Boundary Value Problems for Surfaces of Constant Gauss Curvature

David Hoffman*
Department of Mathematics
University of Massachusetts
Amherst, MA 01003

Harold Rosenberg
Department of Mathematics
Université de Paris 7,
75251 Paris, France

Joel Spruck*
Department of Mathematics
University of Massachusetts
Amherst, MA 01003

1 Introduction

The compact smooth surfaces in R^3 with constant positive Gauss Curvature (*K-surfaces*) form a natural class. A K-surface without boundary is itself the boundary of a convex body, so it must be embedded. The surfaces of interest to us have non-empty boundary and so are not necessarily embedded. A fundamental question is this: given a collection $\gamma = \{C_1 \ldots, C_n\}$ of Jordan curves in R^3, what are the K-surfaces with boundary γ? For example, if γ is a single Jordan curve with no inflection points, does γ bound a K-surface?

When γ is a single *planar* Jordan curve, the simplest case, a great deal can be said. We begin our discussion of this case by recalling several elementary facts. Let P be a plane and $\mathcal{S}$ a smooth surface in R^3. Let γ be a component of $\mathcal{S} \cap P$ that is not a point. The normal curvature (up to sign) of $\mathcal{S} \cap P$ is the projection of the curvature vector of $\mathcal{S} \cap P$, considered as a curve in P, onto the normal line of $\mathcal{S}$ at P. This means that an inflection point of $\gamma \subset P \cap \mathcal{S}$ will have an asymptotic direction

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on $\mathcal{S}$. Also, if $\mathcal{S}$ is tangent to P at a point $p \in \gamma$, then $\mathcal{S}$ has an asymptotic direction at p (in the direction of γ).

We conclude that if a closed curve γ in a plane P bounds a K-surface $\mathcal{S}$, then γ is free of inflection points - i.e., γ is *convex* - and P meets $\mathcal{S}$ transversally. Let G be the Gauss map of $\mathcal{S}$. Since γ is convex, and $\mathcal{S}$ meets P transversally $G(\gamma)$ is a closed curve that meets each longitude exactly once and never passes through a pole. (We can and do assume that P is the (x_1, x_2) plane.) In particular, $G(\gamma)$ is an embedded curve and G is injective in a neighborhood of γ, since G is regular. By the Gauss-Bonnet formula and the fact that γ is a convex Jordan curve,

$$0 < \int_{\mathcal{S}} K dA = \int_{\gamma} k_g ds + 2\pi\,\chi(\mathcal{S}) < 2\pi[1 + \chi(\mathcal{S})]$$

The strict inequality follows from the fact that the total curvature of γ in $P \subset \mathrm{R}^3$ is 2π and its geodesic curvature must be everywhere less than its curvature, since $\mathcal{S}$ is transversal to P. It follows that $\chi(\mathcal{S}) > -1$. Since $\chi(\mathcal{S}) = 1 - 2$ genus$(\mathcal{S})$, we conclude that genus $(\mathcal{S}) = 0$, i.e. $\mathcal{S}$ is a disk, and from the estimate above, $\int_{\mathcal{S}} K dA < 4\pi$. This in turn implies that the Gaussian image of $\mathcal{S}$ is not the entire sphere. Since the Gauss map G is open and G is one-to-one on $\gamma = \partial\mathcal{S}$, we can conclude that G is a diffeomorphism of $\mathcal{S}$ onto one of the components of $S^2 - G(\gamma)$. In particular, G assumes one polar value exactly once, and the other not at all. From this we conclude that the height function x_3 has exactly one critical point, located in $\operatorname{int}\mathcal{S}$ which implies that $\mathcal{S}$ lies in one of the halfspaces determined by P. Moreover $\mathcal{S}$ must be embedded. To see this, notice that near γ, the level curves $\mathcal{S} \cap \{x_3 = c\}$ must be strictly convex Jordan curves. (Without loss of generality, we may assume that the condition that $\mathcal{S} \cap \{x_3 \geq 0\}$ and $c > 0$). The condition that $\mathcal{S} \subset \{x_3 = c\}$ is a strictly convex Jordan curve is an open condition. But since $G|\mathcal{S}$ is injective and takes on only one polar value, $\mathcal{S} \cap \{x_3 = c\}$ cannot grow new components and, as before, $\{x_3 = c\} \cap \mathcal{S}$ is strictly convex. Hence $\mathcal{S}$ is fibred by the strictly convex Jordan curves $\mathcal{S} \cap \{x_3 = c\}$, $0 \leq c < x_3(G^{-1}(0, 0, \pm 1))$. In particular $\mathcal{S}$ is embedded.

If $\mathcal{S}$ is an embedded K-surface whose boundary is a plane curve γ, one can apply the Alexandrov Reflection Principle, using vertical planes, to prove that $\mathcal{S}$ inherits the symmetries of γ. In particular, if γ is a circle, $\mathcal{S}$ must be a spherical cap. It follows immediately in this simple case that if γ is a circle of radius ρ: it bounds no K-surface with curvature $K > 1/\rho^2$; to find a K-surface with a given boundary, it is sometimes necessary to make K small. If $K < \rho^{-2}, \gamma$ bounds a large spherical cap that is not a graph over the disk bounded by γ, as well as a smaller one that is a graph. In general K-surfaces are not unique.

The above discussion addresses the problem of uniqueness and the properties of K-surfaces bounded by a plane curve γ. The existence problem has been solved by Caffarelli, Nirenberg and Spruck [1] for somewhat more general boundaries in the nonparametric setting. They proved that if γ is smooth and projects injectively onto the boundary of a smooth strictly convex planar domain Ω, then γ bounds a K-surface that is a graph over Ω, for K sufficiently small. This K-graph is unique up to reflection in the plane of Ω. We conjecture that given a strictly convex plane curve γ and K sufficiently small, γ bounds exactly two distinct K surfaces, up to multiplicity. (A K-surface $\mathcal{S}$ with boundary γ has multiplicity 2 if any sufficiently small perturbation of γ bounds either 2 or 0 K-surfaces near $\mathcal{S}$. A K-surface $\mathcal{S}$ has multiplicity 1 if any sufficiently small perturbation of γ bounds a single K-surface near $\mathcal{S}$.) In fact the theorem holds in all dimensions and for more general equations of Monge-Ampere type.

Another basic question concerns K-surfaces whose boundary γ consists of the union of two closed convex curves C_1 and C_2 contained in parallel planes P_1 and P_2, respectively. As in the case of a single planar convex curve, it is not difficult to derive some basic properties of solutions. Let $\mathcal{S}$ be such a K-surface. As before, $\mathcal{S}$ must meet each plane P_i transversally, and therefore $\mathcal{S}$ lies locally on one side of each P_i. The intersection of $\mathcal{S}$ with planes sufficiently close to and parallel to the planes P_i must intersect $\mathcal{S}$ transversally in a single convex curve. The Gauss mapping G of $\mathcal{S}$ is open and it is easy to argue that G is a diffeomorphism of $\mathcal{S}$ onto an annular region of S^2 that contains the equator. This implies that $\mathcal{S}$ is an embedded annulus, fibred by convex curves in the slab between P_1 and P_2. Again, Alexandrov Reflection using planes othogonal to the P_i shows that $\mathcal{S}$ inherits the symmetries of $\gamma = \{C_1, C_2\}$.

The existence problem is much more delicate and we will not settle it completely. In this paper we will treat the case when the K-surface bounding γ can be represented as a graph.

Theorem 1.1. *Let Ω be an annulus in R^2 bounded by two strictly smooth convex curves Γ_1, Γ_2. Suppose $\phi = \{\phi_1, \phi_2\}$ are smooth functions on Γ_1, Γ_2 and that there exists a smooth function F on Ω, that agrees with ϕ_i on Γ_i and is strictly convex, i.e., the graph of F has positive Gauss curvature. Then there exists a K-surface that is* *a graph over Ω with the given boundary values ϕ. Moreover, K may be chosen to be any positve constant less than the minimum value of the Gauss curvature of the graph of F.*

A simple case in which we can always construct the strictly convex function F needed in Theorem 1.1 is the following.

Corollary 1.2. *Let C_1 and C_2 be two closed strictly convex curves contained in parallel planes. Assume that C_1 and C_2 orthogonally project onto convex curves Γ_1, Γ_2, respectively, where $\Gamma = \{\Gamma_1, \Gamma_2\}$ is the boundary of an annulus Ω contained in a plane P, which strictly separates the given curves. Then there is a smooth K-surface that is a graph over Ω with boundary $\gamma = \{C_1, C_2\}$.*

Note that in this case the boundary data ϕ of Theorem 1.1 consists of linear functions and the construction of F is fairly straightforward.

We will in fact derive Theorem 1.1 as a corollary of a general theorem about Monge-Ampere equations, in the spirit of [1], in annular domains Ω with strictly convex boundaries.

We seek a strictly convex solution of

$$det(u_{ij}) = \psi(x, \nabla u) \qquad \text{in} \qquad \Omega, \tag{1.1}$$

$$u = \phi \in C^\infty \qquad \text{on} \qquad \Gamma = \partial\Omega, \tag{1.2}$$

where $\psi^{1/n}(x,p)$ is a positive C^∞ function for $x \in \bar{\Omega}$, $p \in R^n$, which is *convex* in p. Here Ω is an annulus in R^n with strictly convex smooth boundaries $\Gamma = \{\Gamma_1, \Gamma_2\}$.

We assume there exists a strictly convex subsolution $\underline{u} \in C^\infty(\bar{\Omega})$, which equals ϕ on Γ and satisfies

$$det(\underline{u}_{ij}) \geq \psi(x, \nabla \underline{u}) + \delta_0 \quad \text{in} \quad \Omega \tag{1.3}$$

Theorem 1.3. *Under condition (1.3), there exists a strictly convex solution $u \in C^\infty(\bar{\Omega})$ of (1.1),(1.2) with $u \geq \underline{u}$. This solution is unique.*

Equations of the Monge-Ampere type have been extensively studied in [1] in strictly convex domains. The novelty here lies in the fact that Ω is an annulus. For the application to Theorem 1.1 (in fact the n dimensional version of Theorem 1.1) we take

$$\psi(x,p) \equiv K(1+p^2)^{\frac{n+2}{2}}.$$

Then $\psi^{1/n}(\cdot, p)$ is easily seen to be convex in p. Also, with F and K as in Theorem 1.1, we may choose $\underline{u} = F$ and condition (1.3) is satisfied for small enough δ_0.

Remark 1. We note that for arbitrary "convex annuli" Ω and data ϕ, it is not always possible to find a K-surface with these boundary values. For example, if ϕ is identically zero on both boundaries, then there is no surface with strictly positive (or strictly negative) curvature with these boundary values. There are other obstructions. We will now present one of them, based on an observation of C. Bonnatti. Let Ω be a convex annulus and Γ_2 its inner boundary. The boundary data on $\partial\Omega = \Gamma = \{\Gamma_1, \Gamma_2\}$ given by $\phi = \{\phi_1, \phi_2\}$ defines two curves γ_i as graphs over $\Gamma_i, i = 1, 2$. Denote by Z_i the vertical cylinder over Γ_i. We may parametrize Γ_2 by the angle θ of its outward normal vector, measured from a fixed direction, and parametrize Γ_1 by letting $\Gamma_1(\theta)$ equal the intersection of Γ_1 and the outward pointing normal by $\Gamma_2(\theta) + t\vec{n}(\theta), t \geq 0$, where $n(\theta)$ is the unit-vector in the θ-direction.

Let $w(\theta)$ be the curvature vector of γ_2 at $\gamma_2(\theta)$. We observe that since γ_2 is a graph over a strictly convex plane curve Γ_2, the component of $w(\theta)$ in the direction $n(\theta)$ is always negative. In particular w points into the cylinder Z_2. Since F is convex, the graph of F lies above the tangent plane to graph F at $\gamma_2(\theta)$ in the simply connected convex region $\Omega_\theta \subset \Omega$ defined by the tangent line to Γ_2 at $\Gamma_2(\theta)$. Since the graph of F is positively curved at $\gamma_2(\theta), w(\theta)$ must point into the the halfspace above the tangent plane to graph F at $\gamma_2(\theta)$. Hence the ray $\gamma_2(\theta) - tw(\theta), t > 0$, strikes the cylinder Z_1 at some point $(\Gamma_1(y(\theta)), h(\theta))$. This relationship defines $h(\theta)$ and forces the inequality

$$h(\theta) < \phi_1(y(\theta)) \,,$$

valid for all θ, $0 \leq \theta \leq 2\pi$. Now in case $\phi_2'(\widehat{\theta}) = 0$, i.e., $\gamma_2'(\widehat{\theta})$ is horizontal, $w(\widehat{\theta})$ must lie in the vertical plane containg $n(\widehat{\theta})$, which means that the ray $\gamma_2(\widehat{\theta}) - tw(\widehat{\theta})$ strikes Z_1 at a point above $\Gamma_1(\widehat{\theta})$, i.e., $y(\widehat{\theta}) = \widehat{\theta}$. From the above inequality, we have

$$h(\widehat{\theta}) < \phi_1(\widehat{\theta}) \,, \tag{1.4}$$

valid whenever $\gamma_2'(\widehat{\theta})$ is horizontal. In particular, if ϕ_2 has a local maximum at $\overline{\theta}$, then $w(\overline{\theta})$ must have a nonpositive vertical component, which means that the $h(\overline{\theta}) > \phi_2(\overline{\theta})$. Combining this with (1.4) we have

$$\phi_2(\overline{\theta}) < \phi_1(\overline{\theta}) \,, \tag{1.5}$$

valid at all local maxima of ϕ_2. In particular

$$\max_{\Gamma_2} \phi_2 = \phi_2(\overline{\theta}) \leq \max_{\Gamma_1} \phi_1 \,. \tag{1.6}$$

Conditions (1.4)–(1.6) constitute obstructions to finding a strictly convex function on Ω with boundary values ϕ. Obstructions to the existence of a strictly concave function follow immediately by applying the previous arguments to $-\phi = \{-\phi_1, -\phi_2\}$, yielding reversed inequalities for (1.4) and (1.5) and

$$\min_{\Gamma_2} \phi_2 = \phi_2(\underline{\theta}) \geq \min_{\Gamma_1} \phi_1 \,. \tag{1.7}$$

Figure 1

Remark 2. One would like to know necessary and sufficient conditions on $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ in order that γ be the boundary of a K-surface, or even a positively curved surface. It is tempting to conjecture that such a condition might be that γ is extreme, that is γ is contained in the boundary of its convex hull.. However there is a simple example that shows that this condition is too strong. Begin with two great circles on the unit sphere that meet orthogonally. Fatten each one a bit to produce two strips that overlap in two small regions near the intersection points. Next, join the strips together near one of the points producing a connected surface of positive curvature which is topologically a torus minus a disk, with a self intersecting region near the other intersection point. One can now perturb this boundary slightly by pulling in one of the strips near the intersection, to produce a smooth Jordan curve γ, which bounds a surface whose curvature is almost exactly equal to 1, but is *not* extreme. It is probably the case that this new contour can be chosen to bound a K-surface.

The remainder of the paper contains the technical proof of Theorem 1.3 and is organized as follows. Section 2 sets up the continuity method and reduces the proof of the theorem to the derivation of suitable a priori estimates up to second derivatives as in [1]. These estimates are obtained in §3. In future work we hope to return to the existence question for convex closed curves contained in parallel planes.

2 The continuity method

Let $\underline{u}$ be a strictly convex, strict subsolution of (1.1),(1.2). That is,

$$\begin{aligned} \det \underline{u}_{ij} &\geq \psi(x, \nabla \underline{u}) + \delta_0 \quad \text{in} \quad \Omega \\ \underline{u} &= \phi \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{2.1}$$

Since ψ is strictly positive we can find (by the Implicit Function Theorem) a strictly convex function $u^0 \in C^\infty(\overline{\Omega})$ with $u^0 = \phi$ on $\partial\Omega$ and satisfying

$$\det \underline{u}_{ij} \geq \det u^0_{ij} + \varepsilon_0\,. \tag{2.2}$$

for some $0 < \varepsilon_0 < \delta_0$. By the maximum principle we also have

$$\underline{u} < u^0 \text{ in } \Omega$$

Set for $0 \leq t \leq 1$

$$\psi^t(x, \nabla u) = t\psi(x, \nabla u) + (1-t) \det u^0_{ij}\,.$$

For each t in $0 \leq t \leq 1$ we wish to find a strictly convex solution u^t in $C^{2+\alpha}(\overline{\Omega})$ of

$$\begin{aligned} \det u^t_{ij} &= \psi^t(x, \nabla u^t) \quad \text{in} \quad \Omega \\ u^t &= \phi \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{2.3}$$

For $t = 0$ we have u^0 as the solution. Note that solutions of (2.3) are unique by the maximum principle since ψ^t does not have any u dependence, and is still convex in p. Using the Implicit Function Theorem and classical Schauder theory one finds that the set of t for which (2.3) is solvable is open. If one can establish the *apriori* estimate

$$|u^t|_{2+\alpha} \leq K$$

independent of t it follows that the set such t is also closed, and hence is the whole unit interval. The function u^1 is then our desired solution of (1.1),(1.2).

It is now well understood, through the work of Evans, Krylov, Trudinger, Caffarelli-Nirenberg-Spruck (see [1, 2] for references) how to derive the estimates for $|u^t|_{2+\alpha}$ once we have derived the apriori estimate

$$|u^t|_2 \leq K \text{ independent of } t\,. \tag{2.4}$$

Thus in the following section we shall derive the estimate (2.4) to complete the proof of Theorem 1.3.

For later reference, we observe that by our construction

$$\det \underline{u}_{ij} = t \det \underline{u}_{ij} + (1-t) \det \underline{u}_{ij}$$

$$\geq t(\psi(x, \nabla \underline{u}) + \delta_0) + (1-t)\,(\det u^0_{ij} + \varepsilon_0)$$

by (2.1) and (2.2). Therefore

$$\det \underline{u}_{ij} \geq \psi^t(x, \nabla \underline{u}) + \varepsilon_0 \qquad \text{in } \overline{\Omega}. \tag{2.5}$$

3 Apriori estimates for derivatives up to second order

To establish the estimates (2.4) we shall consider only the case $t = 1$ since the same arguments hold for any $0 \leq t \leq 1$. We shall follow the arguments of [1, § 7] as closely as possible emphasizing only the new arguments needed to obtain the estimates at the inner convex boundary Γ_2.

Since u is convex, $u \leq \max_{\Gamma_1} \phi$. Furthermore, $u \geq \underline{u}$ by the maximum principle. Thus

$$|u| \leq K_1 \,. \tag{3.1}$$

Again by the convexity of u, $|\nabla u|$ achieves its maximum on $\partial\Omega$, so it suffices to estimate $|u_\nu|$ on $\partial\Omega$ (ν is the exterior unit normal) since the tangential derivatives of u are known on $\partial\Omega$. Since $u \geq \underline{u}$ in Ω and $u = \underline{u} = \phi$ on $\partial\Omega$, we have

$$u_\nu \leq \underline{u}_\nu \,. \tag{3.2}$$

To estimate u_ν from below, choose any point on $\partial\Omega$. We may suppose it is the origin with the x_n axis pointing in the interior normal direction. Let $y \in \partial\Omega$ be the unique point on $\partial\Omega$ where the positive x_n axis leaves Ω *transversally*. By (3.2) and the convexity of u, we have

$$-\, u_\nu(0) = u_n(0) \leq u_n(y) \leq 2|\nabla \underline{u}(y)| \,. \tag{3.3}$$

Thus (3.2), (3.3) imply

$$|\nabla u| \leq K_2 \quad in \ \overline{\Omega} \,.$$

We turn next to second derivative estimates on $\partial\Omega$. Let $0 \in \partial\Omega$ be a point with the x_n axis pointing in the interior normal direction. Near 0, $\partial\Omega$ is represented by

$$x_n = \rho(x') = 1/2 B_{\alpha\beta} x_\alpha x_\beta + 0(|x'|^3) \,. \tag{3.4}$$

If $0 \in \Gamma_1$, $\{B_{\alpha\beta}\}$ is strictly positive definite while if $0 \in \Gamma_2$, $\{B_{\alpha\beta}\}$ is strictly negative definite. In either case, since $(u - \phi)\,(x', \rho) = 0$ near 0, we obtain

$$|u_{\alpha\beta}(0)| = |u_n(0)|\,|B_{\alpha\beta}| \leq K \,. \tag{3.5}$$

As in [1] the plan is to obtain an estimate

$$|u_{\alpha n}(0)| \leq K \,, \qquad \alpha < n \tag{3.6}$$

for the mixed normal tangential derivatives and also an estimate

$$\sum_{\alpha_1\beta<n} u_{\alpha\beta}(0)\xi_\alpha\xi_\beta \geq c_0 \tag{3.7}$$

(for any unit vector $\xi = (\xi_1, \ldots, \xi_{n-1})$) of the strict convexity of u in the tangential directions to $\partial\Omega$ at the origin. From (3.5), (3.6), (3.7), it is easy to estimate [1, p.377]

$$|u_{nn}(0)| \leq K$$

from the equation (1.1).

For $0 \in \Gamma_1$ the estimates (3.6), (3.7) are obtained precisely as in the proof of Theorem 7.1 in [1]. That is, the arguments are essentially local and the presence of Γ_2 has no effect.

So suppose $0 \in \Gamma_2$. To establish (3.7) it suffices by rotating coordinates to show for a single direction, say x_1, that $u_{11} \geq c_0 > 0$. We have

$$(u - \underline{u})\,(x', \rho(x')) = 0 \qquad \text{on } \Gamma_2 \text{ near } 0\,.$$

Hence,

$$(u - \underline{u})_{11}(0) = -(u - \underline{u})_n\, B_{11} \geq 0$$

since $B_{11} < 0$ and $(u - \underline{u})_n \geq 0$ (by (3.2)). Therefore, $u_{11}(0) \geq \underline{u}_{11}(0) \geq c_0$, proving (3.7).

The main step, then, is the estimate for $|u_{\alpha n}(0)|$ with $0 \in \Gamma_2$. Here we make use of the convexity of $\psi^{1/n}(\cdot, p)$ in an essential way.

Rewriting our equation in the form

$$(\det\, u_{ij})^{1/n} = \psi^{1/n}(x, \nabla u) \equiv f(x, \nabla u)$$

let us apply the operator

$$T = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta\partial_n - x_n\partial_\beta)\,, \quad \alpha < n\,. \tag{3.8}$$

Since $x_\beta\partial_n - x_n\partial_\beta$ is the infinitesimal generator of a rotation, we find

$$LTu = Tf(x, \nabla u) = \Sigma f_{p_i}(Tu)_i + 0(1) \tag{3.9}$$

where $0(1)$ represents a term bounded by a constant under control. Here

$$L = \frac{1}{n} f u^{ij} \partial_i\partial_j\,, \tag{3.10}$$

where $\{u^{ij}\}$ is the inverse of the Hessian matrix $\{u_{ij}\}$, is the linearization of the *concave* operator

$$F(D^2u) = (\det\, u_{ij})^{1/n} \tag{3.11}$$

Set

$$\mathcal{L} = L - f_{p_i}\partial_i\,. \tag{3.12}$$

Then (3.9) implies

$$|\mathcal{L}T(u-\phi)| \leq C(1+\Sigma u^{ii})\,. \tag{3.13}$$

Furthermore,

$$\begin{array}{lll} |T(u-\phi)| \leq C|x'|^2 & \text{on } \Gamma_2 & \text{near } 0 \\ |T(u-\phi)| \leq C & \text{in } \Omega\,. & \end{array} \tag{3.14}$$

Lemma 3.1.

$$\begin{array}{lll} \text{i)} & \mathcal{L}(\underline{u}-u) \geq \theta_1 > 0 & \textit{in } \Omega \\ \text{ii)} & (u-\underline{u})\,(x) \geq \theta_2\, d(x,\partial\Omega) & \textit{in } \Omega \\ & (u-\underline{u})_\nu \leq -\theta_3 & \textit{on } \partial\Omega \end{array}$$

for uniform constants $\theta_1, \theta_2, \theta_3$. Here $d(x,\partial\Omega)$ is the distance function to $\partial\Omega$.

Proof. By the concavity of $F(D^2u) = (\det\, u_{ij})^{1/n}$,

$$F(D^2\underline{u}) \leq F(D^2u) + L(\underline{u}-u)\,.$$

Hence,

$$L(\underline{u}-u) \geq F(D^2\underline{u}) - f(x,\nabla u)\,. \tag{3.15}$$

Since $\underline{u}$ satisfies (2.5)

$$F(D^2\underline{u}) \geq (\psi(x,\nabla\underline{u})+\varepsilon_0)^{1/n} \geq f(x,\nabla\underline{u}) + c\varepsilon_0\,.$$

Thus (3.15) yields

$$L(\underline{u}-u) \geq f(x,\nabla\underline{u}) - f(x,\nabla u) + \ c\varepsilon_0\,. \tag{3.16}$$

By assumption, $f(\cdot,p)$ is convex. Therefore

$$f(x,\nabla\underline{u}) \geq f(x,\nabla u) + (\underline{u}_i - u_i) f_{p_i}(x,\nabla u)\,. \tag{3.17}$$

Combining (3.16), (3.17) gives

$$\mathcal{L}(\underline{u}-u) \geq c\varepsilon_0$$

proving (i) with $\theta_1 = c\varepsilon_0$.

To prove (ii), let h be the harmonic extension of the boundary data ϕ to Ω. Since $\underline{u}$ is strictly convex and satisfies (2.5) we can choose $\varepsilon_1 > 0$ so small that

$$w \equiv (1-\varepsilon_1)\underline{u} + \varepsilon_1 h$$

is strictly convex and also a subsolution, i.e.,

$$\det w_{ij} \geq \psi(x, \nabla \underline{w}) .$$

Then $u \geq w$ in Ω, $u_\nu \leq w_\nu$ on $\partial\Omega$ by the maximum principle. In particular,

$$\begin{array}{rcll} u - \underline{u} & \geq & \varepsilon_1 (h - \underline{u}) & \text{in} \quad \Omega \\ (u - \underline{u})_\nu & \leq & \varepsilon_1 (h - \underline{u})_\nu & \text{on} \quad \partial\Omega . \end{array} \tag{3.18}$$

But $\Delta(h - \underline{u}) = -\Delta\underline{u} \leq -\delta < 0$ in Ω for a uniform constant $\delta > 0$, and $h - \underline{u} = 0$ on $\partial\Omega$. A standard barrier argument gives

$$\begin{array}{rcll} h - \underline{u} & \geq & \varepsilon_2 d(x, \partial\Omega) & \text{in} \quad \Omega \\ (h - \underline{u})_\nu & \leq & -\varepsilon_3 & \text{on} \quad \partial\Omega . \end{array} \tag{3.19}$$

The combination of (3.18), (3.19) completes the proof of part (ii). $\square$

Consider now the neighborhood Ω_δ of 0 given by

$$\Omega_\delta = \{ x \in \Omega \mid d(x, \Gamma_2) < \delta, \, x_n > -\delta \} .$$

Define

$$v = A(\underline{u} - u) + Bx_n + C\,\frac{|x|^2}{2}$$

where A, B, C will be chosen so that v is a lower barrier for $\pm T(u - \phi)$ in Ω_δ. Using Lemma 3.1 part (i), we find

$$\mathcal{L}v \geq \theta_1 A + C\Sigma u^{ii} - k(B + \delta C) \qquad \text{in } \Omega \tag{3.20}$$

for a uniform constant $k > 0$. Let us fix C to be the same constant as in (3.13), (3.14). Then for A large as compared with B (that is, $\theta_1 A - k(B + \delta) > C$), (3.20), (3.13) gives

$$\mathcal{L}v \geq |\mathcal{L}T(u - \phi)| \qquad \text{in } \Omega_\delta . \tag{3.21}$$

We make $v \leq -|T(u - \phi)|$ on $\partial\Omega_\delta$ as follows:

Case 1. On Γ_2, $v = Bx_n + C/2|x|^2 \le (-B\theta + C)|x'|^2 \le -C|x'|^2 \le -|T(u-\phi)|$ for B large compared with C.

Case 2. On $x_n = -\delta$, $v \le -B\delta + kC\delta \le -C \le -|T(u-\phi)|$ for B large compared with C.

Now consider B, C fixed to satisfy all our previous requirements.

Case 3. On $d(x, \partial\Omega) = \delta$, $v \le -A\theta_2\delta + (B + kC)\delta \le -C$ (by Lemma 3.1 part (ii)) for A large enough.

Hence by the maximum principle,

$$v \le \pm T(u-\phi) \qquad \text{in } \Omega_\delta$$

and since both sides vanish at 0,

$$v_n(0) \le \pm\, \partial_n T(u-\phi)\,(0)\,.$$

That is,

$$|u_{\alpha n}(0) - \phi_{\alpha n}(0)| \le -v_n(0) = A(u - \underline{u})_n(0) - B \le K\,.$$

This completes the proof of the estimate (3.6) on all of $\partial\Omega$. Consequently,

$$\Sigma|u_{ij}| \,\le\, K \qquad \text{on } \partial\Omega\,. \tag{3.22}$$

Applying the argument of [1, p397(d)], (3.22) implies

$$\Sigma|u_{ij}| \,\le\, K' \qquad \text{in } \Omega\,.$$

Recalling the remarks of § 2, the proof of Theorem 1.3 is complete.

References

[1] L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet Problem for Nonlinear Second-Order Elliptic Equations I. Monge-Ampere Equations, *Comm.in Pure and Applied Math* **37** (1984), 369–402.

[2] L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet Problem for Nonlinear Second-Order Elliptic Equations II, Complex Monge-Ampere and Uniformly Elliptic Equations, *Comm. in Pure and Applied Math* **38** (1985), 209–252.