Embedded Minimal Annuli in R^3 Bounded by a Pair of Straight Lines

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1 Introduction

The subject of this paper is embedded minimal annuli bounded by *two* straight lines. The only known examples of such surfaces are given by subdomains of the singly periodic Riemann examples, $\mathcal{R}$. There is a 1-parameter family of these surfaces. A fundamental domain of a Riemann example consists of a minimal annulus bounded by two straight lines, and a copy of that surface produced by Schwarz reflection about one of the boundary lines. (See Figure 1. and the analytic description of these surfaces in Section 2.)

We will prove that the examples of Riemann constitute *all* of the examples, under certain geometric hypotheses.[1]

Theorem 1 *Suppose $L = L_1 \cup L_2$ is a pair of parallel lines and A is an embedded minimal annulus whose boundary is L. Assume further that A lies between two parallel planes with one line in each plane. Then A extends by Schwarz reflection to a Riemann example $\mathcal{R}$.*

In [13], Shiffman proved that a minimal annulus bounded by circles in parallel planes is fibred by circles in parallel planes. The assumptions of Theorem 1 can be viewed as a limiting case of Shiffman's assumptions, but his proof does not extend. As is well known, the Riemann examples are fibred by circles. In fact, Riemann constructed these surfaces by explicitly determining the coordinate functions, in terms of elliptic integrals, of all minimal surfaces fibred by circles in parallel planes. This was published posthumously [10]. Very soon after the publication date of this paper, Enneper published a work [2] in which he proved that a minimal surface fibred by pieces of circular arcs (not necessarily assumed to lie in parallel planes) was in fact a piece of one of the Riemann examples, or a piece of the catenoid. An excellent summary is given in [9], where we learned about the work of Enneper.

[1]Recently Eric Toubiana has been able to strengthen Theorem 1 significantly by showing that the same conclusion holds even if the lines are not assumed to be parallel ("On the minimal surfaces of Riemann", preprint Université de Bourgogne, Dijon, France.)

From the results cited in the above paragraph, it follows, as observed by Shiffman, that a minimal annulus bounded by circles in parallel planes is a part of one of the Riemann examples. As an immediate corollary of Theorem 1 we have

Corollary 1.1 *Suppose $L = L_1 \cup L_2$ is a pair of parallel lines and A is an embedded minimal annulus whose boundary is L. Assume further that A lies between two parallel planes with one line in each plane. Then A is fibred by round circles in parallel planes.*

We wish to mention some recent papers that are related to our work. Meeks and White have studied minimal annuli bounded by convex curves in parallel planes [8]. In [4], Jagy studies minimal hypersurfaces that are foliated by codimension-2 round spheres in R^n. He proves that when $n > 3$, the only possibilities are hypersurfaces of rotation: generalized catenoids. Thus the phenomenon of singly-periodic minimal embedded surfaces fibred by spheres and planes does not exist in dimensions higher than three. Hoffman and Meeks [3] provide a geometric approach to the Riemann examples that uses the results of [12] and [8] to characterize them by determining their Gauss mapping. Karcher [6] provides a geometric approach to the requisite elliptic function theory for the Riemann examples and related minimal surfaces.

2 A characterization of the Riemann examples

The examples of Riemann can be described easily in terms of their Enneper-Weierstrass Representation. On a rectangular torus, $T_\lambda = C/L$, where L is the lattice generated by $\{\lambda, i\}$, for some real $\lambda \geq 1$, consider the elliptic function P with a double pole at 0, a double zero at $\omega_3 = (\lambda + i)/2$ and no other zeros or poles. The Weierstrass P-function $\mathcal{P}$ has the property that $\mathcal{P} - \mathcal{P}(\omega_3)$ has exactly the same poles and zeros. This determines the function up to a multiplicative complex constant. That is:

$$P = c(\mathcal{P} - \mathcal{P}(\omega_3)) \, .$$

It can be easily checked that this elliptic function has the property that $P(\omega_3/2) = i$, precisely when $c = 1$ and that, when c is real, P is real precisely on the lines

$$Re(z) = 0, \; Re(z) = \lambda/2, \; Im(z) = 0 \;\text{ and }\; Im(z) = 1/2 \, .$$

Another way to produce an elliptic function with these properties is to solve the conformal mapping problem:

$$\begin{gathered} 0 \longrightarrow \infty \\ \tfrac{\lambda+i}{2} \longrightarrow 0 \\ \tfrac{\lambda+i}{4} \longrightarrow i \\ Im(P) \geq 0 \end{gathered}$$

Figure 2

and then extend by Schwarz reflection. The Riemann examples are then given by the Weierstrass Data on $T_\lambda - \{0, \frac{\lambda+i}{2}\}$:

$$g = P, \; \eta = i dz/P \, , \tag{2.1}$$

producing via the Weierstrass Representation the multivalued immersion $X(z)$, whose components are:

$$x_1(z) = Re \int_{\omega_1}^{z} (1 - g^2)\eta = Re \int_{\omega_1}^{z} i \, (P^{-1} - P) \, dz$$

$$
\begin{aligned}
x_2(z) &= Re\int_{\omega_1}^{z} i\,(1+g^2)\eta = -\,Re\int_{\omega_1}^{z}(P+P^{-1})\,dz \\
x_3(z) &= Re\int_{\omega_1}^{z} 2g\eta = Re\int_{\omega_1}^{z} 2\,idz = -2\,Im(z)
\end{aligned}
\tag{2.2}
$$

Here the choice of $\omega_1 = \frac{\lambda}{2}$ as the base point for integration is for convenience.

Remark 2.1 *Since (2.1) defines a minimal surface with a* single *period corresponding to the closed curve on T_λ given by $\mu(t) = \frac{\lambda}{4} + ti,\ 0 \le t \le 1,$ as will be shown in Proposition 2.1, it follows that on $\gamma(t) = \frac{i}{4} + t\lambda,\ 0 \le t \le 1,$*

$$
A \overset{def}{=} Re\int_\gamma Pdz = -\,Re\int_\gamma P^{-1}dz\,.
$$

It follows that if c is any nonzero real number, the data $g = cP$, $\eta = \frac{dz}{cP}$ on T_λ has not only a period corresponding to μ, but also

$$
Re\int_\gamma i(1+g^2)\eta = -\,Re\int_\gamma (c^{-1}P^{-1} + cP)dz = (c^{-1}-c)A\,.
$$

However, we may evaluate A by integrating P along $\widehat{\gamma} = \frac{i}{2} + t\lambda,\ 0 \le t \le 1,$ where Pdz is real and never changes sign. Hence $A \neq 0$, and this period is zero if and only if $c = 1$. We will have use of this observation later when we prove the uniqueness of the Riemann examples.

The following Proposition establishes the properties of the Riemann examples. They are all previously known.

Proposition 2.1 *The Weierstrass data (2.1) and the multivalued immersion (2.2) defines a minimal surface $\mathcal{R}$ that has the following*

a) Geometric Properties:

1. *$\mathcal{R}$ is complete and singly periodic, invariant under a translation T;*
2. *$\mathcal{R}$ is fibred by circles in horizontal planes $x_3 = c \neq 2m,\ m \in Z$. These correspond to the closed curves on T_λ given by $Im(z) = constant \neq 0, 1/2;$*

3. *$\mathcal{R} \cap \{x_3 = 2m\}$, $m \in Z$, are straight lines, parallel to the x_2 axis. These correspond to the lines on T_λ given by $Im(z) = 0, 1/2$;*

4. *$\mathcal{R}$ is embedded;*

5. *$\mathcal{R}$ has an infinite number of flat ends, asymptotic to planes at height $x_3 = 2m$, $m \in Z$;*

6. *$\mathcal{R}$ is invariant under reflection in the x_1, x_3-plane. The intersection of this plane with $\mathcal{R}$ consists of planar geodesics and they correspond to the lines on T_λ given by $Re\{z\} = 0,\ \lambda/2$;*

7. *$\mathcal{R}$ is invariant under rotation about horizontal lines that are parallel to the lines $\mathcal{R} \cap \{x_3 = \lambda m\}$, lie at heights $x_3 = (m + 1/2)\lambda$, and meet the surface orthogonally;*

b) *Uniqueness Properties:*

8. *Any minimal surface that is fibred by circular arcs, not necessarily assumed to be in parallel planes or assumed to be closed, is a subset of either the catenoid or some $\mathcal{R}$;*

9. *Any minimal annulus bounded by circles in parallel planes is either a subset of the catenoid or of some $\mathcal{R}$.*

We give proofs of 1–7 that are self-contained and simpler than those in the literature. Statement 8. is a lengthy computation carried out by Enneper in [2] and outlined in the modern text [9]. Statement 9. follows from Statement 8. and Shiffman's Theorem: If a minimal annulus is bounded by circles in parallel planes, it is fibred by circles [13].

Proof. From (2.2), it is clear that X has a period corresponding to the closed curve $\mu = \frac{\lambda}{4} + ti \;\; 0 \leq t \leq 1$. This period has x_3 component equal to -2. We will call this period vector T; $T = (\alpha, \beta, -2)$.

The conformal metric on T_λ induced by the immersion (2.2) has length element

$$|\eta|(1 + |g|^2) = (|P| + |P^{-1}|)|dz| \,. \tag{2.3}$$

Because of the way that P is constructed in Figure 2, it is evident that reflection in the line $Re\{z\} = 0$, (equivalently $Re(z) \equiv \frac{\lambda}{2}$) or in the line $Im \; z = 0$ (equivalently $Im \; z = i/2$) induces an isometry of the induced metric. Since these four lines are fixed in T_λ by one of these reflections, they are geodesics. The second fundamental form of the immersion (2.2) is given by $Re\{fg' \, dz^2\} = Re\{i(P'/P) \, dz^2\}$. Along all four lines in question, P and $(dz)^2$ will be real, while P' is necessarily real along the horizontal pair and purely imaginary along the vertical pair. This means that $\{fg' \, dz^2\}$ is imaginary along the lines $Im\{z\} = 0, \frac{1}{2}$ and real along $Re\{z\} = 0, \frac{\lambda}{2}$. Thus, this second pair of lines is mapped by X to planar geodesic lines of curvature, while the first pair is mapped into straight lines. (See [6] or [5] for details of this sort of argument.) This proves 3. As a consequence of this, $\mathcal{R}$ is invariant under rotation about these lines and reflections through the planes of these geodesics. It is clear from the formula for the third component of X, that the horizontal line $Im\{z\} = c$ is mapped into the horizontal plane $x_3 = -2c \pmod 2$. In fact these curves are *circles*; in particular, they are closed. We will show this is Lemma 2.1 below. For now we assume it to be true; this gives statement 2. This means there are no periods, except perhaps at the punctures 0 and $\frac{1}{2}(\lambda + i)$. But through each puncture passes a line of rotational symmetry and an orthogonal plane of reflectional symmetry. Since the period of X must be orthogonal to both the plane and the line, it has no period at a puncture. Hence T is the only period and $\mathcal{R}$ is singly periodic. We note that the period vector T must reflect into itself through $\{x_2 = 0\}$. That is $T = (\alpha, 0, -2)$. From (2.3) it is clear that $\mathcal{R}$ is complete since $|P|+|P|^{-1}$ has a double pole at either puncture; the length of any curve diverging to 0 or $\frac{\lambda+i}{2}$ must be infinite. This completes the proof of 1.

Note that from 2. and 3., we know that the intersection of $\mathcal{R}$ with any horizontal plane is an embedded curve. In particular, $\mathcal{R}$ is embedded. This proves 4.

We now look at an end corresponding to 0 or $\frac{1}{2}(\lambda+i)$. It is embedded, contains a line, and the Gauss map has order 2 there. Moreover it is period-free. This forces it to be planar. Since the Gauss map is vertical at the end and the end contains a horizontal line at height $x_3 = 2m$, we have proved statement 5.

Because of the fact that the planar curves

$$X(\gamma_c(t)), \qquad \gamma_c(t) = ci + t, \qquad 0 \le t \le 1\,,$$

are closed circles, the planes which contain the images of $\tilde{\mu}(t) = \frac{\lambda}{2} + ti$ and $\hat{\mu}(t) = ti$ must coincide. Since $g = P$ is real on μ this plane of symmetry of $\mathcal{R}$ is vertical and parallel to the plane $\{x_2 = 0\}$. Since we have chosen to integrate from $\omega_1 = \mu(0)$, this vertical plane of symmetry is exactly the coordinate plane $\{x_2 = 0\}$. This proves 6.

To prove 7., we observe that if

$$Q(z) = P(I(z))\,,$$

where I is inversion at the point $\frac{1}{4}(\lambda+i)$,

$$\begin{aligned} Q(0) &= 0 \\ Q(w_3) &= \infty \\ Q\left(\frac{(\lambda+i)}{4}\right) &= i \end{aligned}$$

Checking that Q solves the same mapping problem as $\frac{-1}{P}$ on the rectangle with vertices $0, w_1, w_3, w_2$ shows that $Q = -\frac{1}{P}$. Therefore $(|P|+|P|^{-1})\circ I = |P|+|P|^{-1}$. Since $|I^*dz| = |dz|$, it follows from (2.3) that I is an isometry in the induced metric on T_λ. In fact I is induced by a symmetry of R^3 consisting of rotation by π about a horizontal line orthogonal to the (x_1, x_3)-plane and bisecting the line segment between $X(w_2)$ and $X(w_1)$. This horizontal line meets $\mathcal{R}$ orthogonally at the points $X(\frac{\lambda+i}{4})$ and $X(\frac{\lambda}{4}+\frac{3}{4}i)$. To see this, simply observe that because $I^*dz = -dz$, and since

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -i(P - P^{-1})\, dz \\ -i(P + P^{-1})\, dz \\ i2dz \end{pmatrix}, \tag{2.4}$$

it follows that

$$I^*\Phi = \begin{pmatrix} -\phi_1 \\ +\phi_2 \\ -\phi_3 \end{pmatrix}.$$

That is I^* acts on the integrands Φ in (2.4) by rotation about the x_2-axis. Since I fixes $\frac{\lambda+i}{4}, \frac{\lambda}{4}+\frac{3}{4}i$, it follows that I is induced by rotation about the line, parallel to the x_2-axis, which passes through $X(\frac{\lambda+i}{4})$ and $X(\frac{\lambda}{4}+\frac{3}{4}i)$. This proves 7. □

Lemma 2.1 *The level curves $x_3 = c \neq m$ in the Riemann example are circles.*

Proof. We first derive for the curvature κ of the level lines $x_3 = constant$ on a minimal surface

$$\kappa = Im\left(\frac{g'}{g}\right)(|g| + |g|^{-1})^{-1}, \tag{2.5}$$

where ' is differentiation with respect to a specially adapted conformal coordinate. Then we conclude $\kappa = constant$ from the differential equation of the elliptic function $g = P$. We choose conformal coordinates $z = u + iv$ so that $v = x_3$, up to an additive constant. If g is the stereographic projection of the Gauss map, we must have $2g\eta = 2idz$. Hence the Weierstrass Representation is given by

$$X(z) = \int_{z_0}^{z} \Phi(\zeta),$$

where

$$\begin{aligned}\Phi &= (1-g^2,\, i(1+g^2),\, 2g)\eta \\ &= (g^{-1}-g,\, i(g^{-1}+g),\, 2)idz\,.\end{aligned}$$

In particular, the conformal metric is given by $\lambda = (|g| + |g|^{-1})$. The level curves of x_3 are of the form

$$c(u) = X(u + iv_0)\,.$$

We observe that if $N(u)$ is the normal to S at $c(u)$, the projection of $N(u)$ onto the plane $x_3 = v_0$ is a scalar multiple of the vector $(Re\, g,\, Im\, g)$. Adopting complex notation, we may then write the normal $n(u)$ to the plane curve $c(u)$ as

$$n(u) = \frac{g}{|g|}\,,$$

and the unit tangent vector to this curve as

$$\frac{dc}{ds}(u(s)) = \frac{ig}{|g|}\,,$$

where s denotes arc length on c. We now compute

$$\begin{aligned}\frac{d^2c(u(s))}{ds^2} &= i\left(\frac{g}{|g|}\right)'\frac{du}{ds} = i\left(\frac{g}{|g|}\right)'\lambda^{-1} \\ &= i\left(\frac{g'}{|g|} - \frac{g}{|g|^3}\langle g, g'\rangle\right)\lambda^{-1}\,,\end{aligned} \tag{2.6}$$

where $\langle f, h\rangle = Re\, f\overline{h}$ is the usual inner product, and λ is, as computed above, equal to $|g| + |g|^{-1}$. Noticing that the second term on the right-hand-side is a multiple of $\frac{dc}{ds}$ (or equivalently that $\langle g, ig\rangle = 0$), we may write the curvature of c in the following form.

$$\kappa = \frac{d^2c}{ds^2}\cdot i\frac{dc}{ds} = \frac{\langle ig'g\rangle}{|g|^2}\lambda^{-1} = \langle i\,\frac{g'}{g}, 1\rangle\,\lambda^{-1}\,.$$

The last equality follows from the general equality

$$\frac{\langle z, \omega\rangle}{|\omega|^2} = \langle\frac{z}{\omega}, 1\rangle\,,$$

valid for any complex numbers $z, \omega,\ \omega \neq 0$. Thus indeed

$$\kappa = \left(Im\, \frac{g'}{g} \right) \left(|g| + |g|^{-1} \right)^{-1} .$$

We wish to show that κ is constant on c. We will do this by showing that $\frac{d\kappa}{du} = 0$. We begin by calculating $\frac{d\lambda^{-1}}{du}$.

$$\begin{aligned} \frac{d\lambda^{-1}}{du} = -\lambda^{-2}\frac{d\lambda}{du} &= -\lambda^{2}\left[\frac{\langle g', g\rangle}{|g|} - \frac{\langle g', g\rangle}{|g|^3} \right] \\ &= -\lambda^{-2}(|g| - |g|^{-1})\frac{\langle g', g\rangle}{|g|^2} \\ &= -\lambda^{-2}(|g| - |g|^{-1})\langle \frac{g'}{g}, 1\rangle \end{aligned}$$

Thus

$$\begin{aligned} \frac{d\kappa}{du} &= \langle i(\frac{g'}{g})', 1\rangle\, \lambda^{-1} - \langle i\frac{g'}{g}, 1\rangle\, \lambda^{-2}\langle \frac{g'}{g}, 1\rangle\, (|g| - |g|^{-1}) \\ &= \lambda^{-1}\left[+ Im\left((\frac{g'}{g})' \right) - \lambda^{-1}(|g| - |g|^{-1})\ Re(\frac{g'}{g})\ Im(\frac{g'}{g}) \right] \qquad (2.7) \\ &= (|g| + |g|^{-1})^{-1}\left[+ Im(\frac{g'}{g})' - \left(\frac{|g|^2 - 1}{|g|^2 + 1} \right) \frac{1}{2}\, Im\left((\frac{g'}{g})^2 \right) \right] . \end{aligned}$$

For the Riemann examples $\mathcal{R}$, $g = P$, as defined in (2.1). This elliptic function on a rectangle satisfies the differential equation

$$(P')^2 = -\mu P(P - \alpha)\,(P + \beta)\,,$$

where $\alpha\beta = 1$ and μ is a real positive constant. In fact $+\alpha$ and $-\beta$ are the values of P at the half periods ω_1 and ω_2 (see Figure 2.1, as well as [6], [5]). From this it follows that

$$\left(\frac{P'}{P} \right)^2 = -\mu(P - P^{-1} - \alpha + \beta)\,, \qquad (2.8)$$

and

$$2\left(\frac{P'}{P} \right) \left(\frac{P'}{P} \right)' = -\mu\left(P' + \frac{P'}{P^2} \right) = -\mu\frac{P'}{P}\left(P + P^{-1} \right) ,$$

or

$$\left(\frac{P'}{P}\right)' = \frac{-\mu}{2}\left(P + P^{-1}\right). \tag{2.9}$$

Using (2.7), (2.8) and (2.9), we have

$$\frac{d\kappa}{du} = \mu(|P| + |P|^{-1})^{-1}\left[-\frac{1}{2}\,Im(P + P^{-1}) + \frac{|P|^2 - 1}{|P|^2 + 1}\left(\frac{1}{2}\,Im(P - P^{-1})\right)\right]$$

But notice that in general if $\omega = u + iv$

$$\begin{aligned} Im(\omega - \omega^{-1}) &= v\left(\frac{u^2 + v^2 + 1}{u^2 + v^2}\right), \\ Im(\omega + \omega^{-1}) &= v\left(\frac{u^2 + v^2 - 1}{u^2 + v^2}\right) = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\,Im(\omega - \omega^{-1}) \\ &= \frac{|\omega|^2 - 1}{|\omega|^2 + 1}\,Im(\omega - \omega^{-1}) \end{aligned}$$

Thus $\frac{d\kappa}{du} = 0$ on the Riemann examples. This shows that the planar curves c are round circles, except when $\kappa \equiv 0$ (and the curve c is a line) which happens precisely when P is real along c. From the behavior of P, this happens precisely when $x_3 = 2m$, $m \in \mathrm{Z}$. □

3 The Proof of Theorem 1

We will use the following result, proved in [1]. See also [7].

Theorem 3.1 *Suppose M is a properly embedded minimal surface with an infinite symmetry group and more than one topological end. Then either M is the catenoid or:*

i) M is invariant under a screw motion T;

ii) M/T has finite topology if and only if the total curvature of M/T is

$$2\pi(\chi(M/T) - r),$$

where r is the number of ends of M/T;

iii) All the annular ends of M are flat ends. If M/T has finite topology, all its ends are flat.

Lemma 3.1 *Suppose A is an embedded minimal surface that is bounded by a pair of lines $L = L_0 \cup L_1$ and lies in a slab between parallel planes, $P = P_0 \cup P_1$ with $L_i \subset P_i$. Then A extends by Schwarz Reflection to a singly-periodic embedded minimal surface $\mathcal{S}$, invariant under a screw motion T, where T is $R_1 \circ R_0$, R_i being rotation by π about L_i. If A has genus k, S/T has genus $2k+1$, two flat ends and total curvature $-4\pi(2k+2)$. Furthermore, T is a pure translation if and only if L_0 is parallel to L_1, and in that case the translation vector lies in the plane containing $L_0 \cup L_1$ and is orthogonal to these lines. If T is a translation, the Gauss map of S descends to a well-defined meromorphic function on $\overline{S/T}$.*

Proof. The hypothesis that the surface lies between two parallel planes and is embedded and minimal means that it extends by Schwarz Reflection about the lines to a complete embedded minimal surface. Let T be the symmetry of the surface produced by the composition of rotations about the two line boundaries of A. T must be the composition of a nontrivial translation, transverse to the planes of the slab containing A, and a (possibly trivial) rotation. Thus A extends to a singly-periodic surface $\mathcal{N}$ that is, modulo T, a twice-punctured genus=$(2k+1)$ surface. Two copies of A having a line in common form a fundamental domain of $\mathcal{N}/T$. The assumption that A is embedded in a slab forces the singly-periodic surface to be embedded. By Theorem 3.1 iii) the two ends of $\mathcal{N}/T$ are flat. By Theorem 3.1 ii) $\mathcal{N}/T$ has finite total curvature equal to $-4k(2k+2)$.

If in addition the lines L are assumed to be parallel, the symmetry T, which is generated by successive rotations about the two distinct lines in the quotient, is a pure translation. (This translation will be orthogonal to the boundary planes of the slab if and only if the lines L both lie in a plane orthogonal to the slab.) In particular, there is no rotational component to

T. This implies that the Gauss map of $\mathcal{N}$ descends to $\mathcal{N}/T$. Since $\mathcal{N}/T$ has finite total curvature and is complete in R^3/T, the Gauss map of $\mathcal{N}/T$ extends to the compactified surface. $\square$

Remark 3.1 *Since $\mathcal{N}$ is embedded, we may assume without loss of generality that the Gauss map of $\mathcal{N}$ is vertical at the two ends. Denote by g the Gauss map of $\overline{\mathcal{N}}/T$. Then $\mathcal{N}$ may be represented as a multivalued conformal immersion of $\mathcal{N}/T$ by using the Enneper-Weierstrass representation with the data g, η, where $\eta = dx_3/2g$. Since the two ends are flat, η must have a double pole at each one. But $\mathcal{N}$ is constructed in a manner that insures that there is a single line diverging into each end, which in this normalization must be horizontal.*

Proof of Theorem 1.

Step 1. By Lemma 3.1, the annulus A extends by Schwarz reflection to a singly-periodic minimal surface S, invariant under a translation T, and S/T a torus with two flat ends and total curvature -8π. Also, the Gauss map of S descends to S/T, and extends to $\overline{S}/T$. Thus g, the stereographic projection of the Gauss map is a degree-two elliptic function on this torus. By Remark 3.1 g may be assumed to have a pole of order two at one end, say $p_1 \in L_1$ and a zero of order two at the other end, say $p_0 \subset L_0$. Because the degree of g is two, it has no other zeros or poles. The lines $L = L_0 \cup L_1$ are horizontal and we may assume without loss of generality that they are parallel to the x_1-axis. This forces g to be real on L. Since g is real along L and has degree equal to two on $\overline{S}/T$, it follows that there is a single simple branch point of g on each of the lines. We will label the branch point of the Gauss map on L_i by $b_i, i = 0, 1$.

Step 2. We will prove that S is a Riemann example by determining its Weierstrass Representation. In this step we will determine $\overline{S}/T$ and the one-form η.

First, we determine the underlying conformal structure of $\overline{S/T}$. The two lines $L = L_0 \cup L_1$ correspond to disjoint closed curves on this torus. Rotation about one of the lines is an order-two isometry, whose fixed-point set is L. Consider this isometry as a conformal involution on the torus. Only rectangular or rhombic tori possess conformal involutions which fix a curve (namely reflections for the flat metric). However a rhombic torus cannot have two such curves in the same homotopy class as is the case for L_0 and L_1. Hence $\overline{S/T}$ is a rectangular torus.

Without loss of generality, we may assume that this torus is T_λ, that is C modulo the lattice determined by $\{i, \lambda\}$, for some real λ, and that the aforementioned involution is induced by complex conjugation. Hence L corresponds to the set $Im\{z\} = 0, \frac{1}{2}$, modulo λ, and we will label $Im\, z = 0$ as L_0, and $Im\, z = \frac{1}{2}$ as L_1. Let $z = u_1 + iu_2$ be the complex parameter on C.

Since the lines L are horizontal and the height function x_3 is harmonic on S, it defines a function on C which is a real multiple of u_2 (up to an additive real constant). Hence $dx_3 = idz$ on C, up to a multiplicative real constant, which by a homothety of R^3 we may assume to be equal to 1. Since $dx_3 = 2g\eta$, where η is the one-form in the Weierstrass representation, we have

$$\eta = \frac{idz}{2g} \tag{3.1}$$

on S and $\overline{S}/T$.

Step 3. It remains to determine the Gauss map g. Recall from Step 2 that we know g is a degree-two elliptic function on the rectangular torus T_λ, which has a double pole at $p_0 = 0$, a finite branch point on $Im\, z = 0$ and a double zero and a finite branch point on $Im\, z = \frac{1}{2}$. We now observe that the branch points of a degree-two elliptic function on a rectangular torus, which has a double pole, must coincide with the branch points of any other such function, up to translation.[2] Taking the function P defined in Figure 2 as a model, we may conclude that the branch points of g, namely p_0, p_1, b_0, b_1 are distributed in one of the following two ways:

[2]Here is a simple proof. Let h be an elliptic function of degree-2 on T_λ. After a translation in $\mathbb{C}$ and composition with a fractional linear transformation on $\mathbb{C} \cup \{\infty\}$ we may assume that h has a double pole at 0, as does the P-function defined in Figure 2 of Section 2. Suppose h is *not* branched at $\omega_3 = \frac{\lambda+i}{2}$, where P has a double zero. Then $\frac{h-h(\omega_3)}{P}$ has a simple pole at ω_3 and no other poles, a contradiction. Hence h must be branched at ω_3. Similar arguments show h is branched at $\omega_1 = \frac{\lambda}{2}$ and $\omega_2 = \frac{i}{2}$, the other branch points of P, and nowhere else. If one only "knows" elliptic functions from their Riemann mapping definition then one needs such an argument. The standard Mittag-Leffler expansions are even with respect to the pole and therefore also give the result quickly.

Figure 3

In Case 1, note that g has exactly the same zeros and poles as P, constructed in Figure 2. Hence $g = cP$ for some nonzero complex constant c. However both g and P are real on $Im\, z = 0$. Hence

$$g = cP \qquad c \text{ real} . \tag{3.2}$$

Thus in Case 1, (3.1) and (3.2) give the Weierstrass data for S on the rectangular torus T_λ. However, in Remark 2.1, we noted that the Weierstrass representation will always have nonzero period in the x_2-coordinate on a horizontal generator of T_λ unless $c = 1$. Since this must be a closed curve or a line parallel to the x_1-axis, we conclude that $c = 1$. Hence

$$g = P,\ \eta = \frac{idz}{P} \ \text{ on } T_\lambda \tag{3.3}$$

are the Weierstrass data in Case 1. This case is exactly the Riemann example given in (2.1) and (2.2).

Remark 3.2 *The fact that $\overline{S}/T$ is a rectangular torus T_λ on which the branch points of g are essentially determined can be proved by working more directly with the minimal surface. Consider the sum of $S/T \subset \mathrm{R}^3/T$ with itself in the sense of* minimal herissons, *according to [11]. At each $\vec{n} \in S^2$ we define $\mathcal{H}(\vec{n})$ to be the sum of all $q \in S/T$ where $G(q) = \vec{n}$, G being the Gauss map. Since S/T has flat ends, it follows from [11] that $\mathcal{H}$ is* constant *on S^2. However, we may arrange things so that $X(b_0)/T = \vec{0} \in S/T$ and b_0 is the* *only point $q \in \overline{S/T}$ where $G(q) = G(b_0)$. Hence $\mathcal{H} \equiv \vec{0}$. Moreover, since G has degree two, inversion about $\vec{0}$ in R^3/T ($X \to -X$ mod T) must be a symmetry of S/T. Hence inversion about $\vec{0}$ is a symmetry of S. Therefore inversion about $\vec{0}$ followed by rotation about $L_0 (= x_1$-axis) is a symmetry of S. This symmetry is precisely reflection in the vertical plane $\{x_1 = 0\}$.*

The symmetry lines L and $\{x_1 = 0\} \cap \overline{S/T}$ divide $\overline{S/T}$ into four rectangles bounded by geodesics, each one congruent to any other one by a rotation, reflection or a composition of the two. Therefore, $\overline{S/T}$ is a rectangular torus. The symmetries force the finite branch points to be located as in Case 1 or 2 above.

Step 4. It remains to show that Case 2 cannot occur. We will do this by determining g explicitly, and then showing that the period problem is not solvable. Recall that in this case g has a double pole at 0 a double zero at $\frac{i}{2}$ and branch points at $\frac{1}{2}$ and $\frac{\lambda+i}{2}$. (See Figure 3.) Let Q be the elliptic function defined by first solving the Riemann mapping problem:

$$\frac{i}{2} \longrightarrow 0$$
$$\frac{\lambda}{2} + \frac{3i}{4} \longrightarrow 1$$
$$\frac{3i}{4} \longrightarrow -1$$

The construction of Q

Figure 4

and then extending to C by reflection. The mapping Q is an elliptic function of degree 2, defined on the torus T_λ. It has a double pole at 0 and a double zero at $\frac{i}{2}$. Noting that $Q(\omega_3)$ is some real number (depending on λ) between 0 and 1, we may write $Q(\omega_3) = \cos\alpha$, for some α, $0 < \alpha < \frac{\pi}{2}$. This implies that $Q(\frac{\lambda}{2} + i) = \sec\alpha$.

The zeros, poles and branch values of Q along with the horizontal generator γ.

Figure 5.

With this information it is straightforward to see that the zeros and poles of $(Q'/Q)^2$ and $Q + \frac{1}{Q} - (\cos\alpha + \sec\alpha)$ coincide. Since both functions are real and positive on the line segment $\frac{i}{2} + t$, it follows that

$$\left(\frac{Q'}{Q}\right)^2 = c^2(Q + \frac{1}{Q} - (\cos\alpha + \sec\alpha))\,, \tag{3.4}$$

for some positive $c \in \mathrm{R}$.

Note that Q is also real on the lines $Re\{z\} = 0, \frac{\lambda}{2}$, and has the same zeros and poles as g. It follows immediately that the Gauss map g is a real multiple of Q. That is:

$$g = AQ, \qquad A \neq 0,\, A \in \mathrm{R}. \tag{3.5}$$

As observed in Remark 3.2, S must have a vertical plane of symmetry that passes through the finite branch points of g. This implies that the lines $Re\, z = 0, \frac{\lambda}{2}$ correspond to planar lines of curvature in vertical planes. Since Q is real on these lines, the vertical plane(s) of symmetry must be parallel to the (x_1, x_3)-plane. *(Of course, we could have deduced this directly from the properties of Q. The metric is $(A|Q| + A^{-1}|Q|^{-1})|dz|^2$, which is invariant under reflection in the lines $Re\, z = 0, \frac{\lambda}{2}$. Hence these lines are geodesics. The second fundamental form is given by $\frac{1}{2}\, Re\{g' f\, dz^2\} = \frac{1}{4}\, Re\{iQ'/Q\, dz^2\}$. On these lines $dz = idv$, Q is real and Q' is imaginary. Hence $\frac{iQ'}{Q} dz^2$ is real, which means that this geodesic is a line of curvature in a vertical plane parallel to the (x_1, x_3) plane.)*

We will now show that along the horizontal curves $Im\, z =$ constant there is always a nonzero period, which means that the Weierstrass data (3.1) (3.5) on T_λ can never produce the required example. Because these curves lie in horizontal planes and are symmetric with respect to the (x_1, x_3)-plane, the only possible period is in the x_2-direction. We will choose our generator to be $\gamma(t) = \frac{3}{4}i + t$, $0 \leq t = \lambda$. The period condition is then

$$Re \int_\gamma i(1 + g^2) f\, dz = -\, Re \int_\gamma (g^{-1} + g)\, dz = 0\,, \tag{3.6}$$

where $\gamma(t) = \frac{3}{4}\, i + t,\ 0 \leq t \leq \lambda$. Using (3.5) we have that condition (3.6) is equivalent to

$$A^{-1}\, Re \int_\gamma \frac{dz}{Q} = -A\, Re \int_\gamma Q\, dz\,. \tag{3.7}$$

But along γ, dz is real and moreover Q is unitary, so $Q^{-1} = \overline{Q}$. Hence (3.6) is equivalent to

$$Re \int_\gamma Q\, dz = 0\,. \tag{3.8}$$

In other words, the parameter A is irrelevant to the closing of the period on γ with the Weierstrass data (3.1) (3.5) on T_λ; either no example exists in Case 2, or there is a one-parameter family of examples for each rectangular torus. We will now show that no example exists. To do this, we will show that (3.8) is false.

By construction, $Q \circ \gamma(t) = Q(\frac{3i}{4} + t)$ is a one-to-one and onto map from the closed curve represented by γ onto the equator $|z| = 1$. Therefore we may write $Q = e^{i\phi}$, $\phi = \phi(t)$. Along γ, we have $\frac{dQ}{Q} = id\phi$ and

$$dz = \frac{dQ}{Q'} = \frac{Q}{Q'}\,\frac{dQ}{Q} = \frac{iQ}{Q'}\,d\phi\,. \tag{3.9}$$

Also note that along $\gamma, Q + Q^{-1} = Q + \overline{Q} = 2\cos\phi$. Hence the right-hand side of (3.4) is real and negative. This means that Q'/Q is purely imaginary on γ; in fact when $Q = e^{i\phi}$,

$$\frac{Q'}{Q} = \pm\, i\, c\sqrt{(\cos\alpha + \sec\alpha) - 2\cos\phi}\,. \tag{3.10}$$

Hence,

$$\begin{aligned} Re\int_\gamma Q\,dz &= \pm\, Re\int_0^{2\pi} e^{i\phi}[c^{-1}(\cos\,\alpha + \sec\alpha - 2\cos\,\phi)^{-1/2}]\,d\phi \qquad (3.11)\\ &= \pm\, c^{-1}\int_0^{2\pi} \frac{\cos\,\phi\,d\phi}{((\cos\,\alpha + \sec\,\alpha) - 2\cos\,\phi)^{1/2}}\,. \end{aligned}$$

The integral in (3.11) is clearly not zero for the following elementary reason. Note that $\displaystyle\int_0^{2\pi} \cos\,\phi\,d\,\phi = 0$. However, when $\cos\,\phi$ is positive the denominator in the integrand of (3.11) is *smaller* than $\cos\,\alpha + \sec\,\alpha$ while it is bigger than $\cos\,\alpha + \sec\,\alpha$ when $\cos\,\phi$ is negative. Hence,

$$\int_0^{2\pi} \frac{\cos\,\phi\,d\,\phi}{(\cos\,\alpha + \sec\,\alpha - 2\cos\,\phi)^{1/2}} > 0\,.$$

Thus condition (3.8) is not satisfied. Since this condition was necessary for the existence of an example with data given in Case 2, we have shown that Case 2 is impossible. $\square$

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