# Some Basic Facts, Old and New, About Triply Periodic Embedded Minimal Surfaces

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## 1 Introduction

In the last twenty years, triply-periodic embedded minimal surfaces have been of great interest and utility to researchers in chemistry, cystallography and material science. The classical examples of Schwarz, Neovious et al., dating from the 19th century were known, but seemed to rest on the periphery of modern mathematical interest in minimal surfaces. The monograph of Schoen, which may fairly be said to be the source of modern interest in the subject, presented many new examples, but this work was much better known to scientists than to mathematicians. Except for the gyroid, few of Schoen's examples have had very much impact, until quite recently, in the field of mathematics proper. This has changed recently due in part to the work of Karcher, Fischer-Koch, Wohlgemuth, Nitsche, and Ross.

Of course, of primary interest to scientists is the fact that these triply-periodic embedded minimal surfaces are the common boundary of two connected intertwined solid regions, referred to as *labyrinths* that are themselves triply-periodic. These labyrinths are defined as the components of the complement of the surface in $\mathrm{R}^3$. Minimal surfaces have nonpositive Gauss curvature and this seems to be an important secondary property of the labyrinthine interface. Such a surface makes very plain the underlying symmetry group of the labyrinths.

In some of the applications of these minimal surfaces in science, the fact that they have zero mean curvature does *not* seem to be a critical property. In those cases, the assumption that the dividing surface is minimal may be a confusing and unnecessary restriction. To illustrate this, consider the following list of questions about triply-periodic embedded minimal surfaces.[1]

1. Is there a triply-periodic embedded minimal surface of arbitrarily high genus?

2. Does there exist a triply-periodic embedded minimal surface with no symmetries other than translations?

3. Does there exist a properly embedded minimal surface of infinite genus that is not periodic, but does separate space into two labyrinths?

4. Does there exist an embedded minimal surface, which is embedded in a ball minus a point $p$, such that every neighborhood of $p$ intersects the surface in a minimal surface of infinite genus?

[1] For definitions of terms used in the remainder of this section, e.g., genus, 3-torus, $P$-surface, see the next section.

The answer to all of these questions are unknown. However if we substitute for "minimal surface" the condition "surface of nonpositive Gaussian curvature", then the answer is well known to be **yes**.

The condition of minimality is a restrictive and rigid one. We shall see two more instances of this below. First, we will discuss triply-periodic embedded minimal surfaces (TPEMS) of *genus three* and show, by an application of classical function theory, that these surfaces possess a symmetry consisting of inversion about any flat point. We will then present a simple proposition, inspired by a calculation of Maggs and Leibler, which shows that the Schwarz P-surface does not possess certain deformations preserving full reflectional symmetry, even though one can easily find nonpositively curved surfaces that do so.

Before beginning this discussion it is worthwhile to ask a basic mathematical question that has physical significance. *Which triply-periodic embedded minimal surfaces are stable?* "Stable" means that the second derivative of area is positive, or at least nonnegative, for all compactly supported variations. It is not hard to see that a sufficiently large but still compact piece of a TPEMS cannot be stable. Consider the P-surface in the unit cube. Clearly there are area-reducing variations as indicated in Figure 1.

However, if one imagines that the surfaces as an interface and considers it in a compact 3-torus $\mathrm{T}^3$, then it is natural to consider the question of stability for only those variations that *preserve the volume* of the regions on either side of the surface. With this restraint imposed, Ross has shown that the Schwarz P-surface and D-surface, are stable [7]. See also the remark at the end of Section 3. The general question of stability for a TPEMS is still very much open.

# 2 Triply periodic embedded surfaces

In this section, we present some basic properties of triply periodic embedded surfaces, not necessarily minimal.

**Definition 2.1** *A surface $\Sigma$ in $\mathrm{R}^3$ is triply-periodic if its group of symmetries contains three linearly independent translations.*

We will be interested in dividing surfaces, i.e., those which are the interface of two intertwined labyrinths, so we will assume that our surfaces are connected, embedded and free of boundary points. We also want to exclude the existence of an infinite labyrinth in a finite volume of $\mathrm{R}^3$. This can be avoided if we insist that the surfaces we consider are *proper*; that is, curves which diverge on the surface also diverge in $\mathrm{R}^3$.

**Definition 2.2** *Let $T$ be the group of translations that leave $\Sigma$ invariant. Then $\mathrm{T}^3 = \mathrm{R}^3/T$ is the compact 3-dimensional manifold produced by identifying points in $\mathrm{R}^3$ that differ by a translation in $T$. Similarly $S = \Sigma/T \subset \mathrm{T}^3$.*

$\mathrm{T}^3$ is usually referred to as a "3-torus" because it is topologically the same as the product of three circles. (A torus or "2-torus" is topologically the product of two circles.) We will insist that $S$ be a compact surface of finite genus. (Recall that the genus of a compact surface $S$ is the number of handles necessary to add to a sphere to produce a surface that is homeomorphic to $S$). The Euler characteristic of a compact surface $S$ is denoted by $\chi(S)$ and defined to be

$$\chi(S) = 2(1-k)\,, \tag{2.1}$$

where $k$ is the genus of $\mathcal{S}$. The Gauss-Bonnet Theorem relates the Gauss curvature of $\mathcal{S}$ to the Euler characteristic:

$$\int_S K dA = 2\pi\chi(S) = 4\pi(1-k)\,. \tag{2.2}$$

For example the Schwarz P-surface (see Figure 1) has genus 3 and, by the above formula, total curvature $-8\pi$.

A properly embedded surface $\Sigma \subset \mathrm{R}^3$ must divide $\mathrm{R}^3$ into two components. As a consequence, $\Sigma$ has two distinct sides and is oriented. However, in the case of a triply-periodic surface $\Sigma$, it is not always the case that $S = \Sigma/T$ is orientable. The group of translations $T$ may contain orientation-reversing translations.

A properly embedded surface may possess an *orientation-reversing* translation, in which case $S \subset \mathrm{T}$ will be an unorientable surface and will not divide $\mathrm{T}$ into two components. For example, in Figure 2, we see on the left a fundamental domain of a triply periodic surface. this fundamental domain $S$ divides its 3-torus into two components, and the surface $S$ is orientable. It possesses an orientation-reversing symmetry induced by a translation. If we further reduce the surface by this translation we produce a new fundamental 3-torus and a new "quotient surface", which is not orientable and does not divide its 3-torus into two components. Moreover, the surface $S$ on the right of Figure 2 has genus 2, while its "double" on the left has genus 3.

To avoid producing a non-dividing, non-orientable surface in the quotient we will make the convention that $T$ is the group of *orientation-preserving translations* of $\Sigma$. This will assure that $S = \Sigma/T \subset \mathrm{T} = \mathrm{R}^3/T$ is an oriented surface that divides $\mathrm{T}$ into two components. Since we are also assuming $\Sigma$, and hence $S$, is connected, it can be shown that this forces the genus of $S$ to be at least 3. We will prove this in the next section under the assumption that $\Sigma$ is minimal.

More important, the fact that $T$ consists of orientation-preserving translations means that the Gauss normal map $N$ of $\Sigma$ to the sphere $S^2$ descends to a well-defined mapping of $S$ to $S^2$. *(If $p$ and $q$ are points of $\Sigma$ that differ by an orientation-preserving translation $\tau$, then $N(p) = N(\tau(p)) = N(q)$.)* Because $KdA$ is the pull-back to $S$ of the area element on $S^2$ we may interpret formula (2.2) as saying that the area of the Gaussian image, measured with a sign, is equal to $4\pi(1-k)$, where $k$ is the genus of $S$. Also, two points $p, q \in \Sigma$ that differ by a translation $\tau$ have coordinates that differ by a constant vector. Thus if $h$ is any coordinate function on $\Sigma$, (i.e., $h(p) = p \cdot \vec{u}$ for some unit vector $\vec{u} \in \mathrm{R}^3$) then $h(q) = h(\tau(p)) = h(p) + c$ for some constant $c$. *It follows that $dh$ is well-defined on $\Sigma/T = S$.*

# 3 Triply-periodic embedded minimal surfaces

As we have seen in the previous section, given a properly embedded triply-periodic surface $\Sigma \subset \mathrm{R}^3$, the Gauss mapping $N$ and height differential $dh$, for any height function $h$, are well-defined on the quotient surface $\Sigma/T = S \subset \mathrm{T} = \mathrm{R}^3/T$, where $T$ is the group of translational symmetries of $\Sigma$ that preserve orientation. If $\Sigma \subset \mathrm{R}^3$ is minimal, so is $S \subset \mathrm{T}$. Minimal surfaces are characterized by the property that their Gauss mappings are meromorphic. It follows that the Gauss map of $N\colon S \to S^2$ is meromorphic.[2]

An immediate consequence of this fact is that we can relate $d \geq 0$, the degree of $N$, to the genus of $S$. Recall that we are assuming that $S$ has finite genus $k$. A meromorphic map from $S$ to $S^2$ of degree $= d$ will cover each point of $S^2$ exactly $d$ times, counting multiplicity. From formula 2.2 we have

$$\int_S K dA = 4\pi(1-k)\,.$$

But the left-hand side is the algebraic area of $N(S)$, which is minus the area of $S^2$ times $d$; i.e., $-4\pi d$. The negative sign arises because $K$ is nonpositive on a minimal surface; $N$ is orientation-reversing. Combining this with the formula above, yields $-4\pi d = 4\pi(1-k)$ or

$$d + 1 = k\,. \tag{3.3}$$

It follows immediately that $k \geq 1$.

If $k = 1$ then $d = 0$, which means that the Gauss map is constant and $S$ is flat. This can only happen if $S$ is a flat torus in T. But such a flat torus does not separate T and lifts to $\mathrm{R}^3$ as a family of parallel planes, *not a connected surface*. This is not a dividing surface between two labyrinths.

If $k = 2$, then $d = 1$, according to (3.3). This is also impossible because $N$ would be a degree $= 1$ meromorphic map from $S$ to the sphere $S^2$. Such a map must be a homeomorphism, an impossibility because of the difference in genus: $S$ has genus two and $S^2$ has genus zero. We may conclude that $k \geq 3$.

For the remainder of this discussion, we will assume that $k$, the genus of $S$, is equal to 3. This is the case for the classical $P$ and $D$ surfaces of Schwarz as well as for the gyroid, which lies in the same associate family of minimal surfaces. This means that $d = 2$.

A compact Riemann surface on which there exists a degree $= 2$ meromorphic mapping to $S^2$ is said to be *hyperelliptic*. (All compact surfaces of genus $= 1$ and genus $= 2$ possess such mappings. For genus $= 3$ surfaces this is not the case.) We will use this fact to show that any genus $= 3$ properly embedded minimal surface is invariant under an inversion about any point $p$ on the surface where Gaussian curvature is zero.

Recall that, in general, a point $p$ on a minimal surface is a point of zero Gaussian curvature $\Longleftrightarrow p$ is an umbilic (i.e., both principal curvatures are equal to 0) $\Longleftrightarrow$ the Gauss map $N$ is singular at $p$ (because $|dNp| \sim |K|$). Since $N$ is meromorphic for minimal surfaces, points where $N$ is singular must be isolated. The singular points of a meromorphic map are referred to as *branch points*, and the value of the map at a branch point is referred to as a *branch value*. In a small enough neighborhood

[2] The minimal surface $\Sigma \subset \mathrm{R}^3$ has harmonic coordinates and therefore, locally, any coordinate function $h$ together with its harmonic conjugate $h^*$ produce conformal coordinates. Letting $z = h + ih^*$ gives a local complex parameter making $\Sigma$ a Riemann surface.

$\mathcal{O}$ of a branch point $b$, all the points except $b$ are regular points, and $N(\mathcal{O})$ is a small neighborhood of the branch value of $N(b)$. The inverse image of a point $\widehat{b} \neq b$ in $N(\mathcal{O})$ will be the image of exactly two points in $\mathcal{O}$, because $N$ has degree 2. Such a branch point is called a *simple branch point.* (If $N$ were 1-1 on $\mathcal{O} - \{b\}$, one could show that $b$ was not a branch point. Thus in our case the branch points of $N$ are as simple as they can be.)

We will now show that $N$ has exactly 8 branch points. Consider a triangulation of $S^2$ chosen so that each of the branch values is at the vertex of some triangle. Of course, there may be other vertices. Lift all the faces, edges and vertices of this triangulation to $S$. If

$$\begin{aligned} F &= \# \text{ faces} \\ E &= \# \text{ edges} \\ V &= \# \text{ vertices} \end{aligned}$$

of the triangulation of $S^2$, then the lifted triangulation of $S$ has $2F$ faces and $2F$ edges. The number of vertices is $2V$ minus the number of branch points, which have a single lift; that is $2V - B$, where $B$ is the number of branch points. Applying the Euler formula, we have

$$F - E + V = \chi(S^2) = 2 - 2 \ (\text{genus} \ \ S^2) = 2$$

$$2F - 2E + 2V - B = \chi(S) = 2 - 2 \ (\text{genus of} \ \ S) = -4\,.$$

Combining we get $B = 8$, as claimed.

Figure 3

$g = \sigma \circ N$,

$\sigma\colon S^2 \to \mathrm{C} \cup \{\infty\}$, is stereographic projection

Letting $g = \sigma \circ N$, where $\sigma\colon S^2 \to \mathrm{C} \cup \{\infty\}$ is stereographic projection, we realize the Gauss map as a two-fold covering of the extended complex plane with eight simple branch points. On $S$, any height function $h$ has a well-defined complex differential $dh$. Since $h$ is harmonic $dh$ is holomorphic. Furthermore, the *sheet interchange* $\alpha\colon S \to S$, which fixes the branch points and interchanges any two points of $S$ with the same $g$ value is a conformal diffeomorphism satisfying

$$g \circ \alpha = g\,. \tag{3.4}$$

Consider $\mu = \alpha^*(dh) + dh$, and notice that $\mu$ is well defined, via projection, not only on $S$ but on $S^2 = \mathrm{C} \cup \{\infty\}$. But $S^2 = \mathrm{C} \cup \{\infty\}$ carries no holomorphic one forms, except the zero form. Hence $\mu \equiv 0$, or

$$\alpha^* dh = -dh\,. \tag{3.5}$$

The two previous formulae have strong consequences. Recall that the Weierstrass representation states that a minimal immersion may be recovered from the Gauss map, the height differential of a coordinate function and the underlying Riemann surface $S$ via the formula

$$X(q) = \frac{\mathrm{Re}}{2} \int_{p_0}^{q} \Phi(z) dz \qquad q \in S\,, \text{ where} \tag{3.6}$$
$$\Phi = \left( \frac{1}{g} - g\,,\, i\left(\frac{1}{g} + g\right)\,, 2 \right) dh\,.$$

In general $X$ is conformal. The choice of base point $p_0$ changes the immersion by a rigid translation. In our case, we are assuming that the surface is triply-periodic. Hence $X/T$ is well-defined in T.

If we choose $p_0$ to be one of the branch points of $g = \sigma \circ N$, then $X(p_0) = \vec{0}$. Notice that it follows from (3.5) and (3.6) that $\alpha^*\Phi = -\Phi$, and hence

$$X \circ \alpha = -X\,.$$

This means that inversion about $\vec{0} = X(p_0)$ in $\mathrm{R}^3/T = \mathrm{T}$ leaves $X(S)$ invariant, and induces the diffeomorphism $\alpha : S \to S$. Thus inversion about $\vec{0}$ fixes all the branch points of $g$, mod $T$. In fact inversion about any branch point of $g$ (considered as a point in $\mathrm{R}^3$) must produce the same mapping, mod $T$. This has the immediate consequence of forcing the eight branch points of the Gauss map of $S$ to be located at the eight half-periods of a fundamental domain for $\mathrm{R}^3/T$, a rather rigid situation that must hold for every triply-periodic embedded minimal surface of genus=3.

This analysis of symmetry in the genus = 3 case is an application of Abel's Theorem in complex analysis to minimal surfaces. It is due to Bill Meeks. You can read more about it in [4, 5].

**Remark 3.1** *The result of Ross [7] mentioned in Section 1 seems to indicate that if a TPEMS of genus 3 has its branch values sufficiently near those of the P-surface, then it is stable with respect to volume-preserving variations.*

# 4 An example of symmetry reduction

In the previous section, we have explained triply-periodic embedded minimal surfaces with the intention of showing that any such surface of genus = 3 must have forced symmetries of inversion about its umbilic points. We will now move in the other direction and present a situation in which expected symmetry is lacking. Consider the classical P-surface of Schwarz (see Figure 1,top). This surface is a genus-3, triply-periodic, embedded minimal surface invariant under the integer lattice of translations; its fundamental 3-torus is the cube with opposite sides identified. The P-surface has all the symmetries of reflection of the cube.

It is known that the P-surface can be deformed through a two parameter family of triply periodic minimal surfaces by using the Weierstrass-Enneper Representation. (See, for example [1, 2].) In general these surfaces have fundamental domains that are not cubes. We wish to pose the problem somewhat differently.

**Problem 4.1** *Consider a fundamental domain* $\mathrm{T} = \mathrm{R}^3/T$*. Find a triply periodic embedded minimal surface* $\Sigma$ *in* $\mathrm{R}^3$*, invariant under* $T$*, such that* $S = \Sigma/T$ *has genus = 3 and all the symmetries of* T.

One specific case of this problem is to determine deformations of the P-surface, beginning with the unit cube through rectangular solids of dimension $1 \times 1 \times \lambda$. The symmetry requirement will be to retain all the reflectional symmetries of the rectangular solid. This question was motivated by a computation of Maggs and Leibler, who attempted to solve the problem numerically under the symmetry assumptions of full symmetry. (The solution surface was assumed to have all the symmetries of the rectangular solid.) They did this by a scheme that was a discretization of the problem of minimizing $\int H^2 dA$ for fundamental (simply-connected) patches of the surface. This worked well for $\lambda$ small, but for values of $\lambda$ somewhat greater than 1, it failed. It was unclear whether this was a computation problem, or a real obstruction.

In fact, there is a real obstruction. We will show

**Proposition 4.1** *For $\lambda > 1$ sufficiently large, there is no minimal deformation of the P-surface whose fundamental domain is the rectangular solid $\mathrm{T}_\lambda$ with sides $1, 1, \lambda$, that preserves the symmetries of reflection of $\mathrm{T}_\lambda$.*

To prove this Proposition, we are going to show that any solution must develop a long segment whose boundary is contained in two separated spheres that are far apart. This turns out to be impossible for a minimal surface, as we will now explain.

Consider a connected minimal surface $M$ whose boundary $\partial M$ lies in the union of two unit balls $B_1, B_2$ with centers separated by a distance $L$. Consider a circle of diameter $L$ and vertical translation of it by 1.5. For $L$ large enough, these two circles bound a stable catenoid and a second unstable catenoid $C_L$.

L

Profiles of the stable and unstable
catenoids bounded by the circles
Figure 4

For $L$ large enough $C_L$ will not intersect any solid ball of unit radius centered at any point $(L/2 \cos\theta\,,\ L/2 \sin\theta\,, .75)$. See Figure 5. In particular

The unstable catenoid is disjoint from the balls.

Figure 5

if $\mathcal{M}$ is a connected minimal surface with boundary in the union of the two unit balls centered at $(L/2, 0, 3/4)$ and $(-L/2, 0, 3/4)$, then $\partial M$ is disjoint from $C_L$. Not only that, $\partial M$ is disjoint from any horizontal translate of $C_L$. A sufficiently large horizontal translate of $C_L$ will make it disjoint from $\mathcal{M}$, which is, after all, compact. Now take this translated copy of $C_L$ that is disjoint from $M$ and move it slowly toward $\mathcal{M}$, always moving horizontally, until the first contact with $\mathcal{M}$. At the point, or points, where this happens, $\mathcal{M}$ will lie locally on one side of $C_L$. This is a violation of the maximum principle for minimal surfaces: *Suppose two minimal surfaces $M_1$ and $M_2$ are tangent at a point $p$ interior to both surfaces. Suppose further that they lie (weakly) on opposite sides of the tangent plane in some neighborhood of $p$. Then $M_1 = M_2$.* (You can find a proof of the maximum principle in [3] or [6]. Essentially it comes from the fact that coordinate functions of a minimal surface are harmonic.)

Since $M \neq C_L$, we have arrived at a contradiction. To summarize, we have proved the following Lemma.

**Lemma 4.1** *Suppose $M$ is a connected minimal surface whose boundary lies in the union of two disjoint unit balls. Then the distance between the center of the balls is strictly less than a fixed constant $L$, independent of $M$.*

Since the catenoid is known explicitly, $L$ may be estimated explicitly by letting the distance between the circles (chosen above to be 1.5) decrease to 1.

Now we will apply this to prove Proposition 4.1. Suppose first that the closed curves on the $1 \times \lambda$ faces of the rectangular solid have a *bounded width*, independent of $\lambda > 1$. Because of our symmetry assumptions, all four of these curves are congruent. The closed curve on the square $1 \times 1$ face always lies inside a ball of radius $\frac{\sqrt{2}}{2}$. As is indicated in Figure 7, a long "neck" must develop. This is a connected minimal surface whose boundary lies in the union of spheres of radius $\frac{\sqrt{2}}{2}$. By assumption, their centers are becoming arbitrarily far apart as $\lambda$ increases. This contradicts Lemma 4.1. The only other possibility is that the width of the closed curves on the $1 \times \lambda$ faces grows without bound. But this produces an arbitrarily long neck with boundary in spheres of radius $\frac{\sqrt{2}}{2}$, again a contradiction. This completes our discussion of the proof of the Proposition. We may conclude that if a deformation of the P-surface exists (as an embedded minimal surface) it *must* lose some symmetry. Our argument shows that at the very least one of the vertical planes of symmetry is gone and the closed curves on adjacent $1 \times \lambda$ sides of the rectangular solid must be of differing type (one bounded in width, the other unbounded).

**Acknowledgements.** I wish to thank Bill Meeks for useful conversations concerning genus-three triply-periodic embedded minimal surfaces and Jim Hoffman at GANG, University of Massachusetts, who prepared the illustrations for this paper. At the conference itself, conversations with participants greatly influenced the form and content of this paper. I wish to thank J. P. Bourguignon, S. Leibler, A. Maggs and P. Pansu, in particular.

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