoting the outward unit normal vector field to Γ_0 (extended smoothly to some neighborhood of Γ_0). Inequality (7.14) says that Γ_0 has positive mean curvature with respect to the outer unit normal field. Consequently, if Γ_0 evolves according to mean curvature, we see from (2.2) that initially at least the motion is directed into U.

We show now that in fact Γ_t lies in U for all $t \geq 0$, and that Γ_t continues to have positive mean curvature, this last statement interpreted in an appropriate weak sense.

Informally our idea is to solve (5.4)–(5.6) by separating variables. Indeed we will show

$$u(x,t) \equiv v(x) - t \qquad (x \in U,\, t > 0), \tag{7.15}$$

where v is the (unique) weak solution of the stationary problem

$$-\,(\delta_{ij} - \frac{v_{x_i} v_{x_j}}{|Dv|^2})\, v_{x_i x_j} = 1 \ \text{ in } \ U \tag{7.16}$$

$$v = 0 \ \text{ on } \ \partial U = \Gamma_0. \tag{7.17}$$

We will show that

$$\Gamma_t = \{x \in U \mid v(x) = t\} \qquad (t \geq 0), \tag{7.18}$$

so that $\Gamma_t \subset U\,(t \geq 0)$ and $\Gamma_t = \emptyset$ for $t \geq t^* \equiv \| v \|_{L^\infty(U)}$. Note also that in any open region where v is smooth and $|Dv| \neq 0$, we can rewrite (7.16) to read

$$div(\nu) = \frac{1}{|Dv|} > 0,$$

for

$$\nu \equiv -\frac{Dv}{|Dv|}$$

As ν is the outward pointing unit normal field along $\Gamma_t \equiv \{v = t\}$, we can informally interpret our PDE (7.16) as implying Γ_t to have positive mean curvature for $0 \leq t < t^*$.

To carry out the foregoing program rigorously, let us first define $v \in C(\overline{U})$ to be a weak solution to (7.16) provided for each $\phi \in C^\infty(\ ^n)$ if

$$u - \phi \ \text{ has a local maximum (minimum) at a point } \ x_0 \in U,$$

then

$$\begin{cases} -(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2})\,\phi_{x_i x_j} \leq (\geq)\, 1 \ \text{ at } \ x_0 \\ \\ \text{if } \ D\phi\,(x_0) \neq 0 \end{cases} \tag{7.19}$$

and

$$\begin{cases} -(\delta_{ij} - \eta_i \eta_j)\,\phi_{x_i x_j} \leq (\geq)\, 1 \ \text{ at } \ x_0 \\ \\ \text{for some } \ \eta \in \ ^n \ \text{ with } \ |\eta| \leq 1, \ \text{ if } \ D\phi\,(x_0) = 0. \end{cases} \tag{7.20}$$

Theorem 7.4 *There exists a unique weak non-negative solution v of (7.16), (7.17). Furthermore, there exist constants $A, a > 0$ so that*

$$\begin{array}{ll} a\, dist(x, \Gamma_0) \leq v(x) \leq \sqrt{n}\, A\, dist(x, \Gamma_0) & \quad x \in U \\ \qquad\qquad |Dv(x)| \leq \sqrt{n}\, A & \end{array} \tag{7.21}$$

Proof. 1. Similarly to §4, we approximate (7.16), (7.17) by the uniformly elliptic PDE

$$-\,(\delta_{ij} - \frac{v^\epsilon_{x_i} v^\epsilon_{x_j}}{|Dv^\epsilon|^2 + \epsilon^2})\, v^\epsilon_{x_i x_j} = 1 \qquad \text{in } \ U \tag{7.22}$$

$$v^\epsilon = 0 \qquad\qquad \text{on } \ \partial U = \Gamma_0 \tag{7.23}$$

for $0 < \epsilon \leq 1$. We will construct upper an lower barriers for (7.22), (7.33) of the form

$$\overline{v}(x) = \lambda g(d(x)), \qquad\qquad d(x) = dist(x, \Gamma_0)$$

in a neighborhood $U_0 = \{0 < d(x) < 2\delta_0\}$ of Γ_0 on which $d(x)$ is smooth. Owing to the mean curvature condition (7.14), $d(x)$ satisfies

$$-c_2 \le \Delta\, d \le -c_1 < 0 \qquad d_{x_i} d_{x_j} d_{x_i x_j} \equiv 0 \tag{7.24}$$

in this region. We then compute using (7.21)

$$\begin{aligned} M\overline{v} \equiv (\delta_{ij} - \tfrac{\overline{v}_{x_i}\overline{v}_{x_j}}{|D\overline{v}|^2+\epsilon^2})\,\overline{v}_{x_i x_j} &= \lambda(g'\Delta\, d + g'') - \tfrac{\lambda^3}{\lambda^2 g'^2+\epsilon^2}\, g'^2 g'' \\ &= \lambda g' \Delta\, d + \tfrac{\epsilon^2 \lambda g''}{\lambda^2 g'^2+\epsilon^2} \end{aligned} \tag{7.25}$$

Choosing $g(t) = \delta_0^2 - (t-\delta_0)^2$, we find from (7.24), (7.25) that $M\overline{v} \ge -c\lambda\delta_0 - 2\lambda > -1$ for λ sufficiently small. Since $\overline{v} = 0$ on ∂U_0, $\overline{v} < v^\epsilon$ in U_0 by the maximum principle. In particular, $v^\epsilon > \lambda\delta_0^2$ on $\{d(x) = \delta_0\}$ and so $v^\epsilon > \lambda\delta_0^2$ on $\{d(x) > \delta_0\}$. These estimates give the lower bound

$$v^\epsilon(x) \ge a\, d(x) \qquad \text{in } U$$

with a independent of ϵ. To obtain the corresponding upper bound, we choose

$$g(t) = \ln(2\delta_0/2\delta_0 - t)$$

Then $g(t)$ is convex on $[0, 2\delta_0)$ and satisfies

$$g(0) = 0,\ g' \ge \frac{1}{2\delta_0},\ g'' = g'^2,\ g'(2\delta_0) = +\infty \tag{7.26}$$

Again using (7.24)–(7.26), we find

$$M\overline{v} \le -c\lambda + \frac{\epsilon^2}{\lambda} < -1$$

for λ sufficiently large. Since $\frac{\partial \overline{v}}{\partial v} = +\infty$ on $\{d(x) = 2\delta_0\}$ (ν the exterior normal to U_0) we find that $v^\epsilon < \overline{v}$ in U_0 by a simple variant of the maximum principle. This gives the estimate

$$v^\epsilon \le A\, d(x) \qquad \text{for} \qquad 0 < d(x) < \delta_0. \tag{7.27}$$

To complete our preliminary estimates, we observe that (7.27) implies $|Dv^\epsilon| \leq A$ on Γ_0. By differentiating (7.22) with respect to x_ℓ, we see that any derivative $v^\epsilon_{x_\ell}$ achieves its maximum and minimum on Γ_0. Thus

$$|v^\epsilon_{x_\ell}| \leq A \text{ in } U \text{ and so } |Dv^\epsilon| \leq \sqrt{n}\, A \text{ in } U.$$

In particular $v^\epsilon \leq \sqrt{n}\, A\, d(x)$ in U_0.
2. As a consequence of step 1., we derived the uniform bounds

$$\sup_{0<\epsilon\leq 1} \| v^\epsilon \|_{c^1(U)} < \infty$$

Hence we may extract a subsequence $\{v^{\epsilon_k}\}_{k=1}^\infty \subset \{v^\epsilon\}_{0<\epsilon\leq 1}$ so that $\epsilon_k \to 0$ and $v^{\epsilon_k} \to v$ uniformly on $\overline{U}$. Now as in the proof of Theorem 4.2, we verify that v is a weak solution of (7.16).
3. The uniqueness of this weak solution v will follow from the characterization of $\{\Gamma_t\}_{t\geq 0}$ below. □

Theorem 7.5 *Let $\{\Gamma_t\}_{t\geq 0}$ denote the generalized evolution by mean curvature starting with Γ_0. Then*

$$\Gamma_t = \{x \in U \mid v(x) = t\}$$

for each $t \geq 0$.

Proof. 1. Define $u(x,t) \equiv v(x) - t$ for $x \in U$, $t > 0$. It is then straightforward to verify that u is a weak solution of

$$u_t = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2})\, u_{x_i x_j} \qquad \text{in } U \times (0,\infty) \tag{7.28}$$

set

$$\widehat{\Gamma}_t \equiv \{x \in U \mid v(x) = t\} = \{x \in U \mid u(x,t) = 0\} \qquad (t > 0) \tag{7.29}$$

2. Now let

$$\widetilde{u}(x,t) = |u(x,t)| = |v(x) - t| \qquad\qquad (x \in U,\, t > 0)$$

In view of Theorem 2.3 $\widetilde{u}$ is a weak solution of

$$\begin{cases} \widetilde{u}_t = (\delta_{ij} - \frac{\widetilde{u}_{x_i}\widetilde{u}_{x_j}}{|D\widetilde{u}|^2})\, \widetilde{u}_{x_i x_j} & \text{in } U \times (0,\infty) \\ \widetilde{u} = t \ \text{ on } \ \partial U \times [0,\infty) & \\ \widetilde{u} = v \ \text{ on } \ \overline{U} \times \{t = 0\}. & \end{cases} \tag{7.30}$$

3. Choose any smooth function $g\colon \mathbb{R}^n \to \mathbb{R}$ so that

$$\begin{cases} \Gamma_0 = \{g = 0\},\ g \geq 0 \\ g \ \text{ is constant on } \ \mathbb{R}^n\{|x| \geq S\} \ \text{ for some } \ S > 0. \end{cases} \tag{7.31}$$

Let $w \geq 0$ be the unique weak solution of

$$\begin{cases} w_t = (\delta_{ij} - \frac{w_{x_i} w_{x_j}}{|Dw|^2})\, w_{x_i x_j} \ \text{ in } \ \mathbb{R}^n \times (0,\infty) \\ w = g \ \text{ on } \ \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{7.32}$$

so that

$$\Gamma_t = \{x \in \mathbb{R}^n \mid w(x,t) = 0\} \qquad (t \geq 0). \tag{7.33}$$

According to our construction in §4, w is Lipschitz in t, and so

$$|w(x,t)| \leq c\,t \qquad\qquad (x \in \Gamma_0,\, t > 0). \tag{7.34}$$

for some constant c.

4. Employing now (7.21), (7.27) and (7.30), we see that

$$\underline{w} = \alpha\, w$$

satisfies

$$\begin{cases} \underline{w} \leq v \ \text{ on } \ \overline{U} \times \{t = 0\} \\ \underline{w} \leq t \ \text{ on } \ \partial U \times (0,\infty) \end{cases} \tag{7.35}$$

if $\alpha > 0$ is sufficiently small.

Now the proof of our Comparison Theorem 3.2 can be straightforwardly modified to show from (7.26), (7.28), (7.31) that

$$\underline{w} \leq \tilde{u} \qquad \text{in } U \times [0, \infty).$$

Thus $x \in \widehat{\Gamma}_t$ implies $x \in \Gamma_t$, and so

$$\widehat{\Gamma}_t \subseteq \Gamma_t \qquad \text{for } t \geq 0.$$

Similarly selecting

$$\overline{w} = \beta\, w$$

for some $\beta >> 1$, we deduce

$$\tilde{u} \leq \overline{w} \quad \text{in } U \times [0, t_0).$$

Thus

$$\Gamma_t \subseteq \widehat{\Gamma}_t \qquad \text{for } t \geq 0.$$

$\square$

7.4 Convexity

Theorem 7.6 *Assume Γ_0 is the boundary of a smooth convex bounded open set U. Then there exists $t^* > 0$ such that Γ_t is the boundary of a convex, nonempty open set for $0 \leq t < t^*$ and Γ_t is empty for $t > t^*$.*

Proof. 1. Because of §7.3 it suffices to consider the stationary PDE

$$(7.36) \qquad (\delta_{ij} - \frac{v_{x_i}\, v_{x_j}}{|Dv|^2})\, v_{x_i x_j}, = -1 \quad \text{in } U, \qquad v = 0 \quad \text{on } \Gamma_0 = \partial U$$

We will show that $\{x \in U \mid v(x) > A\}$ is convex for $0 \leq t < t^*$, for $t^* = \sup v$. In fact, we will show that $\sqrt{v(x)}$ is concave using ideas of Korevaar

[] and Kennington []. The motivation for considering $\sqrt{v}$ comes from the corresponding linear problem $\Delta v = -1$.

Formally, if $u = \sqrt{v}$, with v satisfying (7.36), than u satisfies

$$(7.37) \qquad (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2})\, u_{x_i x_j} = -\frac{1}{2u} \qquad \text{in } U$$

This suggests we consider approximations $u^\epsilon = \sqrt{v^\epsilon}$ satisfying

$$(7.38) \qquad \begin{aligned} Mu^\epsilon = (\delta_{ij} - \tfrac{u^\epsilon_{x_i} u^\epsilon_{x_j}}{|Du^\epsilon|^2+\epsilon^2})\, u^\epsilon_{x_i x_j} &= -\tfrac{1}{2u^\epsilon} && \text{in } U \\ u^\epsilon &= 0 && \text{on } \Gamma_0 \end{aligned}$$

$$(7.39) \qquad \left(\delta_{ij} - \frac{v^\epsilon_{x_i} v^\epsilon_{x_j}}{|Dv^\epsilon|^2 + 4\epsilon^2 v^\epsilon}\right) v^\epsilon_{x_i x_j} = \frac{1}{2} \frac{\epsilon^2\, |Dv^\epsilon|^2}{|Dv^\epsilon|^2 + 4\epsilon^2 v^\epsilon} - 1$$

Because the convexity arguments are very sensitive to the form of the equation, we are forced into making a nice approximation u^ϵ to (7.37) and then making due with nastier approximations v^ϵ to (7.36).

2. We next demonstrate the existence of a solution $u^\epsilon \in C^2(U) \cap C^{1/2}(\overline{U})$ to (7.38). Consider

$$(7.40) \qquad \begin{aligned} Mu^{\epsilon,\delta} \equiv \left(\delta_{ij} - \tfrac{u^{\epsilon,\delta}_{x_i} u^{\epsilon,\delta}_{x_j}}{|Du^{\epsilon,\delta}|^2+\epsilon^2}\right) u^{\epsilon,\delta}_{x_i x_j} &= -\tfrac{1}{2(u^{\epsilon,\delta}+\delta)} && \text{in } U \\ u^{\epsilon,\delta} &= 0 && \text{on } \Gamma_0 \end{aligned}$$

Equation (7.40) being uniformly elliptic and strictly coercive in $u^{\epsilon,\delta}$, the existence of a solution $u^{\epsilon,\delta} \in C^{2+\alpha}\overline{(U)}$ is well-known [GT].

Choose a large ball $B_R(P)$ containing U with $dist(P, U) \geq R/2$, and let $r = |x - P|$.

Set $w = (2R - r)$. Then

$$Mw + \frac{1}{2(w+\delta)} = -\frac{(n-1)}{r} + \frac{1}{2(2R-r)} \leq -\frac{(n-1)}{R} + \frac{1}{3R} < 0$$

Since $w > 0$ on ∂U, $w > u^{\epsilon,\delta}$ in U by the maximum principle. Hence

$$(7.41) \qquad 0 \leq u^{\epsilon,\delta} < 2R \qquad \text{in } U$$

with R independent of ϵ, δ.

Next, let $w = \lambda\sqrt{d(x)}$ in $\{0 < d(x) < \delta_0\}$. Using formula 7.25 of §7.3 (with $g(t) = \sqrt{t}$) we find

$$Mw + \frac{1}{2(w+\delta)} \leq -\frac{c\lambda}{\sqrt{d(x)}} + \frac{1}{2(\lambda\sqrt{d}+\delta)} < 0$$

for λ sufficiently large. If in addition, we choose λ so that $\lambda\sqrt{\delta_0} \geq 2R$, then $w \geq u^{\epsilon,\delta}$ on $\partial\{0 < d(x) < \delta_0\}$, and thus $w \geq u^{\epsilon,\delta}$ on $\{0 < d(x) < \delta_0\}$ by the maximum principle. In particular

$$0 \leq u^{\epsilon,\delta} \leq A\sqrt{d(x)} \qquad \text{in } U \tag{7.42}$$

with A independent of ϵ, δ.

Estimate (7.42) implies that

$$|u^{\epsilon,\delta}(x) - u^{\epsilon,\delta}(y)| \leq c|x-y|^{1/2} \qquad if\, x \in U,\, y \in \Gamma_0 \tag{7.43}$$

with c independent of ϵ, δ. We show that (7.43) holds for all $x, y \in \Omega$ by the following well known argument (see e.g. [G]). Given $x, y \in \Omega$ we set $\tau = y - x$, $\Omega_\tau = \{z \in \ .\ ^n | z - \tau \in \Omega\}$ and $u_\tau^{\epsilon,\delta}(z) = u^{\epsilon,\delta}(z-\tau)$. Note that Ω_τ is open and non-empty since $y \in \Omega_\tau$. On $\Omega \cap \Omega_\tau$, both $u^{\epsilon,\delta}$ and $u_\tau^{\epsilon,\delta}$ satisfy (7.40) and hence the difference $w = u^{\epsilon,\delta} - u_\tau^{\epsilon,\delta}$ satisfies a linear elliptic equation of the form $Lw + c(x)w = 0$ with $c(x) \leq 0$. Hence by the maximum principle,

$$|w(y)| \leq \max_{z \in \partial(\Omega\cap\Omega_\tau)} |w(z)|$$

Since for $z \in \partial(\Omega \cap \Omega_\tau)$ either $z \in \partial\Omega$ or $z - \tau \in \partial\Omega$ we have by (7.43) that

$$|u^{\epsilon,\delta}(y) - u^{\epsilon,\delta}(x)| = |u^{\epsilon,\delta}(y) - u_\tau^{\epsilon,\delta}(y)| \leq c|x-y|^{1/2} \tag{7.44}$$

Finally, in order to pass to the limit for a sequence $\delta_k \searrow 0$, we need to establish interior estimates for $u_k = u^{\epsilon,\delta_k}$.

$$||u_k||_{C^{2+\alpha}(U')} \leq M(\epsilon, dist(U', \Gamma_0)) \tag{7.45}$$

with M independent of δ_k. By Schauder theory, (7.45) follows from an interior gradient estimate

$$||Du_k||_{L^\infty(U')} \leq C(\epsilon, dist(U', \Gamma_0))$$

which in turn follows from Theorem 15.5 of [GT]. Therefore, we have established the existence of a (unique) solution u^ϵ of (7.38), and in addition

$$\begin{aligned} &0 \leq u^\epsilon \leq A\sqrt{d(x)} \\ &|u^\epsilon(x) - u^\epsilon(y)| \leq c|x-y|^{1/2} \\ &0 \leq u^\epsilon \leq 2R \end{aligned} \tag{7.46}$$

with A, C, R independent of ϵ.

3. Before we proceed to the proof of the concavity of u^ϵ, we shall need to establish the lower bound

$$u^\epsilon \geq a\, d(x) \tag{7.47}$$

with a independent of ϵ.

Consider $w = \lambda g(d(x))$ in $\{0 < d(x) < 2\delta_0\}$ with $g(t) = (\delta_0^2 - (t-\delta_0)^2)^{1/2}$. Then from formulas (7.24), (7.25) we find

$$\begin{aligned} Mw \ &\geq -\tfrac{\lambda\delta_0}{g}\left(c + \tfrac{\epsilon^2\delta_0}{\lambda^2(d-\delta_0)^2 + \epsilon^2(\delta_0^2 - (d-\delta_0)^2)}\right) \\ &\geq -\tfrac{\lambda\delta_0}{g}\,(c + \tfrac{1}{\delta_0}) \qquad\quad \text{for } \ \lambda \geq \epsilon \end{aligned}$$

and so

$$Mw + \frac{1}{2w} \geq -\frac{\lambda}{g}\,(\delta_0 c + 1) + \frac{1}{2\lambda g} \geq 0$$

for $\epsilon^2 \leq \lambda^2 = 2(\delta_0 c + 1)$. With this choice, $u^\epsilon \geq w$ in $\{0 < d(x) < 2\delta_0\}$, and as in §7.3 the estimate (7.47) follows easily.

4. We can now show that u^ϵ is concave. For $x, y \in \overline{U}$ set $z = \lambda x + (1-\lambda)y$, $\lambda \in (0,1)$ fixed. The concavity function of u^ϵ, $\mathcal{C}^\lambda \colon U \times U \to$. is defined by

$$\mathcal{C}^\lambda(x,y) = u^\epsilon(z) - \lambda u^\epsilon(x) - (1-\lambda)u^\epsilon(y)$$

The fundamental concavity maximum principle for $\mathcal{C}$ was established by Korevaar [] for a large class of equations $a_{ij}(Du)u_{ij} = b(x,u,Du)$. His result states that for b strictly monotone in u and jointly convex in (x,u), then $\mathcal{C}^\lambda$ cannot have a negative interior minimum on $U \times U$. In the case at hand, $b = -\frac{1}{2u}$ fails to satisfy Korevaar's condition. However, Kennington's improved concavity maximum principle [, Theorem 3.1] works under the assumption that $-\frac{1}{b}$ is convex. In our case $-\frac{1}{b} = u$, so the inf of $\mathcal{C}^\lambda$ is not attained on $U \times U$.

To complete the proof we must essentially show that u^ϵ is concave near Γ_0. Since $u^\epsilon = \sqrt{v^\epsilon}$ satisfies (7.38) (7.39), it is straightforward to see that $v^\epsilon \in C^{2+\alpha}(\overline{U})$ and $Dv^\epsilon \cdot \nu \geq a > 0$ for ν the interior normal to Γ_0. Using the strict convexity of U it is easy to check that

$$u^\epsilon_{\ell\ell} = \frac{1}{2\sqrt{v^\epsilon}}\, v^\epsilon_{\ell\ell} - \frac{(v^\epsilon_\ell)^2}{4(v^\epsilon)^{3/2}}$$

is strictly negative near Γ_0. It follows easily that $\mathcal{C}^\lambda \geq 0$ on $U \times U$ (for complete details, see Lemma 2.4 of [ ] or the proof of Theorem 3.1 of [CS]). This completes the proof that u^ϵ is concave.

5. Since u^ϵ is concave, it follows that $|Du^\epsilon| \neq 0$ on each levelset of u^ϵ below the maximum of u^ϵ. It follows that all these levelsets are smooth convex hypersurfaces.

We claim that these levelsets have uniformly bounded principal curvatures. To see this, it suffices because of the convexity of these levelsets to know that the mean curvature $\mathcal{H}$ with respect to the inward normal is uniformly bounded. But

$$-\mathcal{H}|Du^\epsilon| = (\delta_{ij} - \frac{u^\epsilon_{x_i}u^\epsilon_{x_j}}{|Du^\epsilon|^2})\, u^\epsilon_{x_ix_j} \;\; = \;\; -\frac{1}{2u^\epsilon} - u^\epsilon_{x_i}u^\epsilon_{x_j}u^\epsilon_{x_ix_j}(\frac{1}{|Du^\epsilon|^2} - \frac{1}{|Du^\epsilon|^2 + \epsilon^2}).$$

Since u^ϵ is concave we conclude that

$$0 \leq \mathcal{H} \leq \frac{1}{2u^\epsilon|Du^\epsilon|}$$

and therefore $\mathcal{H}$ is uniformly bounded on each of the levelsets below the maximum of u^ϵ.

6. We complete the proof of Theorem 7.6 by showing that $v^\epsilon \to v$ uniformly on $\overline{U}$, where v is the unique solution of (7.36) constructed in Theorem 7.4.

Since u^ϵ satisfies (7.38), v^ϵ satisfies

$$|v^\epsilon(x) - v^\epsilon(y)| \le 4RC|x-y|^{1/2} \quad x, y \in U.$$

Hence, we may choose a sequence $\epsilon_k \to 0$ with

$$v^{\epsilon_k} \to v \text{ uniformly on } \overline{U}.$$

We assert that v is a weak solution of (7.36). As before, it suffices to consider $\phi \in C^\infty(\ ^n)$ with $v - \phi$ having a *strict* local maximum at a point $x_0 \in U$. As $v^{\epsilon_k} \to v$ uniformly near x_0, $v^{\epsilon_k} - \phi$ has a local maximum at a point x_k with $x_k \to x_0$ as $k \to \infty$.

Since v^{ϵ_k} and ϕ are smooth, we have

$$Dv^{\epsilon_k} = D\phi, \ D^2(v^{\epsilon_k} - \phi) \le 0 \quad \text{at } x_k.$$

Thus (7.39) implies

$$-\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2 + 4\epsilon_k^2 v^{\epsilon_k}}\right) \phi_{x_i x_j} \le -\frac{\epsilon_k^2}{2} \frac{|D\phi|^2}{|D\phi|^2 + 4\epsilon_k^2 v^{\epsilon_k}} + 1 \tag{7.48}$$

at x_k. Suppose first $D\phi(x_0) \neq 0$. Then $D\phi(x_k) \neq 0$ for large k. Consequently we may pass to the limit in (7.48) (since $0 \le v^{\epsilon_k} \le 4R^2$) to deduce,

$$-\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right) \phi_{x_i x_j} \le +1 \qquad \text{at } x_0.$$

Next, assume instead $D\phi(x_0) = 0$ and set

$$\eta^k \equiv \frac{D\phi(x_k)}{(|D\phi(x_k)|^2 + 4\epsilon_k^2 v^{\epsilon_k})^{1/2}}$$

so that (7.48) becomes

$$-\left(\delta_{ij}-\eta_i^k\eta_j^k\right)\phi_{x_ix_j} \le -\frac{\epsilon_k^2}{2}\,\frac{|D\phi|^2}{|D\phi|^2+4\epsilon_k^2 v_k^\epsilon}+1 \qquad \text{at } x_k. \tag{7.49}$$

Since $|\eta^k| \le 1$ we may pass to a subsequence and reindex if necessary and assume

$$\eta_k \to \eta \quad \text{in } {}^n$$

for some $|\eta| \le 1$. Sending k to infinity in (7.49) we discover

$$-\left(\delta_{ij}-\eta_i\eta_j\right)\phi_{x_ix_j} \le 1 \qquad \text{at } x_0. \tag{7.50}$$

Consequently v is a weak subsolution. Similarly, we find that v is a weak supersolution and the proof of Theorem 7.6 is complete. $\square$

8 Examples, Pathologies, Conjecture

In this concluding section, we note various odd behavior allowed by our generalized mean curvature flow

$$\Gamma_0 \longmapsto \mathcal{M}(t)\Gamma_0 = \Gamma_t \qquad (t \geq 0)$$

8.1 Instantaneous extinction

Suppose Σ_0 is the smooth, connected boundary of a bounded open subset $U_0 \subset \mathbb{R}^n$, and let Γ_0 be a compact subset of Σ_0. If $\Gamma_0 = \Sigma_0$, then we know from § 6.1 that at least for small times $t > 0$ Γ_t is the classical evolution via mean curvature.

What happens if Γ_0 is a proper subset of Σ_0?

Theorem 8.1 *Assume that Γ_0 is compact, $\Gamma_0 \subseteq \Sigma_0$, $\Gamma_0 \neq \Sigma_0$. Then*

$$\Gamma_t = \emptyset \qquad \text{for each } t > 0 \tag{8.1}$$

If we take Γ_0 to be, say, Σ_0 with a small disk removed, we may informally regard (8.1) as asserting Γ_0 "pops" instantly. In this heuristic interpretation, we may think of Γ_0 as somehow having so much mean curvature concentrated along its boundary within Σ_0 that the hole then widens infinitely fast

The proof of Theorem 8.1 will be given later, after the next assertion, which is of independent interest. Assume now that $\widehat{\Sigma}_0$ is the smooth connected boundary of a bounded open set $\widehat{U}_0 \subset \mathbb{R}^n$ and that

$$\widehat{\Sigma}_0 \subset \overline{U}_0, \tag{8.2}$$

where Σ, U are as above. Thus the surface $\widehat{\Sigma}_0$ lies within the closed region $\overline{U}$ enveloped by Σ_0. Suppose further that

$$\widehat{\Sigma}_0 \neq \Sigma_0 \tag{8.3}$$

Choose then a time $t_o > 0$ so small that the classical evolutions $\{\Sigma_t\}$ and $\{\widehat{\Sigma}_t\}$ starting at Σ_0 and $\widehat{\Sigma}_0$, respectively, exist at least for times $0 \leq t \leq t_0$.

Theorem 8.2 *We have*

$$\Sigma_t \cap \widehat{\Sigma}_t = \emptyset \qquad \textit{for } 0 < t \leq t_0 \tag{8.4}$$

We are thus asserting that even if Σ_0 and $\widehat{\Sigma}_0$ coincide except for a very small region:

then for any positive time $0 < t \leq t_0$ the subsequent evolutions will have completely broken apart:

The point is that the PDE describing evolution by mean curvature is "uniformly parabolic along the surface" and thus admits infinite propagation speed for disturbances.

We will give the proof of Theorem 8.2 (as well as an interesting new proof of the short time existence of classical mean curvature flow) in a separate paper [] using the signed distance function $d(x,t)$ that we have introduced in § 6. Suffice it to say here that one can also prove this result by covering Σ_0 and $\widehat{\Sigma}_0$ by overlapping balls small enough so that restricted to each ball ,Σ_t and $\widehat{\Sigma}_t$ can be written as a graph. Since the mean curvature evolution equation is uniformly parabolic for small t_0 and $\widehat{\Sigma}_0 \neq \Sigma_0$, in at least one of the balls the surfaces Σ_t and $\widehat{\Sigma}_t$ must instantly separate. Thus in each ball the surfaces must also separate.

Proof of Theorem 8.1. Given Γ_0 and Σ_0 as in Theorem 8.1 we may choose a smooth, nearby surface $\widehat{\Sigma}_0$ to Σ_0 satisfying (8.2) (8.3) and

$$\Gamma_0 \subset \widehat{\Sigma}_0$$

Owing then to Theorem 7.2 we have

$$\Gamma_t \subseteq \Sigma_t \cap \widehat{\Sigma}_t$$

for small $t > 0$. Assertion (8.1) now follows from (8.4). □

8.2 Development of an interior

The forgoing demonstrates that a "large" initial set Γ_0 can instantly vanish under the generalized mean curvature flow. An opposite and perhaps more surprising phenomenon is that the set Γ_t for $t > 0$ may develop an interior, even if Γ_0 had none.

The simplest example occurs if we take Γ_0 to be the union of the coordinate axes in the plane . ^{2}:

(Ignore for the moment that Γ_0 is not compact and so our theory in § 2.7 in not really applicable.) To discover heuristically at least the subsequent evolution of Γ_0, consider instead the simpler figure as drawn.

As for instance in Brakke [, Figure 3 in the Appendix] we expect this corner to evolve into this shape

for times $t > 0$. Since Γ_0 is composed of four rotated copies of this corner, we expect from Theorem 7.2 that Γ_t will look like:

This assertion is at variance wtih Brakke [, Figure 5 in the Appendix]. Our Γ_t presumably contains the set

which he draws as one of the (nonunique!) evolutions for Γ_0. *We conjecture that our Γ_t contains all of the evolutions for Γ_0 allowed by Brakke.*

The discussion above can be modified to apply to various compact figures Γ_0, to which our theory does apply. We leave it to the reader to provide at least a heuristic proof that the set $\Gamma_0 \subset {}^2$ as drawn below will develop an interior.

We conjecture that if $\Gamma_0 = \Sigma_0$ is, as above, the boundary of a smooth open set, then Γ_t will never have an interior.