Variational Problems with Critical Sobolev Growth and Positive Dirichlet Data

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1 Introduction

In this paper, we consider the Dirichlet problem for the conformally invariant model problem of critical Sobolev growth:

$$\begin{array}{rclll} -\Delta u & = & \lambda u^{\frac{n+2}{n-2}} & \text{in} & \Omega \\ u & = & \phi & \text{on} & \partial\Omega \\ u & \geq & 0 & & \end{array} \tag{1.1}$$

Problem (1.1) is formally the Euler-Lagrange equations for the variational problem

$$\int_\Omega |\nabla u|^2 \to \min \tag{1.2}$$

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for u in the admissible class

$$\mathcal{A} = \left\{ u \in H^1(\Omega) \colon u - h \in H_0^1(\Omega), \int_\Omega u \frac{2n}{n-2} = \gamma \right\}$$

where h is the harmonic extension of $\phi \geq 0$.

It is well known that for $\phi \equiv 0$, the existence of nontrivial solutions is often a very subtle question. For example, if Ω is starshaped there is no solution of (1.1) while if Ω has "nontrivial topology" then there do exist solutions [B].

The purpose of this paper is to show that for positive $C^{1+\beta}$ data ϕ, Problem (1.2) is well posed for any C^2 domain Ω. More precisely, we have the following

Theorem 1.1 *Let $\phi \in C^{1+\beta}(\partial\Omega) \geq 0$ be positive somewhere, $\partial\Omega \in C^2$. Assume that $\gamma > \int_\Omega h \frac{2n}{n-2}$ where h is the harmonic extension of ϕ. Then there is a positive minimizer $u \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ of Problem (1.2) satisfying (1.1) for a positive constant $\lambda > 0$.*

More precise estimates on the multiplier λ are given in Theorem 2.1.

More generally, we can consider the variational problem

$$\int_\Omega |\nabla u|^2 \to \min \tag{1.3}$$

for u in the admissible class

$$\mathcal{A} = \left\{ u \in H^1(\Omega) \colon u - h \in H_0^1(\Omega), \int_\Omega G(u) = \gamma \right\}$$

with (formal) Euler-Lagrange equations

$$\begin{array}{rcll} -\Delta u & = & \lambda g(u) & \text{in} \quad \Omega \\ u & = & \phi & \text{on} \quad \partial\Omega \end{array} \tag{1.4}$$

Then we have the following

Theorem 1.2 *Let $\phi \in C^{1+\beta}(\partial\Omega) \geq 0$ be positive somewhere, $\partial\Omega \in C^2$ and $\gamma > \int_\Omega G(h)$ where h is the harmonic extension of ϕ.*

Assume that the nonlinearity $g(t)$ is a locally Lipschitz function satisfying

$$
(1.5) \qquad \begin{array}{ll}
\text{(i)} & g(t) \text{ is non-decreasing, } g(0) = 0 \\
\text{(ii)} & t^{-\dfrac{n+2}{n-2}} g(t) \text{ is non-increasing} \\
\text{(iii)} & g(t) \geq \kappa t, \text{ for } t \geq t_0 > 0, \text{ where } \kappa > \lambda_1(\Omega)
\end{array}
$$

Then there is a positive minimizer $u \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ of Problem (1.4) satisfying (1.4) for a positive constant $\lambda > 0$. Here $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω.

Some estimates for λ are given in Proposition 5.1.

We should remark that the existence of minimal or "small norm" solutions of (1.1) is completely standard and the main point of our work is the non-existence of the blow-up of minimizing sequences for the variational problem (1.2). This fact is based on strong apriori estimates that we derive in Sections 3 and 4. Here, the positivity and smoothness of ϕ is essential. These estimates rely on the "asymptotic symmetry" method that we developed in our paper [C-G-S]. For the convenience of the reader we have outlined the main steps in this method in Section 4.

One consequence of these strong estimates is the existence of a continuum $\mathcal{S}$ of solutions (u, λ) of (1.1) containing $(h, 0)$ which is unbounded (but $\{(u, \lambda) \in \mathcal{S} : \lambda \geq \varepsilon\}$ is bounded) and such that λ decreases to zero as $\gamma = \int_\Omega u^{2n/n-2} \to \infty$.

An outline of the paper is as follows. Section 2 contains preliminary existence and multiplier estimates for the subcritical approximations to problem (1.2). In Section 3, we prove a preliminary apriori supnorm estimate for solutions of (1.4). These estimates allow us to use the method of [C-G-S] to prove uniform $C^{1+\beta}$ estimates. This is carried out in Section 4. Finally, in Section 5 we prove Theorem 1.1 and then go to prove Theorem 1.2.

2 Multiplier Estimates for the Approximating Problems

In this section we will consider subcritical approximations to the variational problem (1.2) and estimate the multipliers associated to solutions.

To fix the ideas, let $\phi \in C^{1+\beta}(\partial\Omega) \geq 0$, $\phi \not\equiv 0$ and consider the variational problem

$$\int_\Omega |\nabla u|^2 \to \min \tag{2.1}$$

for u in the admissible class

$$\mathcal{A}_\gamma = \left\{ u \in H^1(\Omega)\colon u - \phi \in H_0^1(\Omega), \int |u|^{\alpha+1} = \gamma \right\}$$

where $1 < \alpha < \frac{n+2}{n-2}$, and $\gamma > \int_\Omega h^{\alpha+1}$, for h the harmonic extension of ϕ.

The main result of this section may be stated as follows.

Theorem 2.1 *There is a positive solution $u \in C^2(\Omega) \cap C^{1+\beta}(\bar{\Omega})$ to problem (2.1) satisfying*

$$-\Delta u = \lambda u^\alpha \textit{ in } \Omega$$

$$u = \phi \textit{ on } \partial\Omega$$

where $\lambda > 0$ is a positive constant. In terms of $\theta =: \gamma - \int_\Omega h^{\alpha+1}, \lambda$ satisfies the asymptotic estimates

$$\begin{array}{ll} (i) & c_1\theta \leq \lambda \leq c_2\theta \textit{ for } \theta > 0 \textit{ small} \\ (ii) & c_3\theta^{\left(\frac{1-\alpha}{\alpha+1}\right)} \leq \lambda \leq c_4\theta^{\left(\frac{1-\alpha}{\alpha+1}\right)} \textit{ for } \theta > 0 \textit{ large} \end{array}$$

for absolute constants c_1, c_2, c_3, c_4.

Proof. The existence of a classical solution u to problem 2.1 is well-known. That is, u satisfies

$$\begin{aligned} -\Delta u &= \lambda |u|^{\alpha-1} u \quad \text{in } \Omega \\ u &= \phi \quad \text{on } \partial\Omega \end{aligned} \tag{2.2}$$

$$\int_\Omega |u|^{\alpha+1} = \gamma > \int_\Omega h^{\alpha+1}.$$

We claim that $\lambda > 0$ and so $u > h$ in Ω. For using $u-h$ as a test function in (2.2), we obtain

$$\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u \cdot \nabla h = \lambda \int_\Omega (u-h)|u|^{\alpha-1}u = \lambda(\gamma - \int_\Omega h|u|^{\alpha-1}u) \tag{2.3}$$

By Hölder,

$$\int_\Omega h|u|^{\alpha-1}u \le (\int_\Omega h^{\alpha+1})^{\frac{1}{\alpha+1}}(\int_\Omega |u|^{\alpha+1})^{\frac{\alpha}{\alpha+1}} \le \gamma \tag{2.4}$$

Now observe that

$$\int_\Omega \nabla u \cdot \nabla h = \int_\Omega |\nabla h|^2$$

and so

$$\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u \cdot \nabla h = \int_\Omega |\nabla (u-h)|^2 > 0 \tag{2.5}$$

It follows from (2.3)-(2.5) that $\lambda > 0$ and thus $u > h$ by the maximum principle.

We can obtain a sharper estimate for λ from below as follows. By convexity,

$$u^{\alpha+1} - h^{\alpha+1} \le (\alpha+1)u^\alpha(u-h).$$

Therefore,

$$\int_\Omega (u^{\alpha+1} - h^{\alpha+1}) \le (\alpha+1)\int_\Omega u^\alpha(u-h) \le (\alpha+1)(\int_\Omega u^{\alpha+1})^{\alpha/\alpha+1}(\int_\Omega (u-h)^{\alpha+1})^{\frac{1}{\alpha+1}}$$

and so,

$$\gamma - \int_\Omega h^{\alpha+1} \le (\alpha+1)\gamma^{\frac{\alpha}{\alpha+1}}(\int_\Omega (u-h)^{\alpha+1})^{\frac{1}{\alpha+1}} \tag{2.6}$$

Using Sobolev's inequality, (2.6) implies

$$(\gamma - \int_\Omega h^{\alpha+1})^2 \le (\alpha+1)^2\gamma^{\frac{2\alpha}{\alpha+1}}c^2 \int_\Omega |\nabla(u-h)|^2$$

$$= (\alpha+1)^2\gamma^{\frac{2\alpha}{\alpha+1}}c^2\lambda(\gamma - \int_\Omega hu^\alpha)$$

$$\le (\alpha+1)^2 \gamma^{\frac{2\alpha}{\alpha+1}} c^2 \lambda(\gamma - \int_\Omega h^{\alpha+1})$$

Therefore,

$$\gamma - \int_\Omega h^{\alpha+1} \le c^2(\alpha+1)^2 \gamma^{\frac{2\alpha}{\alpha+1}} \lambda$$

or

$$\lambda \ge (\gamma - \int_\Omega h^{\alpha+1})/c^2(\alpha+1)^2\gamma^{\frac{2\alpha}{\alpha+1}} \tag{2.7}$$

From our earlier discussion,

$$\int_\Omega |\nabla(u-h)|^2 = \lambda(\gamma - \int_\Omega u^\alpha h) \ge \frac{\lambda}{\alpha+1}(\gamma - \int_\Omega h^{\alpha+1})$$

or

$$\lambda \le (\alpha+1)\int_\Omega |\nabla(u-h)|^2/(\gamma - \int_\Omega h^{\alpha+1}) \tag{2.8}$$

We can use (2.8) to estimate λ from above by estimating

$$\int_\Omega |\nabla(u-h)|^2 = \int_\Omega |\nabla u|^2 - \int_\Omega |\nabla h|^2.$$

For example, let $\eta \ge 0$ be a fixed nontrivial $C_0^1(\bar{\Omega})$ function and set $v = h + \varepsilon\eta$. Then there is a unique $\varepsilon > 0$ so that $\int_\Omega v^{\alpha+1} = \gamma$. This choice of ε makes $v \varepsilon \mathcal{A}_\gamma$ and so

$$\int_\Omega |\nabla u|^2 \le \int_\Omega |\nabla v|^2 = \int_\Omega |\nabla h|^2 + \varepsilon^2 \int_\Omega |\nabla \eta|^2 \tag{2.9}$$

Again by convexity, we can estimate ε by

$$\gamma = \int_\Omega (h+\varepsilon\eta)^{\alpha+1} \ge \int_\Omega h^{\alpha+1} + (\alpha+1)\varepsilon \int_\Omega h^\alpha \eta$$

so that

$$\varepsilon \le C(\gamma - \int_\Omega h^{\alpha+1}) \tag{2.10}$$

The combination (2.9) (2.10) gives

$$\int_\Omega |\nabla(u-h)|^2 \le C(\gamma - \int_\Omega h^{\alpha+1})^2 \tag{2.11}$$

Inserting (2.11) into (2.8) gives the upper bound

$$\lambda \leq C(\gamma - \int_\Omega h^{\alpha+1}) \tag{2.12}$$

The estimates (2.7) and (2.12) give the required asymptotics for λ as stated in part i) of Theorem 2.1.

To get the correct asymptotics for λ as $\gamma \to \infty$, we let v be a minimizer to the problem

$$\int |\nabla v|^2 \to \min = \mu,\ v \varepsilon H_0^1(\Omega),\ \int_\Omega |v|^{\alpha+1} = 1$$

and set $u = (1-\varepsilon)\gamma^{\frac{1}{\alpha+1}} v + h$. As before, ε is chosen so that $\int_\Omega u^{\alpha+1} = \gamma$. Then

$$\int_\Omega |\nabla u|^2 = (1-\varepsilon)^2 \gamma^{\frac{2}{\alpha+1}} \mu + \int_\Omega |\nabla h|^2$$

so from (2.8),

$$\lambda \leq (\alpha+1)(1-\varepsilon)^2 \gamma^{\frac{2}{\alpha+1}} / (\gamma - \int_\Omega h^{\alpha+1}) \leq C\gamma^{-(\frac{\alpha-1}{\alpha+1})} \tag{2.13}$$

for γ large. The estimate (2.13) together with (2.7) completes the proof of the estimates stated in part ii) of Theorem 2.1. $\square$

3 Apriori Estimates near the Boundary

In this section, we will derive the following apriori sup norm estimate near the boundary.

Theorem 3.1 *Suppose Ω is a domain with $\partial\Omega \in C^2$ and $\phi \in C^{1+\beta}(\partial\Omega) \geq 0, \partial\phi \not\equiv 0$. Let $u \in C^2(\Omega) \cap C^{1+\beta}(\bar{\Omega})$ be a solution to the problem*

$$\begin{array}{rcllr} -\Delta u &=& g(u) & \text{in} & \Omega \\ u &=& \phi & \text{on} & \partial\Omega \end{array} \tag{3.1}$$

where $g(t)$ is a locally Lipschitz function satisfying

$$\begin{array}{ll} (i) & g(t) \text{ is non-decreasing, } g(0) = 0 \\ (ii) & t^{-(\frac{n+2}{n-2})} g(t) \text{ is non-increasing} \\ (iii) & g(t) \geq Kt, \text{ for } t \geq t_0 > 0, \text{ where } K > \lambda_1(\Omega). \end{array} \tag{3.2}$$

Then $u(x) \leq C$ in a neighborhood of $\partial\Omega$, $0 < d(x) < d_0$ ($d(x) =$ distance $(x, \partial\Omega)$) where C, d_o depend only upon ϕ, Ω and $(K - \lambda_1(\Omega))^{-1}$.

We first estimate u in $L^1_{loc}(\Omega)$ in a standard way [F-L-N].

Lemma 3.2 *Let u satisfy (3.1), (3.2). Then $\int_\Omega uv_1 \leq C$, where $v_1 \geq 0$ is the first Dirichlet eigenfunction of Ω.*

Proof. Let h be the harmonic extension of ϕ and let $v_1 \geq 0$ be the first (normalized) Dirichlet eigenfunction of Ω. Then

$$\int_\Omega v_1 \Delta(h - u) - (h - u)\Delta v_1 = 0, \text{ and so}$$

$$\int_\Omega v_1 g(u) + \lambda_1(h - u)v_1 = 0$$

Using 3.2 (iii), this gives

$$\int_\Omega v_1 u \leq \frac{\lambda_1}{K - \lambda_1} \int_{\{u < t_0\}} v_1 u \leq \frac{\lambda_1 t_0}{K - \lambda_1} |\Omega|^{\frac{l}{2}} = C.$$

Our aim is to use the method of moving planes and inversion to show that the Kelvin transform of u is monotone. For boundary date $\phi \equiv 0$, this was done in [?] for $n = 2$ and in [?] for $n > 2$.

Let us fix the notation as follows. For a point $P \in \partial\Omega$, we will choose a ball $B = B_R$ of fixed radius R externally tangent to $\partial\Omega$ at P. For convenience we will assume that $B = B_R(0)$ and that $P = Re_n$. We perform an inversion in B

$$y = Ix = \frac{R^2 x}{|x|^2} \tag{3.3}$$

Let κ denote a principal curvature at $X \in \partial D$ with respect to the interior normal $\nu(X)$ at X. Then [?, ?], the principal curvature $\widetilde{\kappa}$ of $I(\partial\Omega)$ at $I(X)$ (with respect to the interior normal) is given by

$$\widetilde{\kappa} = \kappa \frac{|X|^2}{R^2} + \frac{2}{R^2} X \cdot \nu$$

In particular, we may choose $\delta_0 \in (0,1)$ so small that if $X \in \partial\Omega \cap B_{\delta 0R}(P)$, $X \cdot \nu \geq \frac{1}{2}$, so that $I(\partial\Omega \cap B_{\delta 0R}(P))$ is strictly convex for R sufficiently small, say $\widetilde{\kappa} \geq \frac{1}{2R^2}$.

For any function ψ in Ω, the Kelvin transform of ψ is defined by

$$(K\psi)(y) = \frac{R^{n-2}}{|y|^{n-2}}\ \psi(x) = \frac{R^{n-2}}{|y|^{n-2}}\ \psi(\frac{R^2 y}{|y|^2}),\ x \in \Omega \tag{3.4}$$

and satisfies

$$-\Delta(K\psi) = \frac{R^{n+2}}{|y|^{n+2}}\ \Delta\psi\ (\frac{R^2 y}{|y|^2}) \tag{3.5}$$

$\square$

Lemma 3.3 *Let h be the harmonic extension of ϕ in Ω. Then for δ_0 and R small enough, Kh is monotone on $I(\partial\Omega \cap B_{\delta_0}(P))$, i.e. $\nabla(Kh) \cdot \tau > 0$ for τ in a cone of directions near $-e_n$.*

Proof. Using (3.4), we compute

$$(\nabla Kh)_i(y) = -\frac{n-2}{|y|^n} R^{n-2} y_i h(x) + \frac{R^n}{|y|^n}(\delta_{ij} - \frac{2y_i y_j}{|y|^2}) h_j(x)$$

This gives

$$\nabla(kh)\cdot\tau = \frac{R^{n-2}}{|y|^n}(-(n-2)(y\cdot\tau)h(x) + O(R^2)O(|\nabla h|)) = \tag{3.6}$$

$$\frac{R^{n-2}}{|y|^n}((n-2)R\tau\cdot -e_n + O(\delta_0 R) + O(R^2)O(|\nabla h|))$$

Since $h \in C^{1+\beta}(\bar{\Omega})$, (3.6) implies that

$$\nabla(kh)\cdot\tau(y) > 0 \text{ if } \tau\cdot -e_n > O(\delta_0) + O(R).$$

$\square$

Corollary 3.4 *Ku is monotone on $I(\partial\Omega)\cap B_{\delta_0 R}(P))$ for τ in the cone of directions near $-e_n$ given in lemma 3.3.*

Proof. Since $u \geq h$ in Ω, $u = h$ on $\partial\Omega$, we have that $Ku \geq Kh$ in $I(\Omega)$ and $Ku = Kh$ on $I(\partial\Omega)$. Therefore,

$$\nabla(Ku)\cdot\tau \geq \nabla(Kh)\cdot\tau.$$

$\square$

We can now prove

Proposition 3.5 *Let u satisfy the hypotheses of Theorem 3.1. Then $\nabla(Ku)\cdot\tau > 0 \in K\Omega\cap B_{\delta_0 R/4}(P)$, for y in a cone of directions of aperture angle $\alpha(\tau, -e_n) \leq \delta_0/8R$, for δ_0, R sufficiently small.*

Proof. We may assume that $K(\partial\Omega)\cap B_{\delta_0 R}(P)$ is a graph in any of the directions τ. For any allowable τ, we denote by y_λ the reflection of y in the plane $T_\lambda = \{y\cdot\tau = \lambda\}$, and set $v = Ku$. Since $K(\partial\Omega)\cap B_{\delta_0 R}(P)$ is strictly convex, there is a unique value λ_0, such that the plane T_{λ_0} is tangent to $K(\partial\Omega)$ (at a point near P) and then for $\lambda > \lambda_0$ cuts off an open cap Σ_λ from $K\Omega$. For $y \in \Sigma_\lambda$ we define $v^\lambda(y) = v(y_\lambda)$.

By our choice of δ_0, R and the cone aperture α, we may increase λ to a first value λ_1 so that the cap Σ_{λ_1} contains a $\frac{\delta_0^2}{16}$ neighborhood of $K(\partial\Omega)\cap B_{\delta_0 R/4}(P)$ and also is contained in $B_{\delta_0 R}(P)$. This implies that $v \not\equiv v^\lambda$ for $\lambda_0 < \lambda < \lambda_1$. For otherwise, $\nabla v \cdot \tau = 0$ on $T_\lambda \cap K(\partial\Omega)$, contradicting Corollary 3.4. Using Corollary 3.4 we see that $v(y) < v^\lambda(y)$ for $\lambda > \lambda_0, \lambda$ close to λ_0. According to [**?**] [**?**] the inequalities $v < v^\lambda, \nabla v \cdot \tau > 0$ must continue to hold for $\lambda_0 < \lambda \leq \lambda_1$, since we have ruled out the possibility $v \equiv v^\lambda$. □

Proof of Theorem 3.1. Let Q be any point in $K(\Omega)\cap B_{\delta_0 R/8}(P)$ at distance $c\delta_0^2$ from $k(\partial\Omega)$, where c is a small constant independent of δ_0, R chosen so that the cone Γ_Q of aperture α and height $c\delta_0^2$ with vertex at Q is contained in a $\delta_0^2/16$ neighborhood of $K(\partial\Omega) \cap B_{\delta_0 R/4}(P)$. Let $v = Ku$. According to Lemma 3.2 we can estimate

$$\int_{\Gamma_Q} v \leq C \tag{3.7}$$

for a constant C uniformly controlled. Then using Proposition 3.5, $v \geq v(Q)$ in Γ_Q, and so from (3.7) we obtain

$$v(Q) \leq \frac{c}{|\Gamma_Q|} = C_1 \tag{3.8}$$

In particular, using Proposition 3.5 once more, we can conclude that

$$v(y) \leq C_1$$

for y in a $c\delta_0^2$ neighborhood of $K(\partial\Omega) \cap B_{\delta_0 R/16}$. Finally, using the compactness of $\partial\Omega$, and that $v = Ku$ we conclude that $\varepsilon_0 \leq u(x) \leq c_2$ in a uniform neighborhood $0 < d(x) < d_0$ of $\partial\Omega$. This completes the proof of Theorem 3.1. □

4 Global Estimates via the Asymptotic Symmetry Method

The apriori near the boundary estimates of the previous section allow us to apply the "asymptotic symmetry method" that we developed in [C-G-S] to prove interior Lipschitz estimates. In this section we will outline the main steps in the proof of the following crucial result.

Theorem 4.1 *Suppose Ω is a domain with $\partial\Omega \in C^2$ and $\phi \in C^{1+\beta}(\partial\Omega), \phi \not\equiv 0$. Let $u \in C^2(\Omega) \cap C^{1+\beta}(\bar{\Omega})$ be a solution to the problem*

$$-\Delta u = g(u) \text{ in } \Omega \quad u = \phi \text{ on } \partial\Omega \tag{4.1}$$

where $g(t)$ is a locally Lipschitz function satisfying the conditions (3.2). Then $|u(x) - u(z)| \leq K\,|x - z|$ for $x, z \in \Omega$ where K depends only on ϕ, Ω and $(K - \lambda_1(\Omega))^{-1}$.

Combining Theorems 3.1 and 4.1 and using standard elliptic regularity theorem, we obtain the following global apriori estimate.

Corollary 4.2 *Let Ω, ϕ, u satisfy the hypothesis of Theorem 4.1. Then $||u||_{C^{1+\beta}(\Omega)} \leq C$, where C depends only on ϕ, Ω and $(K - \lambda_1, (\Omega))^{-1}$.*

We first observe that from Theorem 3.1, it follows in a standard way that

$$||u||_{C^{1+\beta}(\Omega_0)} \leq C \tag{4.2}$$

where $\Omega_0 = \{x \in \Omega\colon 0 < d(x) < \frac{3}{4}d_o\}$, and C depends only upon ϕ, Ω and $(K - \lambda_1(\Omega))^{-1}$. Therefore, in proving Theorem 4.1, it suffices to consider points x, z with $d(x)$, $d(z) \geq \frac{d_o}{2}$. Since $u > h > 0$ in Ω, and u is under control in Ω_0, we may always reduce Theorem 4.1 to the case $\phi \geq \varepsilon_0$ (i.e. $u \geq \varepsilon_0$ in $\overline{\Omega}$).

Fix $x_0 \in \Omega$ such that $d(x_0) \geq \frac{d_o}{2}$ and set $R = \frac{d_0}{8}$. Let $B = B_R(x_0)$. For convenience, we assume $x_0 = 0$. We perform an inversion in B, $y = Ix$ and set $v = Ku$ as in Section 3, formulas (3.3)-(3.5).

We denote by $\Gamma_1, \ldots \Gamma_N$ the connected components of $\partial\Omega$ with Γ_N the finite boundary component of $R^n - \Omega$. Let $\widetilde{\Gamma}_i = I(\Gamma_i)$, $i = 1, \ldots, N$. Then v is well-defined in the unbounded domain $\widetilde{\Omega} = I\Omega$. Note that $\widetilde{\Omega} = R^n \setminus \cup_{i=1}^N H_i$ where the H_i are "holes", that is the H_i are disjoint connected domains bounded by $\widetilde{\Gamma}_i$, and $x_0 \in \widetilde{\Gamma}_N$.

Note that in $\widetilde{\Omega}$, v satisfies

$$-\Delta v = \frac{R^{n+2}}{|y|^{n+2}}\, g\Big(\frac{|y|^{n-2}}{R^{n-2}} v(y)\Big) \equiv f(y,\, v) \tag{4.3}$$

and because of (4.2),

$$||v||_{C^{1+\beta}(I\Omega_o)} \leq C_1, \quad 0 < \delta_0 \leq v \leq \frac{1}{\delta_0} \text{ in } I\Omega_0 \tag{4.4}$$

where C_1, δ_0 are under control.

Step 1. *Extension Lemma.* There exists $\sigma = \sigma(\delta_0,\, C_1,\, f)$ such that for any open set $A \subset \cup_{i=1}^N H_i$ with $|A| < \sigma$, we can extend v to a Lipschitz function $\bar{v}$ in R^n satisfying

$$\begin{aligned} &\text{(i)} \quad \bar{v} \geq \delta_0/2 \text{ in } R^n \\ &\text{(ii)} \quad -\Delta\bar{v} \geq f(y,\, \bar{v}) \text{ in } [\overline{\widetilde{\Omega} \cup A}]^0 \end{aligned} \tag{4.5}$$

Our aim is to show that v is asymptotically radial at infinity in the sense that

$$v(y) = (\inf v)_{\partial B_{|y|}}\Big(1 + 0\big(\frac{1}{|y|}\big)\Big) \text{ as } |y| \to \infty \tag{4.6}$$

Step 2. *Reflection Theorem.* Assume there exists a set $A' \subset \{(y',\, 0) \colon |y'| < R\}$, $|A'| < \frac{\sigma}{2}$ and a positive number $M > R$ such that if $y = (y',\, y_n)$ with $y' \notin A'$ and $y_n \geq M$ then $v(y) \leq \frac{\delta_0}{4}$. Let $\bar{v}$ be the extension of v to R^n given by the extension lemma (Step 1) corresponding to $A = \{y = (y',\, y_n) \colon |y| < R,\, y' \in A'\}$ Then

$$\bar{v}(y) \leq \bar{v}(y_\lambda) \text{ for } y_n > \lambda \geq M \tag{4.7}$$

(Here $y = (y', y_n)$ and $y_\lambda = (y', 2\lambda - y_n)$ is the reflection of y in the plane $y_n = \lambda$)

A direction (which for convenience we have taken to be e_n) for which the hypothesis of the Reflection Theorem is satisfied is said to be an *admissible direction*. The essential meaning of admissibility is that v decays to zero uniformly, on rays parallel to the reflection direction, except possibly for a certain exceptional set of rays which hit B_R. The set A is the intersection of the exceptional rays with B_R and is required to have small measure.

The next step is to show that for M large enough the set of admissible directions covers almost all of the unit sphere Σ_1. The precise statement is as follows.

Step 3. Let $v \geq 0$ be superharmonic in $R^n \backslash B_R$ and satisfy

$$\int_{R^n \backslash B_R} \frac{v^p}{|x|^\beta} < c_0, \text{ for some } p \geq 1, \ \beta < n. \tag{4.8}$$

Then the set of admissible directions for a given $M = 2^k$ and $\sigma = 2^{-\varepsilon k} = M^{-\varepsilon}$ has measure $|\Sigma_1| -_c M^{-\delta}$ for $\varepsilon = \frac{n-\beta}{2}$, $\delta = \frac{n-\beta}{4}$ and c depending only on c_0 and n.

Note that because of Lemma 3.2, $\int_{B_{2R(x_0)}} u \leq C$ and since $v = Ku$ is given by (3.4), v satisfies (4.8) with $p = 1$ and $\beta = n - 2$ for suitable c_0.

The final step asserts that when the Reflection Theorem holds for a sufficiently large set of directions then asymptotic symmetry (4.6) holds.

Step 4. Let $v \geq 0$ be superharmonic on $R^n - B_R$ with the property that for some $M > 0$ and $\tau \in \mathcal{A}$ (the admissible set) $\subset \Sigma_1$,

$$v(y) \leq v(y_\lambda) \text{ if } y \cdot \tau \geq \lambda > M$$

Then

$$v(y) = \inf{}_{|z|=|y|} v(z)\,(1 + 0(\frac{1}{|y|)}) \text{ for } |y| \to \infty \tag{4.9}$$

where all error estimates are uniformly controlled.

The detailed proofs of Steps 1-4 are given in [**?**]. From Step 4, Theorem 4.1 follows easily. For from (4.9) it follows that

$$u(x) = \inf_{|x|=r} u(1+0(r)) \text{ for } |x| = r \text{ as } r \to 0. \tag{4.10}$$

Formula (4.10) implies, by a simple chaining argument that u is bounded. For we can connect any $x,\ d(x) \geq \frac{d_0}{2}$ to a point $\bar{x},\ d(\bar{x}) < d_0$ in a finite number of steps k (with k under control) so that $x_1 = \bar{x},\ x_k = x$ and

$$u(x_{i+1}) \leq c\, u(x_i), \quad i = 1, \ldots, k$$

Thus, $u(x) = u(x_k) \leq c^{k-1}\, u(\bar{x}) \leq C$ by Theorem 3.1.

Now that u is bounded, choosing the center point x_0 and radius r to be $x_0 = \frac{x+z}{2}$ and $r = |x-z|/2$, (4.10) gives

$$|u(x) - u(z)| \leq C\, |x - z|$$

for $|x-z|$ sufficiently small. This completes the proof of Theorem 4.1.

5 The Existence of Smooth Minimizers

In this section we will use the apriori estimates of Section 4 to pass from approximating minimizers to the limit case.

We proceed directly to the

Proof of Theorem 1.1. Let $M = \inf \int_\Omega |\nabla u|^2$ for u in the admissible class

$$\mathcal{A} = \left\{ u - h \in H_0^1(\Omega) \colon \int_\Omega |u| \frac{2n}{n-2} = \gamma > \int_\Omega h \frac{2n}{n-2} \right\}$$

Given any $\varepsilon_k > 0$ there exists $\eta \in \mathcal{A}$ such that

$$\int_\Omega |\nabla \eta|^2 \leq M + \varepsilon_k$$

By Lebesgues' theorem,

$$\int_\Omega |\eta|^{\alpha_j + 1} \to \int_\Omega |\eta| \frac{2n}{n-2} = \gamma > \int_\Omega h \frac{2n}{n-2}$$

for any sequence $\alpha_j \nearrow \frac{n+2}{n-2}$. Let us fix such a sequence and assume that for j large

$$\gamma_j =: \int_\Omega |\eta|^{\alpha_j+1} > \int_\Omega h^{\alpha_j+1}$$

Using the results of Sections 2–4, there exists a positive minimizer $u_{jk} \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ in the admissible class $\mathcal{A}_{\gamma j}$ to the subcritical variational problem (2.1) with $\alpha = \alpha_j$, j large. Moreover (from Corollary 4.2),

$$\|u_{jk}\|_{C^{1+\alpha}(\Omega)} \leq C \tag{5.1}$$

where C is a uniform constant. Since u_{jk} is a minimizer, $\int_\Omega |\nabla u_{jk}|^2 \leq \int_\Omega |\nabla\eta|^2 \leq M + \varepsilon_k$.

Choosing a diagonal subsequence, $u_{j_i j_i}$ converges to a limit minimizer $u \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ satisfying

$$\begin{array}{rcll} -\Delta u & = & \lambda u^{\frac{n+2}{n-2}} & \text{in} \quad \Omega \\ u & = & \phi & \text{on} \quad \partial\Omega \\ u & > & 0 & \text{in} \quad \Omega \end{array}$$

where $\lambda > 0$ satisfies the estimates of theorem 2.1. □

We now consider a more general variational problem

$$\int |\nabla u|^2 \to \min \tag{5.2}$$

$$u \in \mathcal{A}_\gamma = \left\{ u \in H^1(\Omega)\colon u - h \in H_0^1(\Omega), \int_\Omega G(u) = \gamma > \int_\Omega G(h) \right\}$$

Here $G(t)$ is the primitive of $g(t)$, with $g(t)$ satisfying the hypotheses of Theorem 1.2 and extended to $g \equiv 0$ for $t \leq 0$. As before, $\phi \in C^{1+\beta}(\partial\Omega) \geq 0$ is positive somewhere and $\partial\Omega \in C^2$.

We will assume, for the moment, that $g(t)$ satisfies the subcritical growth condition

$$g(t) \leq At^\alpha \quad \alpha \in \left[1, \frac{n+2}{n-2}\right), \; t > T \tag{5.3}$$

for some A, T, α.

We note that given an arbitrary $g(t)$ satisfying the conditions (1.5), we can construct subcritical approximations $g_j(t)$ satisfying the same conditions by choosing $t_j > t_0$ and setting

$$g_j(t) = \begin{cases} g(t) & t \leq t_j \\ \frac{g(t_j)}{t_j} t & t > t_j \end{cases} \tag{5.4}$$

Preliminary to proving Theorem 1.2, we have the following

Proposition 5.1 *Let $g(t)$ satisfy the hypotheses of Theorem 1.2 and the subcriticality condition (5.4). Then there exists a positive minimizer $u \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ of Problem (5.2) satisfying*

$$\begin{aligned} -\Delta u &= \lambda g(u) \quad \text{in} \quad \Omega \\ u &= \phi \qquad\;\; \text{on} \quad \partial\Omega \end{aligned}$$

where $0 < \lambda < C(\gamma - \int_\Omega G(h))$. If $\frac{g(t)}{t} \to \infty$ as $t \to \infty$, $\lambda \to 0$ as $t \to \infty$.

Proof. The existence of a classical solution to Problem (5.2) is well-known.

That is, u satisfies

$$\begin{aligned} -\Delta u &= \lambda g(u) \quad \text{in} \quad \Omega \\ u &= \phi \qquad\;\; \text{on} \quad \partial\Omega \end{aligned} \tag{5.5}$$

We claim that $\lambda > 0$ and so $u > h$ in Ω by the maximum principle. For using $u - h$ as a test function in (5.5).

$$\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u \cdot \nabla h = \lambda \int_\Omega (u-h) g(u) \tag{5.6}$$

Since G is convex, we have

$$0 < \gamma - \int_\Omega G(h) = \int_\Omega (G(u) - G(h)) \leq \int_\Omega (u-h) g(u) \tag{5.7}$$

Observing that $\int_\Omega \nabla u \cdot \nabla h = \int_\Omega |\nabla h|^2$, we conclude from (5.6) (5.7) that $\lambda > 0$, as claimed. We also obtain the estimate

$$0 < \lambda \leq \left(\int_\Omega |\nabla u|^2 - \int_\Omega |\nabla h|^2 \right) / \, (\gamma - \int_\Omega G(h)) \tag{5.8}$$

Using (5.8) we can estimate λ from above. For example, let $\eta \geq 0$ be a fixed nontrivial $C_0^1(\overline{\Omega})$ function and set $v = h + \varepsilon\eta$. There is a unique $\varepsilon > 0$ so that $v \in \mathcal{A}_\gamma$. Then

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla v|^2 = \int_\Omega |\nabla h|^2 + \varepsilon^2 \int_\Omega |\nabla\eta|^2 \tag{5.9}$$

Again by the convexity of G,

$$\gamma = \int_\Omega G(h + \varepsilon\eta) \geq \int_\Omega G(h) + \varepsilon \int_\Omega \eta g(h)$$

so that

$$\varepsilon \leq C(\gamma - \int_\Omega G(h)) \tag{5.10}$$

The combination (5.8)–(5.10) gives

$$0 < \lambda \leq C(\gamma - \int_\Omega G(h)) \tag{5.11}$$

To get a better upper bound as $\gamma \to \infty$ let $v = h + t\eta$ for a fixed nontrivial $\eta \geq 0$ $\eta \in C_0^1(\Omega)$. Choosing t so that $v \in \mathcal{A}_\gamma$, we have

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla v|^2 = \int_\Omega |\nabla h|^2 + t^2 \int_\Omega |\nabla\eta|^2$$

Therefore, from (5.8)

$$\lambda \leq C \frac{t^2}{\gamma - \int G(h)} \tag{5.12}$$

By convexity, we can estimate

$$\gamma - \int_\Omega G(h) \geq t \int_\Omega \eta g(t\eta) \tag{5.13}$$

If we assume that $\frac{g(t)}{t} \to \infty$ as $t \to \infty$, we can conclude from (5.12), (5.13) that

$$\lambda \to 0 \text{ as } \gamma \to \infty \tag{5.14}$$

□

We are now in a position to give the

Proof of Theorem 1.2:. Choose a sequence $t_j \nearrow +\infty$ with $t_j > \sup_{\partial\Omega} h$. As in the proof of Theorem 1.1, given any $\varepsilon_k > 0$, there exists

$$\eta \in \mathcal{A} = \left\{ u - h \in H_0^1(\Omega) \colon \int_\Omega G(u) = \gamma > \int_\Omega G(h) \right\}$$

such that

$$\int_\Omega |\nabla \eta|^2 \leq M + \varepsilon_k$$

Here, $M = \inf_{u \in \mathcal{A}} \int |\nabla u|^2$. By Lebesgues' theorem,

$$\int_\Omega G_j(\eta) \to \int_\Omega G(\eta) = \gamma > \int_\Omega G(h)$$

where $G_j^{'}|t| = g_j(t)$ and $g_j(t)$ is given by (5.4). Using Proposition 5.1 and Corollary 4.2, there exists a positive minimizer $u_{jk} \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ in the admissible class $\mathcal{A}_{\gamma_j}$, where

$$\gamma_j =: \int_\Omega G_j(\eta) > \int_\Omega G_j(h) = \int_\Omega G(h).$$

Since u_{jk} is a minimizer,

$$\int_\Omega |\nabla u_{jk}|^2 \leq \int_\Omega |\nabla \eta|^2 \leq M + \varepsilon_k$$

Choosing a diagonal subsequence, $u_{j_i j_i}$ converges to a limit minimizer $u \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ satisfying

$$\begin{array}{rcll} -\Delta u & = & \lambda g(u) & \text{in} \quad \Omega \\ u & = & \phi & \text{on} \quad \partial\Omega \\ u & > & 0 & \end{array}$$

where $\lambda \geq 0$. Since $\int_\Omega G(u) = \gamma > \int_\Omega G(h)$, $0 < \lambda < C(\gamma - \int_\Omega G(h))$ by the arguments of Proposition 5.1. If in addition $\frac{g(t)}{t} \to \infty$ then $\lambda \to 0$ as $\gamma \to \infty$. The proof of Theorem 1.2 is complete. $\square$