Motion of Level Sets by Mean Curvature II

L. C. Evans*
Department of Mathematics
University of California
Berkeley, CA 94720

J. Spruck†
Department of Mathematics
University of Massachusetts
Amherst, MA 01003

Abstract

1 Introduction

We present in this paper a new, elementary, and fairly concise proof of short time existence for the classical motion of a smooth hypersurface evolving according to its mean curvature. In this problem we are given initially a smooth connected hypersurface Γ_0 which is the boundary of a bounded open set $U \subset {}^n$. We then allow Γ_0 to evolve in time into a family of surfaces $\{\Gamma_t\}_{t>0}$ by moving each point on Γ_t $(t \geq 0)$ in the opposite direction to its mean curvature vector, at a velocity equal to $(n-1)$ times the absolute value of the mean curvature. Our intent is to verify that for small times at

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least, the classical motion as envisioned in fact exists and is unique. This assertion was first proved by R. Hamilton [4], and we discuss below the relation of our work to his.

This is a companion to our paper [2], wherein we defined and then studied a generalized notion of evolution via mean curvature, existing for all times and agreeing with the classical motion, if and so long as the latter exists. This generalized evolution is constructed in [2] by first building an appropriately defined unique weak solution of the PDE

$$u_t = (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}) u_{x_i x_j} \qquad \text{in } {}^n \times (0, \infty) \tag{1.1}$$

$$u = g \qquad \text{on } {}^n \times \{t = 0\}. \tag{1.2}$$

Equation (1.1) says that *each* level set of u evolves according to its mean curvature, at least in regions when u is smooth and $|Du| \neq 0$. Given then Γ_0 as above, we select a smooth function $g\colon {}^n \to$ for which

$$\Gamma_0 = \{x \in {}^n \mid g(t) = 0\}. \tag{1.3}$$

We next write

$$\Gamma_t \equiv \{x \in {}^n \mid u(x,t) = 0\} \qquad (t \geq 0), \tag{1.4}$$

and call the family of sets $\{\Gamma_t\}_{t \geq 0}$ so defined the *generalized evolution* by mean curvature starting from Γ_0. The rest of [2] is devoted to deducing from the PDE (1.1), (1.2) various elementary geometric properties of the flow $\Gamma_0 \longmapsto \Gamma_t (t \geq 0)$.

In this paper we return again to the idea of studying a nonlinear PDE, a level set of whose solution evolves via mean curvature. Our idea is first to assume that Γ_0 develops by classical mean curvature motion, at least for times $0 \leq t \leq t_0$, and then to derive the PDE verified by d, the signed distance function to the surface at each time. This turns out to be a fully nonlinear, uniformly parabolic equation: see (1.9) below. Next we construct for short times a smooth solution of this equation, subject to nonlinear boundary conditions in an appropriate region, and then finally verify that our solution is in fact the signed distance function to a family of smooth surfaces evolving from Γ_0 by mean curvature motion.

We proceed now to the heuristic derivation of our PDE. Suppose therefore we are given the smooth hypersurface $\Gamma_0 = \partial U$ as above, a time $t_0 > 0$, and a classical evolution $\{\Gamma_t\}_{0 \le t \le t_0}$ of surfaces developing from Γ_0 by mean curvature flow. Then for each time t in $[0, t_0]$, Γ_t is the smooth boundary of a bounded open set U_t, which is diffeomorphic to $U = U_0$. Define the *signed distance function*

$$
d(x,t) \equiv \begin{cases} \text{dist } (x, \Gamma_t) & (x \in {}^n - \overline{U}_t) \\ -\text{dist } (x, \Gamma_t) & (x \in U_t) \end{cases} \tag{1.5}
$$

for $x \in {}^n$, $0 \le t \le t_0$. By assumption $\Gamma \equiv \bigcup_{0 \le t \le t_0} \Gamma_t \, x\{t\}$ is smooth, and thus d is smooth in

$$
Q^+ \equiv \{(x,t) \mid 0 < t < t_0, \; 0 < d(x,t) < \delta_0\}
$$

and in

$$
Q^- = \{(x,t) \mid 0 < t < t_0, \; -\delta_0 < d(x,t) < 0\}
$$

for $\delta_0 > 0$ small enough. Fix any point $(x,t) \in Q^+$. Then provided $\delta_0 > 0$ is sufficiently small, there exists a unique point $y \in \Gamma_t$ for which $d(x,t) = |x - y|$. Let $\nu = Dd$ be the smooth unit normal vector field pointing from Γ into Q^+. As $\{\Gamma_t\}_{0 \le t \le t_0}$ moves via mean curvature, we have

$$
d_t(x,t) = \text{div}(\nu) \; (y,t).
$$

On the other hand the (unordered) eigenvalues of the matrix $D^2 d(x,t)$ are

$$
\begin{aligned}
&\lambda_i = \lambda_i(D^2 d(x,t)) = -\kappa_i / 1 - \kappa_i d(x,t) \qquad (1 \le i \le n-1) \\
&\lambda_n = \lambda_n(D^2 d(x,t)) = 0,
\end{aligned} \tag{1.6}
$$

$\{\kappa_1, \dots, \kappa_{n-1}\}$ denoting the principle curvatures of Γ_t at y, computed with respect to the normal field ν. Inverting (1.6) we compute

$$\kappa_i = \frac{\lambda_i}{\lambda_i d(x,t) - 1} \qquad (1 \le i \le n-1).$$

Since

$$\operatorname{div}(\nu) = -(\kappa_1 + \cdots + \kappa_{n-1}),$$

we deduce finally

$$d_t(x,t) = f(\lambda_1(D^2 d(x,t)), \cdots, \lambda_n(D^2 d(x,t)),\, d(x,t)) \tag{1.7}$$

for

$$f(\lambda_1, \cdots, \lambda_n, z) \equiv \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_i z} \tag{1.8}$$

The same formula results if $(x,t) \in Q^-$.

Now our PDE (1.7) has the general form

$$d_t = F(D^2 d, d)\,; \tag{1.9}$$

and, as f is symmetric in the variables $\lambda_1, \dots \lambda_n$, F is smooth ([1]). Since

$$\frac{\partial f}{\partial \lambda_i} = \frac{1}{(1 - \lambda_i d)^2} > 0 \qquad (1 \le i \le n)\,, \tag{1.10}$$

equation (1.9) is uniformly parabolic ([1]).

Our plan is to study directly the PDE (1.7), (1.8). For this suppose now Γ_0 is the smooth connected boundary of a bounded open set $U \subset {}^n$, and let

$$g(x) \equiv \begin{cases} \ \ \text{dist}\ (x, \Gamma_0) & (x \in {}^n - \overline{U}) \\ -\text{dist}\ (x, \Gamma_0) & (x \in U) \end{cases} \tag{1.11}$$

be the signed distance function. Fix then $\delta_0 > 0$ so small that g is smooth within

$$V \equiv \{x \in {}^n \mid -\delta_0 < g < \delta_0\}. \tag{1.12}$$

and write

$$Q \equiv V \times (0, t_0),\ \Gamma \equiv \partial V \times [0, t_0].$$

In § 2 we construct a smooth solution v to the PDE

$$\begin{cases} v_t = F(D^2 v, v) & \text{in } Q \\ |Dv|^2 = 1 & \text{on } \Gamma \\ v = g & \text{on } V \times \{t = 0\} \end{cases} \tag{1.13}$$

for some small time $t_0 > 0$, our argument being in effect a special case of the Inverse Function Theorem. We then verify in § 3 that

$$|Dv|^2 = 1 \qquad \text{in } Q\,. \tag{1.14}$$

Setting

$$\Gamma_t \equiv \{x \in V \mid v(x,t) = 0\} \qquad (0 \le t \le t_0)\,,$$

we deduce from (1.13) and (1.14) that $\{\Gamma_t\}_{0 \le t \le t_0}$ is a smooth evolution by mean curvature and $v = d$ is the corresponding signed distance function.

Section 4 utilizes the PDE (1.7), (1.8) to prove an "instantaneous tearing apart" assertion for close-by evolving surfaces. Some consequences are discussed in [2, § 8].

As noted, R. Hamilton in [4, § 5] has previously established the short-time existence of the classical mean curvature evolution by studying the degenerate parabolic system describing the parametrized surface. As this system is degenerately parabolic ([4, p. 261]), Hamilton is forced to employ fairly complicated techniques related to the Moser-Nash Implicit Function Theorem. See also Gage-Hamilton [3, § 2]. Our methods are simpler.

2 Solving the Nonlinear PDE

Our goal in this section is to construct a smooth solution of the PDE (1.13) for $\delta_0 > 0$ and $t_0 > 0$ small enough. Let us first of all select δ_0 so small that

$$M\delta_0 \leq \frac{1}{4} \tag{2.1}$$

for

$$M \equiv \max_V |D^2 g|. \tag{2.2}$$

Set

$$G \equiv \{R \in S^{n\times n}, z \in \ \mid |z| < \delta_0, \ |R| < 2M\}.$$

Then since (2.1) implies

$$|\lambda_i(R)z| \leq |R||z| \leq \frac{1}{2} \qquad (1 \leq i \leq n)$$

if $(R, z) \in G$,

$$\begin{aligned} F(R,z) &= f(\lambda_1(R), \cdots, \lambda_n(R), z) \\ &= \textstyle\sum_{i=1}^n \frac{\lambda_i(R)}{1-\lambda_i(R)z} \end{aligned} \tag{2.3}$$

is defined and smooth on G. Arbitrarily extend F off G to be smooth on all of $S^{n\times n} \times$, with $|F|, |DF|, |D^2F|$ bounded.

We now check that our PDE is uniformly parabolic near g.

Lemma 2.1 *Then exists a constant $\theta > 0$ so that*

$$\frac{\partial F}{\partial r_{ij}}(R,z)\xi_i\xi_i \geq \theta|\xi|^2 \qquad (\xi \in {}^n) \tag{2.4}$$

in V for each $(R, z) \in G$.

Proof. Fix $\xi \in {}^n$ and choose then $t > 0$ so small that $(R + t\,\xi \otimes \xi, z) \in G$. Using Courant's minimax characterization we deduce that the eigenvalues can be ordered so that

$$\lambda_k(R + t\,\xi \otimes \xi) \geq \lambda_k(R) \qquad (k = 1, \cdots n).$$

Consequently

$$
\begin{aligned}
&F(R+t\,\xi\otimes\xi,z)-F(R,z)\\
&\quad = f(\cdots,\lambda_k(R+t\,\xi\otimes\xi),\cdots,z)-f(\cdots,\lambda_k(R)\cdots,z)\\
&\quad = \sum_{i=1}^{n}\int_0^1 \frac{\partial f}{\partial \lambda_k}(\cdots,s\lambda_k(R+t\,\xi\otimes\xi)+(1-s)\lambda_k(R),\cdots,z)ds\,[\lambda_k(R+t\,\xi\otimes\xi)-\lambda_k(R)]\\
&\quad \geq \theta \text{ trace } (R+t\,\xi\otimes\xi-R)
\end{aligned}
$$

for some $\theta>0$ by (1.10). Dividing by $t>0$ and sending $t\to 0^+$, we arrive at inequality (2.4). □

We will seek our solution v of the PDE (1.13) in the form

$$v=g+th+w, \tag{2.5}$$

where

$$h\equiv F(D^2g,g)\,. \tag{2.6}$$

Then plugging into (1.13) we compute

$$w_t-a_{ij}\,w_{x_ix_j}+cw=A(D^2w,w,x,t)\qquad \text{in } Q$$

for

$$a_{ij}\equiv\frac{\partial F}{\partial r_{ij}}\,(D^2g,g),\ c\equiv\frac{-\partial F}{\partial z}\,(D^2g,g) \tag{2.7}$$

and

$$
\begin{aligned}
A(R,z,x,t)\ &\equiv F(D^2g+t\,D^2h+R,\ g+th+z)\\
&\quad -F(D^2g,g)-\tfrac{\partial F}{\partial r_{ij}}(D^2g,g)\,r_{ij}-\tfrac{\partial F}{\partial z}\,(D^2g,g)z.
\end{aligned}
\tag{2.8}
$$

We next insert (2.5) into the nonlinear boundary condition from the PDE (1.13). Recalling that $|Dg|^2=1$ and Dg is normal to ∂V we discover

$$\frac{\partial w}{\partial \nu}=a\,(Dw,x,t)\qquad \text{on } \Gamma\,,$$

when ν is the outer unit normal vector field along V and

$$(2.9) \qquad a\,(p,x,t) \equiv \begin{cases} -\frac{1}{2}\,|t\,Dh+p|^2 - t\,\frac{\partial h}{\partial \nu} & \text{on } \{d=\delta_0\} \\ \\ \frac{1}{2}\,|t\,Dh+p|^2 - t\,\frac{\partial h}{\partial \nu} & \text{on } \{d=-\delta_0\}. \end{cases}$$

Combining everything above, we hereafter seek a smooth function w satisfying

$$(2.10) \begin{cases} w_t - a_{ij}\,w_{x_ix_j} + cw = A(D^2w,w,x,t) & \text{in } Q \\ \frac{\partial w}{\partial \nu} = a\,(Dw,x,t) & \text{on } \Gamma \\ w = 0 & \text{on } V \times \{t=0\}. \end{cases}$$

We intend to solve (2.10) by finding a fixed point of the mapping T defined by inserting a given function into the nonlinear terms A, a and solving the resulting linear PDE. We will work in certain standard parabolic-type Hölder spaces. Fix $0<\alpha<1$.

We define several norms, following Ladyzenskaja, Solonnikov, Ural'ceva [5, p.7-8]. Set

$$\begin{aligned} |u|^{(0)} &\equiv \sup\{|u(x,t)| \mid (x,t)\in Q\} \\ |u|^{(1)} &\equiv |u|^{(0)} + |Du|^{(0)} \\ |u|^{(2)} &\equiv |u|^{(1)} + |D^2u|^{(0)} + |u_t|^{(0)} \\ \langle u\rangle_x^{(\beta)} &\equiv \sup\left\{\tfrac{|u(x,t)-u(y,t)|}{|x-y|^\beta} \mid (x,t),(y,t)\in Q, x\neq y\right\} \\ \langle u\rangle_t^{(\beta)} &\equiv \sup\left\{\tfrac{|u(x,t)-u(x,s)|}{|t-s|^\beta} \mid (x,t),(x,s)\in Q, t\neq s\right\}, \end{aligned}$$

for $0<\beta<1$. Then

$$\begin{aligned} \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} &\equiv |u|^{(0)} + \langle u\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\frac{\alpha}{2})} \\ \|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q})} &\equiv |u|^{(1)} + \langle Du\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\frac{1+\alpha}{2})} + \langle Du\rangle_t^{(\frac{\alpha}{2})} \\ \|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} &\equiv |u|^{(2)} + \langle Du\rangle_t^{(\frac{1+\alpha}{2})} + \langle D^2u\rangle_x^{(\alpha)} + \langle u_t\rangle_x^{(\alpha)} + \langle D^2u\rangle_t^{(\frac{\alpha}{2})} + \langle u_t\rangle_t^{(\frac{\alpha}{2})}. \end{aligned}$$

Note

$$\|Du\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q})}, \|D^2u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})}, \|u_t\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leq \|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})}.$$

Finally define

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma)} \equiv \inf\left\{\|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q})} \mid v = u \text{ on } \Sigma\right\}.$$

We consider now the linear, uniformly parabolic PDE

$$\left\{\begin{array}{ll} w_t - a_{ij}\, w_{x_i x_j} + cw = B & \text{in } Q \\ \frac{\partial w}{\partial \nu} = b & \text{on } \Sigma \\ w = 0 & \text{on } V \times \{t = 0\}. \end{array}\right. \tag{2.11}$$

Suppose as well

$$b = 0 \text{ on } \partial V \times \{t = 0\}. \tag{2.12}$$

Lemma 2.2 *Assume $B \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}), b \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma)$, and the zeroth order compatibility condition (2.12) holds. Then there exists a unique solution $w \in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})$ of (2.11), with the estimate*

$$\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq C\left(\|B\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} + \|b\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma)}\right). \tag{2.13}$$

The constant C does not depend on t_0.

Proof. Except for the last assertion, this is Theorem IV.5.3 in Ladyzenskaja, Solonnikov, Ural'ceva [5, p. 320–321]. To obtain the statement that the constant C remains bounded even for small $0 < t_0 < 1$, we define

$$\widetilde{Q} = V \times (0,1),\ \widetilde{\Gamma} = \partial V \times (0,1),$$

$$\widetilde{B}(x,t) = \left\{\begin{array}{ll} B(x,t) & \text{if } 0 \leq t \leq t_0 \\ B(x,t_0) & \text{if } t_0 \leq t \leq 1, \end{array}\right.$$

$$\widetilde{b}(x,t) = \left\{\begin{array}{ll} b(x,t) & \text{if } 0 \leq t \leq t_0 \\ b(x,t_0) & \text{if } t_0 \leq t \leq 1. \end{array}\right.$$

Then

$$\|\widetilde{B}\|_{C^{\alpha,\frac{\alpha}{2}}(\tilde{Q})} = \|B\|_{C^{\alpha,\frac{\alpha}{2}}(Q)}$$
$$\|\tilde{b}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\widetilde{\Gamma})} = \|b\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma)}.$$

Let $\tilde{w} \in C^{2+\alpha,\frac{2+\alpha}{2}}(\tilde{Q})$ solve the PDE (2.11) with $\widetilde{Q}, \widetilde{\Gamma}, \widetilde{B}, \tilde{b}$ replacing Q, Γ, B, b. The estimate (2.13) then is valid for $\tilde{w}$ with $\widetilde{Q}, \widetilde{\Gamma}$ replacing Q, Γ. But $\tilde{w} = w$ on Q, by uniqueness for parabolic equations. □

We will henceforth work in the Banach space

$$X \equiv \{w \in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q}) \mid w = 0 \text{ on } V \times \{t = 0\}\}.$$

Given $\hat{w} \in X$, we set

$$\left\{ \begin{array}{ll} B(x,t) & \equiv A(D^2\hat{w}, \hat{w}, x, t) \\ b(x,t) & \equiv a(D\hat{w}, x, t) \end{array} \right. \tag{2.14}$$

for A, a defined by (2.8), (2.9). Now write

$$T(\hat{w}) = w,$$

when $w \in X$ solves the linear PDE (2.11), for B, b as in (2.14). We seek a fixed point of the mapping

$$T: X \to X.$$

Give $r_0 > 0$, set

$$Y \equiv \{w \in X \mid \|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q)} \le r_0\}.$$

Lemma 2.3 *If $t_0, r_0 > 0$ are sufficiently small, then*

$$T: Y \to Y \tag{2.15}$$

Proof. 1. Choose any function $\widehat{w} \in Y$, define B, b by (2.14) and let $w \in X$ solve (2.11). We must show the inequality

$$\|\widehat{w}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq r_0 \tag{2.16}$$

implies

$$\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq r_0, \tag{2.17}$$

provided $r_0, t_0 > 0$ are adjusted sufficiently small.

2. Now (2.8) implies

$$\begin{aligned} A(R,z,x,t) &= t\frac{\partial F}{\partial r_{ij}}(D^2g,g)h_{x_ix_j} + t\frac{\partial F}{\partial z}(D^2g,g)h \\ &+ \int_0^1 (1-s)\frac{\partial^2 F}{\partial r_{ij}\partial r_{k\ell}}(D^2g + stD^2h + sR, g + sth + sz)ds \\ &\quad \times (th_{x_ix_j} + r_{ij})(th_{x_kx_\ell} + r_{k\ell}) \\ &+ 2\int_0^1 (1-s)\frac{\partial^2 F}{\partial r_{ij}\partial z}(D^2g + stD^2h + sR, g + sth + sz)ds \\ &\quad \times (th_{x_ix_j} + r_{ij})(th + z) \\ &+ \int_0^1 (1-s)\frac{\partial^2 F}{\partial z^2}(D^2g + stD^2h + sR, g + sth + sz)ds \times (th+z)^2 . \end{aligned} \tag{2.18}$$

Recall also that

$$\|uv\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leq C\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})}\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \tag{2.19}$$

for all $u, v \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$, and

$$\|t\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leq Ct_0^{1-\frac{\alpha}{2}}. \tag{2.20}$$

Then (2.14), (2.18), (2.20) imply

$$\|B\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leq C(r_0^2 + t_0^{1-\frac{\alpha}{2}}). \tag{2.21}$$

Similarly formula (2.9) implies

$$\|b\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma)} \leq C(r_0^2 + t_0^{\frac{1-\alpha}{2}}) \tag{2.22}$$

3. Inserting estimates (2.21), (2.22) into (2.13) we discover

$$\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq C(r_0^2 + t_0^{\frac{1-\alpha}{2}}) , \tag{2.23}$$

the constant C independent of r_0 and t_0. Let

$$r_0 \leq \frac{1}{2C} .$$

Then (2.23) gives

$$\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq \frac{r_0}{2} + C\, t_0^{\frac{1-\alpha}{2}} \leq r_0$$

if $t_0 = t_0(r_0)$ is small enough. This verifies inequality (2.17). □

Finally we verify that $T\colon Y \to Y$ is a strict contraction.

Lemma 2.4 *If $r_0, t_0 > 0$ are small enough,*

$$\|T(\widehat{w}_1) - T(\widehat{w}_2)\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq \frac{1}{2}\|\widehat{w}_1 - \widehat{w}_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \tag{2.24}$$

for all $\widehat{w}_1, \widehat{w}_2 \in Y$.

Proof. 1. Choose $\widehat{w}_1, \widehat{w}_2 \in Y$; so that

$$\|\widehat{w}_1, \widehat{w}_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq r_0 .$$

Set

$$\begin{cases} B_1(x,t) \equiv A(D^2\widehat{w}_1, \widehat{w}_1, x, t), & B_2(x,t) \equiv A(D^2\widehat{w}_2, \widehat{w}_2, x, t) \\ b_1(x,t) \equiv a(D\widehat{w}_1, x, t), & b_2(x,t) \equiv a(D\widehat{w}_2, x, t) , \end{cases} \tag{2.25}$$

and write

$$w_1 \equiv T(\widehat{w}_1),\ w_2 \equiv T(\widehat{w}_2)$$

Thus w_1 solves the linear PDE (2.11) with B_1, b_1 replacing B, b, and w_2 solves (2.11) with B_2, b_2 replacing B, b. According to Lemma 2.2

$$\|w_1 - w_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})} \leq C(\|B_1 - B_2\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} + \|b_1 - b_2\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma)}) . \tag{2.26}$$

2. Recall that

$$\text{(2.27)} \qquad \|\Phi(u)-\Phi(v)\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leq C(\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})}+\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})}+1)\|u-v\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})}$$

if $u, v \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$ and Φ is smooth, with $D\Phi, D^2\Phi$ bounded.

Utilizing then formulas (2.18)–(2.20), (2.27) we compute

$$\text{(2.28)} \qquad \|B_1 - B_2\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leq C(r_0 + t_0^{1-\frac{\alpha}{2}})\|\widehat{w}_1 - \widehat{w}_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})}$$

Similarly

$$\text{(2.29)} \qquad \|b_1 - b_2\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma)} \leq C(r_0 + t_0^{\frac{1-\alpha}{2}})\|\widehat{w}_1 - \widehat{w}_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q})}.$$

Combining (2.26), (2.28), (2.29), we obtain estimate (2.24), provided $r_0, t_0 > 0$ are small. $\square$

We at last apply Banach's Fixed Point Theorem to establish

Theorem 2.5 *If $\delta_0, t_0 > 0$ are sufficiently small,there exists a unique solution $v \in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{Q}) \cap C^{\infty}(Q)$ of the PDE (1.13).*

In particular $(D^2v, v) \in G$ in Q.

3 Motion of the Zero Level Set by Mean Curvature

This section we devote to proving that the sets

$$\Gamma_t \equiv \{x \in V \mid v(x,t) = 0\} \qquad (0 \le t \le t_0) \tag{3.1}$$

are in fact smooth hypersurfaces evolving by mean curvature.

Theorem 3.1 *We have*

$$|Dv|^2 = 1 \qquad \text{in } Q\,. \tag{3.2}$$

Proof. 1. Let $w \equiv |Dv|^2 - 1 \in C^1(\bar{Q}) \cap C^\infty(Q)$. Then

$$w = 0 \qquad \text{on } \Gamma \tag{3.3}$$

and

$$w = 0 \qquad \text{on } V \times \{t = 0\}, \tag{3.4}$$

according to (1.11) and the PDE (1.13).

2. Differentiating (1.13) we compute

$$v_{t\,x_k} = \frac{\partial F}{\partial r_{ij}}\,(D^2v, v)\, v_{x_i x_j x_k} + \frac{\partial F}{\partial z}\,(D^2 v, v)\, v_{x_k}$$

Thus

$$\begin{aligned} w_t &= 2 v_{x_k} v_{x_k t} \\ &= 2\tfrac{\partial F}{\partial r_{ij}}\,(D^2v,v)\, v_{x_k} v_{x_k x_i x_j} + 2\tfrac{\partial F}{\partial z}\,(D^2v,v)\,|Dv|^2 \qquad (3.5) \\ &= \tfrac{\partial F}{\partial r_{ij}}\,(D^2v,v)\, w_{x_i x_j} - 2\tfrac{\partial F}{\partial r_{ij}}\,(D^2v,v)\, v_{x_k x_i} v_{x_k x_j} + 2\tfrac{\partial F}{\partial z}\,(D^2v,v)\,|Dv|^2 \end{aligned}$$

Now

$$F(D^2v, v) = f(\cdots, \lambda_i\,(D^2v), \cdots, v) \qquad \text{in } Q.$$

Thus

$$
\begin{aligned}
(3.6) \qquad \frac{\partial F}{\partial z}\left(D^2 v, v\right) &= \frac{\partial f}{\partial z}\left(\cdots, \lambda_i\left(D^2 v\right), \cdots, v\right) \\
&= \sum_{i=1}^n \frac{\lambda_i\left(D^2 v\right)^2}{\left(1-\lambda_i\left(D^2 v\right) v\right)^2}
\end{aligned}
$$

On the other hand (see [1])

$$
\begin{aligned}
\frac{\partial F}{\partial r_{ij}}\left(D^2 v\right) v_{x_k x_i} v_{x_k x_j} &= \sum_{i=1}^n \frac{\partial f}{\partial \lambda_i}\left(\cdots, \lambda_i\left(D^2 v\right), \cdots, v\right) \lambda_i\left(D^2 v\right)^2 \\
&= \sum_{i=1}^n \frac{\lambda_i\left(D^2 v\right)^2}{\left(1-\lambda_i\left(D^2 v\right) v\right)^2} \\
&= \frac{\partial F}{\partial z}\left(D^2 v, v\right) \qquad by\,(3.6)\,.
\end{aligned}
$$

Thus (3.5) transforms to read

$$
(3.7) \qquad w_t = \frac{\partial F}{\partial r_{ij}}\left(D^2 v, v\right) w_{x_i x_j} + 2 \frac{\partial F}{\partial z}\left(D^2 v, v\right) w \ \text{ in } \ Q.
$$

In view of Lemma 2.1 this is a uniformly parabolic equation. As (3.3), (3.4) assert $w = 0$ on the parabolic boundary of Q, we deduce $w = 0$ everywhere within Q. □

Owing to (3.1) we see that

$$
\Gamma = \{(x,t) \in \overline{Q} \mid v = 0\}
$$

is a smooth hypersurface in $^{n+1} \cap Q$ and each slice Γ_t is a smooth hypersurface in V.

Theorem 3.2 *The surfaces* $\{\Gamma_t\}_{0 \le t \le t_0}$ *comprise a classical motion by mean curvature starting from* Γ_0.

Proof. For each fixed t in $[0, t_0]$, $\nu \equiv Dv$ is a normal unit vector field to Γ_t in ^{n}. In addition the PDE (1.13) implies

$$
v_t = \Delta v = \operatorname{div}(\nu) \qquad \text{on } \ \Gamma_t.
$$

However

$$\mathrm{div}(\nu) = -(n-1)H = -(\kappa_1 + \cdots + \kappa_{n-1}) \,,$$

where $\kappa_1, \cdots \kappa_{n-1}$ are the principal curvatures of Γ_t computed with respect to ν and

$$H = \frac{1}{n-1}\,(\kappa_1 + \cdots \kappa_{n-1})$$

is the mean curvature.

Now fix $0 \le t \le t_0$, $x \in \Gamma_t$, and evolve the point x according to the nonautonomous ODE

$$\begin{cases} \dot{x}(s) = -[div(\nu)\nu]\,(x(s), s) \quad (s > t) \\ x(t) = x \,. \end{cases}$$

Then

$$\frac{d}{ds}\,v(x(s), s) = -[(Dv \cdot \nu)\,div(\nu)]\,(x(s), s) + v_t(x(s), s) = -div(\nu) + v_t = 0.$$

Thus

$$v(x(s), s) = 0 \quad (s > t)$$

and this implies $\{\Gamma_t\}_{0 \le t \le t_0}$ is evolving by mean curvature. □

4 Instantaneous Tearing Apart

As a further application of the PDE (1.7)–(1.9) we prove in this section that two distinct smooth surfaces evoloving by mean curvature must instantly tear completely apart, even if they initially touched on a large set.

More precisely, suppose as above Γ_0 is the smooth connected boundary of the bounded open set $U \subset {}^n$. Assume also that $\widehat{\Gamma}_0$ is the smooth connected boundary of a bounded open subset $\widehat{U} \subset {}^n$ and that

$$\widehat{\Gamma}_0 \subset \overline{U}. \tag{4.1}$$

Thus the surface $\widehat{\Gamma}_0$ lies within the closed region $\overline{U}$ bounded by Γ_0. Suppose further that

$$\widehat{\Gamma}_0 \neq \Gamma_0. \tag{4.2}$$

Choose next $t_0 > 0$ so small that the classical evolutions $\{\Gamma_t\}$ and $\{\widehat{\Gamma}_t\}$ starting at Γ_0 and $\widehat{\Gamma}_0$, respectively, exist at least for times $0 \leq t \leq t_0$.

Theorem 4.1 *We have*

$$\Gamma_t \cap \widehat{\Gamma}_t = \emptyset \qquad \textit{for } 0 < t \leq t_0. \tag{4.3}$$

We see therefore that the two evolutions by mean curvature instantly completely "tear apart". The point is that the PDE describing evolution by mean curvature is "uniformly parabolic along the surface" and this admits infinite propagation speed for disturbances. See [2, §8] for some geometric consequences.

Proof. 1. Fix δ_0 so small that the signed distance function

$$d(x,t) \equiv \begin{cases} -\mathrm{dist}(x,\Gamma_t) & \text{if } x \in U_t \\ \ \ \mathrm{dist}(x,\Gamma_t) & \text{if } x \in {}^n - U_t \end{cases}$$

is smooth in

$$Q \equiv \{(x,t) \mid -\delta_0 \leq d(x,t) \leq \delta_0,\ 0 \leq t \leq t_0\}.$$

(Here, U_t is the bounded region enveloped by Γ_t). As above we have

$$d_t = F(D^2 d, d) \qquad \text{in } Q. \tag{4.4}$$

Similarly the signed distance

$$\widehat{d}(x,t) \equiv \begin{cases} -\mathrm{dist}(x, t\widehat{\Gamma}_t) & \text{if } x \in \widehat{U}_t \\ \mathrm{dist}(x, \widehat{\Gamma}_t) & \text{if } x \in {}^n - \widehat{U}_t \end{cases}$$

is smooth in Q, provided $\delta_0, t_0 > 0$ are sufficiently small, and $\widehat{\Gamma}_0$ is close enough to Γ_0. (We denote by $\widehat{U}_t$ the region bounded by $\widehat{\Gamma}_t$). Also

$$\widehat{d}_t = F(D^2\widehat{d}, \widehat{d}) \qquad \text{in } Q. \tag{4.5}$$

as well.

2. Now since (4.1) holds, we have

$$\widehat{\Gamma}_t \subseteq \overline{U}_t$$

for each $0 \le t \le t_0$. Thus

$$\widehat{d} \ge d \qquad \text{in } Q.$$

3. Set

$$w \equiv \widehat{d} - d.$$

Then

$$w \ge 0 \text{ in } Q,\, w \ne 0 \text{ on } Q \cap \{t = 0\}, \tag{4.6}$$

and from (4.4), (4.5) it follows that

$$w_t = a_{ij}\, w_{x_i x_j} + cw \qquad \text{in } Q \tag{4.7}$$

for

$$a_{ij} \equiv \int_0^1 \frac{\partial F}{\partial r_{ij}} (t\, D^2\widehat{d} + (1-t)\, D^2 d,\, t\widehat{d} + (1-t)d) dt$$

and

$$c \equiv \int_0^1 \frac{\partial F}{\partial z}\,(t\,D^2\widehat{d} + (1-t)\,D^2 d,\; t\widehat{d} + (1-t)d)dt\,.$$

According to Lemma 2.1 the coefficients $((a_{ij}))$ are uniformly positive definite if δ_0, t_0 are small enough and $\widehat{\Gamma}_0$ is close enough to Γ_0. From (4.6), (4.7) and the strong maximum principle we deduce that

$$w > 0 \text{ in the interior of } Q. \tag{4.8}$$

Assertion (4.3) follows.

4. In the general case that $\widehat{\Gamma}_0$ verfies (4.1), (4.2), but is not necessarily everywhere close to Γ_0, we interpolate a smooth surface $\widetilde{\Gamma}_0$ between Γ_0 and $\widehat{\Gamma}_0$, so that $\widetilde{\Gamma}_0$ satisfies (4.1), (4.2) and is close to Γ_0. Then, by steps 1–3,

$$\widetilde{\Gamma}_t \cap \Gamma_t = \emptyset$$

for all small $t > 0$, $\{\widetilde{\Gamma}_t\}$ denoting the classical evolution from $\widetilde{\Gamma}_0$. As $\widetilde{\Gamma}_t$ lies within the region bounded by Γ_t and $\widehat{\Gamma}_t$ lies in the region bounded by $\widetilde{\Gamma}_t$, assertion (4.3) follows in this case as well. $\square$

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