On Multivortices in the Electroweak Theory II: Existence of Bogomol'nyi solutions in R^2

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Abstract

1 Introduction

In Part I of this paper [11], we have proven the existence of Abrikosov like periodic vortices for the bosonic sector models proposed by Ambjorn and Olesen [4] of the full Glashow-Salam-Weinberg electroweak theory, where the gauge group is $SU(2)\times U(1)$. These solutions were found from a Bogomol'nyi system of first order equations which take on a more complicated form than in the classical abelian case due to the anti-screening of the magnetic field. As a result, this system further reduces to a semilinear elliptic system of nonstandard type and we showed in Part I that the number of such vortices is bounded above in terms of the relevant physical parameters (although the locations may be prescribed arbitrarily).

The goal of the present paper is to study this Bogomol'nyi system for the self-dual electroweak interactions in the full space . ^{2}. These solutions are necessarily of infinite energy and thus the method of Part I cannot be directly applied. Our main strategy then, is to combine the method of weighted Sobolev spaces used by McOwen [8] in his study of conformal deformation equations, with the crucial change of variables introduced in Part I to reduce our elliptic system to a lower diagonal form. As a result, we are able to show (Theorem 3.3) that for any distribution of vortex locations there is a two parameter family of gauge-distinct solutions.

In order to fix the ideas, we first illustrate this method applied to the simplified $SO(3)$ theory of Ambjorn and Olesen [6] (see also Yang [12]) in which the W-bosons

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acquire mass through a Higgs mechanism but the Higgs fields are neglected from the Lagrangian. Here the system of Bogomol'nyi equations can be reduced to a single semilinear elliptic equation very closely related to the equation of prescribed Gaussian curvature. Thus in Section 2, we apply McOwen's method to study the existence of massive $SO(3)$ vortices. We then go on in Section 3 to study the full electroweak theory and prove our main results.

2 The Massive $SO(3)$ Gauge Theory

According to the discussion of Yang [12], Ambjorn and Olesen [6] , the reduced energy density for vortex-line solutions of the massive $SO(3)$ gauge field theory is given by

$$\begin{aligned} \mathcal{E} = \tfrac{1}{2} f_{12}^2 \quad & + |D_1 W + i D_2 W|^2 + 2 m_W^2 |W|^2 \\ & - 2 e f_{12} |W|^2 + 2 e^2 |W|^4, \end{aligned} \tag{2.1}$$

where W is a complex scalar field, A_j $(j = 1, 2)$ is a vector field, $f_{12} = \partial_1 A_2 - \partial_2 A_1$, and

$$D_j W = \partial_j W - ie A_j W \, .$$

By virtue of the relation $(D_j D_k - D_k D_j) W = -ie f_{jk} W$, the Euler-Lagrange equations associated with (2.1) may be written as

$$\left\{ \begin{array}{lcl} D_j D_j W & = & 2 m_W^2 W - 3 e f_{12} W + 4 e^2 |W|^2 W, \\ \partial_j f_{jk} & = & ie(W^\dagger (D_k W) - W (D_k W)^\dagger) \\ & & +3e \varepsilon_{jk} (W^\dagger (D_j W) + W (D_j W)^\dagger) \, . \end{array} \right. \tag{2.2}$$

By rewriting the the energy density $\mathcal{E}$ as

$$\mathcal{E} = |D_1 W + i D_2 W|^2 + \frac{1}{2} (f_{12} - [\frac{m_W^2}{e} + 2e|W|^2])^2 + \frac{m_W^2}{e} (f_{12} - \frac{m_W^2}{2e}) \tag{2.3}$$

the Bogomol'nyi equations associated to (2.1), (2.2) are

$$\left\{ \begin{array}{l} D_1 W + i D_2 W = 0 \, , \\ f_{12} - (\frac{m_W^2}{e} + 2e|W|^2) = 0 \, . \end{array} \right. \tag{2.4}$$

It can be shown that (2.4) implies (2.2). We are interested in solutions of (2.4) over the full ^{2}. What is the energy of such solutions? Using (2.4) in (2.3) we have

$$\mathcal{E} = \frac{m_W^2}{e} (f_{12} - \frac{m_W^2}{2e}) = \frac{1}{2} \frac{m_W^4}{e^2} + 2 m_W^2 |W|^2 \geq \frac{m_W^4}{2e^2} \, .$$

Therefore the total energy $\int_{R^2} \mathcal{E} dx$ is necessarily infinite.

Let z_0 be a zero of W. Equation (2.3a) implies that in a neighborhood of $z = z_0$,

$$W(z) = (z - z_0)^{n_0} h_0(x_1, x_2)\,,$$

where n_0 is an integer and h_0 is a smooth nonvanishing function. Thus the zero set $Z(W)$ of W is discrete. If $Z(W) = \{z_1, \ldots, z_m\}$ is finite and the multiplicity of $z = z_\ell$ is n_ℓ, then the replacement $u = \ln |W|^2$ reduces (2.3) to

$$\Delta u = -2m_W^2 - 4e^2 e^u + 4\pi \sum_{\ell=1}^{m} n_\ell \delta(z - z_\ell)\,. \tag{2.5}$$

Define

$$u_0 = \sum_{\ell=1}^{m} \ln |z - z_\ell|^{2n_\ell} - m_W^2 (x_1^2 + x_2^2)\,.$$

Then

$$\Delta u_0 = 4\pi \sum_{\ell=1}^{m} n_\ell \delta(z - z_\ell) - 2m_W^2$$

and

$$u_1 = u - u_0$$

satisfies

$$\Delta u_1 = -4e^2 U_0 e^{u_1}$$

where

$$U_0 = e^{u_0} = \Pi_{\ell-1}^{m} |z - z_\ell|^{2n_\ell} e^{-m_W^2 r^2}\ , \quad r = |z|\,. \tag{2.6}$$

We now introduce the functions $u_2 \in C^\infty(\ ^2)$ so that

$$u_2 = -\alpha \ln r\,, \quad r \geq 1$$

where $\alpha > 0$ is a constant. Let $\eta = u_1 - u_2$. Then (2.5) is reduced to

$$\Delta \eta + K e^\eta = -\Delta u_2 \equiv f \tag{2.7}$$

where

$$K = 4e^2 U_0 e^{u_2}\,.$$

Because of (2.6), the function K satisfies:

$$K \geq 0\,, \qquad K = 0(e^{-r}) \ \text{ for large } \ r > 0\,. \tag{2.8}$$

It is easily seen that f is of compact support. Also,

$$\begin{aligned} \int_{R^2} f\,dx &= \int_{|x|\leq 1} f\,dx = -\int_{|x|\leq 1} \Delta u_2\,dx \\ &= -\int_{|x|=1} \frac{\partial u_2}{\partial r}\,ds \ = \ 2\pi\alpha \end{aligned} \tag{2.9}$$

As in McOwen [8] we define the functionals

$$I(\eta) = \int_{R^2} \left[\frac{1}{2} |\nabla \eta|^2 + f\eta \right] dx, \ J(\eta) = \int_{R^2} K e^{\eta} dx \,.$$

In order that these functionals be defined properly, we need to consider a suitable weighted Sobolev space. Let $d\mu = h dx$, where h is a pointwise C^∞ function with

$$h(r) = r^{-4} \ \text{ for } \ r = |z| \geq 1 \,.$$

Let $\mathcal{H}$ denote the Hilbert space of $L^2_{\ell oc}$ functions for which

$$\| \eta \|^2_{\mathcal{H}} = \| \nabla \eta \|^2_{L^2(dx)} + \| \eta \|^2_{L^2(d\mu)} < \infty \,.$$

Notice that $\mathcal{H}$ contains the constants and thus

$$\eta \longmapsto \int_{R^2} \eta d\mu$$

is a continuous linear functional on $\mathcal{H}$ so that

$$\widetilde{\mathcal{H}} = \{ \eta \in \mathcal{H} \colon \ \int \eta d\mu = 0 \}$$

is a closed subspace of $\mathcal{H}$. Therefore we have for each $\eta \in \mathcal{H}$ the decomposition:

$$\eta = \overline{\eta} + \eta' \,, \ \ \overline{\eta} = \ \text{constant}, \ \eta' \in \widetilde{\mathcal{H}} \,. \tag{2.10}$$

The following results may be found in McOwen [8]:

Lemma 2.1. *For any* $0 < \varepsilon < 4\pi$, *there is* $C(\varepsilon) > 0$ *so that*

$$\int_{R^2} e^{a|\eta|} d\mu \leq C(\varepsilon) \exp \left[\frac{a^2}{4(4\pi - \varepsilon)} \| \nabla \eta \|^2_{L^2(dx)} \right]$$

for any $a \in$ . .

Lemma 2.2. *The Poincaré inequality holds on* $\widetilde{\mathcal{H}}$: *there is a constant* $C > 0$ *so that*

$$\| \eta \|^2_{L^2(d\mu)} \leq C \| \nabla \eta \|^2_{L^2(dx)} \,, \ \eta \in \widetilde{\mathcal{H}} \,.$$

Lemma 2.3. *The injection* $\widetilde{\mathcal{H}} \longrightarrow L^2(d\mu)$ *is a compact embedding.*

Thus we see that both $I(\eta)$ and $J(\eta)$ are well defined on $\mathcal{H}$. Consider now the optimization problem

$$\min \{ I(\eta) \mid J(\eta) = 2\pi\alpha \,, \, \eta \in \mathcal{H} \} \,. \tag{2.11}$$

Lemma 2.4. *The problem (2.11) has a solution provided* $0 < \alpha < 4$.

Proof. For $\eta \in \mathcal{H}$, let us use the decomposition (2.10). If $J(\eta) = 2\pi\alpha$, then

$$e^{\overline{\eta}} \int_{R^2} K e^{\eta'} dx = 2\pi\alpha\,,$$

or

$$\overline{\eta} = \ln 2\pi\alpha - \ln \left[\int_{R^2} K e^{\eta'} dx \right]. \tag{2.12}$$

As a consequence,

$$\begin{aligned} I(\eta) &= \int_{R^2} \frac{1}{2} |\nabla \eta'|^2 dx + \int_{R^2} (f\overline{\eta} + f\eta') dx \\ &= \tfrac{1}{2} \parallel \nabla \eta' \parallel^2_{L^2(dx)} + \int_{R^2} f\eta' dx + 2\pi\alpha \left[\ln 2\pi\alpha - \ln \left(\int_{R^2} K e^{\eta'} dx \right) \right] \end{aligned} \tag{2.13}$$

On the other hand using Lemma 2.2 we find

$$\begin{aligned} \int_{R^2} K e^{\eta'} dx &= \int_{R^2} K h^{-1} e^{\eta'} h dx \le C_1 \int_{R^2} e^{\eta'} d\mu \\ &\le C_1' \exp \left(\tfrac{1}{4(4\pi - \varepsilon)} \parallel \nabla \eta' \parallel^2_{L^2(dx)} \right)\,, \end{aligned} \tag{2.14}$$

and

$$\left| \int_{R^2} f\eta' dx \right| = \left| \int_{R^2} f h^{-1/2} \eta' h^{1/2} dx \right| \le \varepsilon^{-1} C_2 + \varepsilon \parallel \eta' \parallel^2_{L^2(d\mu)}\ . \tag{2.15}$$

Substituting (2.14)–(2.15) into (2.13) yields the lower bound :

$$I(\eta) \ge \frac{1}{2} \left(1 - \frac{\pi\alpha}{4\pi - \varepsilon} - \varepsilon C' \right) \parallel \nabla \eta' \parallel^2_{L^2(dx)} - C''(\varepsilon) \tag{2.16}$$

where C' is a constant independent of $\varepsilon, \alpha > 0$.

Since $0 < \alpha < 4$, we can fix $\varepsilon > 0$ sufficiently small to make

$$\sigma \equiv 1 - \frac{\pi\alpha}{4\pi - \varepsilon} - \varepsilon C' > 0\,.$$

Let $\{\eta_j\}$ be a minimizing sequence of (2.11). Then (2.16) says that

$$\parallel \nabla \eta' \parallel^2_{L^2(dx)} \le M\,,\ j = 1, 2, \ldots\,.$$

where $M > 0$ is a constant.

By virtue of (2.12) and (2.14), it is seen that $\{\overline{\eta}_j\}$ is bounded as well. So we may assume

$$\begin{aligned} &\eta_j' \to \eta' \in \widetilde{\mathcal{H}} \ \text{ weakly}\,, \\ &\overline{\eta}_j \to \overline{\eta} \in\ . \end{aligned}$$

Hence from Lemma 2.3, we may assume that $\eta_j \to \eta = \overline{\eta} + \eta' \in \mathcal{H}$ strongly in $L^2(d\mu)$. Therefore,

$$
\begin{aligned}
&|\textstyle\int_{R^2} f\eta_j\, dx - \int_{R^2} f\eta\, dx| \le \int_{R^2} |f| h^{-1/2} |\eta_j - \eta|\, h^{1/2} dx \\
&\le C \parallel \eta_j - \eta \parallel_{L^2(d\mu)} \to 0 \ \text{ as } \ j \to \infty\,.
\end{aligned}
$$

and

$$
\begin{aligned}
&|\textstyle\int_{R^2} K e^{\eta_j} dx - \int_{R^2} K e^{\eta} dx| \le \int_{R^2} K e^{|\eta_j| + |\eta|} |\eta_j - \eta| dx \\
&\le C \textstyle\int_{R^2} K e^{|\eta_j'|} h^{-3/4} h^{1/4} |\eta_j - \eta| h^{1/2} dx \\
&\le C \left(\textstyle\int_{R^2} K h^{-3}\right)^{1/4} \left(\textstyle\int_{R^2} e^{4|\eta_j'|} d\mu\right)^{1/4} \left(\textstyle\int_{R^2} |\eta_j - \eta|^2 d\mu\right)^{1/4} \\
&\le C' \exp\left[\tfrac{1}{4\pi - 1} \parallel \nabla \eta_j \parallel^2_{L^2(dx)}\right] \parallel \eta_j - \eta \parallel_{L^2(d\mu)} \to 0 \ \text{ as } \ j \to \infty\,.
\end{aligned}
$$

Thus $I(\eta) \le \liminf\limits_{j\to\infty} I(\eta_j)$ and $J(\eta) = \lim\limits_{j\to\infty} J(\eta_j) = 2\pi\alpha$. In other words, η solves (2.11). □

Lemma 2.5. *The minimizer η of (2.11) obtained in Lemma 2.4 is a solution to (2.7)*

Proof. By the Lagrange multiplier rule, $\exists\, \lambda \in$. so that

$$
\int_{R^2} (D\eta \cdot D\phi + f\phi) dx = \lambda \int_{R^2} K e^{\eta} \phi\, dx, \ \forall \phi \in \mathcal{H}\,. \tag{2.17}
$$

Taking the test function $\phi \equiv 1$ in (2.17), we get

$$
2\pi\alpha = \lambda J(\eta) = 2\pi\alpha\lambda\,.
$$

Hence $\lambda = 1$, and η is a weak solution of (2.7). Then elliptic regularity theory implies that η is a C^∞ solution of (2.7).

Of course, different values of α corresponds to different solutions of (2.7). Do these different solutions give rise to different (gauge-distinct) solutions of the Bogomol'nyi system (2.4)? To answer this, let us recall that

$$
u - u_0 + u_1 \ = \ u_0 + u_2 + \eta
$$

and

$$
K = 4e^2 U_0 e^{u_2} = 4e^2 e^{u_0} e^{u_2}\,.
$$

Hence

$$
\begin{aligned}
2\pi\alpha \ &= \int_{R^2} K e^{\eta} dx = \int_{R^2} 4e^2 e^{u_0 + u_2 + \eta} dx \\
&= 4e^2 \int_{R^2} e^{u} dx = 4e^2 \int_{R^2} |W|^2 dx\,.
\end{aligned} \tag{2.18}
$$

But (2.18) is invariant under gauge transformations ($W \mapsto W e^{i\omega}$) and thus different α gives rise to gauge-distinct solutions of the Bogomol'nyi system (2.4). We have thus shown

Theorem 2.6. *Let* $z_1, \ldots, z_m \in \ \ = \ ^2$ *and* $n_1, \ldots, n_m \in \ _+$. *Then, for any* $0 < \alpha < 4$, *the Bogomol'nyi equations (2.4) have a solution* $(W^{(\alpha)}, A^{(\alpha)})$ *satisfying*

$$\int_{\ ^2} |W^{(\alpha)}| dx = \frac{\pi \alpha}{2e^2} \,,$$

$Z(W^{(\alpha)}) = \{z_1, \ldots, z_m\}$, *and the multiplicity of the zero* $z = z_\ell$ *of* $W^{(\alpha)}$ *is* n_ℓ $(\ell = 1, \ldots, m)$. *In other words, for any distribution of zero locations* $z_1, \ldots z_m \in \ ^2$, *(2.4) have a continuous family of gauge-distinct solutions labeled by the parameter* $0 < \alpha < 4$ *which realize these zeros.*

$\square$

We now turn to the full electroweak theory.

3 The Full $SU(2) \times U(1)$ Electroweak Theory

We have seen in Section 3 of Part I that the Bogomol'nyi equations for the bosonic sector of the $SU(2) \times U(1)$ gauge invariant electroweak theory take the form

$$\left\{ \begin{array}{rcl} D_1 W + i D_2 W &=& 0 \\ P_{12} &=& \frac{g}{2\sin\theta}\phi_0^2 + 2g\sin\theta |W|^2 \,, \\ Z_{12} &=& \frac{g}{2\ln\theta}(\phi^2 - \phi_0^2) + 2g\ln\theta |W|^2 \,, \\ Z_j &=& -\frac{2\ln\theta}{g}\varepsilon_{jk}\partial_k \ln\phi \,, \end{array} \right. \tag{3.1}$$

where W is a complex field, ϕ is a real field, $P_{12} = \partial_1 P_2 - \partial_2 P_1$ (P is the gauge photon vector field), $Z_{12} = \partial_1 Z_2 - \partial Z_1$ (Z is the massive Z meson vector field), $D_j W = \partial_j W - ig\ (\ P_j \sin\theta + Z_j \ln\theta) W$. Recall that, in the theory on a periodic cell,

$$\int_\Omega P_{12} = \frac{2\pi N}{e} \qquad \text{and} \qquad \int_\Omega Z_{12} = 0 \,.$$

We now consider the model on the full R^2.

The energy density of the electroweak theory (in the unitary gauge) is:

$$\begin{array}{rl} \mathcal{E} = & |D_1 W + i D_2 W|^2 + \frac{1}{2}P_{12}^2 + \frac{1}{2}Z_{12}^2 \\ & -2g(Z_{12}\ln\theta + P_{12}\sin\theta)|W|^2 + 2g^2|W|^4 \\ & +(\partial_j\phi)^2 + \frac{1}{4\ln^2\theta}\, g^2\phi^2 Z_j^2 + g^2\phi^2|W|^2 + \lambda(\phi_0^2 - \phi^2)^2 \,. \end{array} \tag{3.2}$$

Let (ϕ, W, P, Z) be a solution to (3.1). Then

$$\begin{array}{rl} \mathcal{E} \geq & \frac{1}{2}P_{12}^2 + \frac{1}{2}Z_{12}^2 - 2g(Z_{12}\ln\theta + P_{12}\sin\theta)|W|^2 \\ & +2g^2|W|^4 + g^2\phi^2|W|^2 + \lambda(\phi_0^2 - \phi^2)^2 \\ & = \frac{1}{2}(\frac{g^2}{4\sin^2\theta}\phi_0^4 + 2g^2\phi_0^2|W|^2 + 4g^2\sin^2\theta|W|^4) \\ & +\frac{1}{2}(\frac{g^2}{4\ln^2\theta}(\phi^2 - \phi_0^2)^2 + 4g^2\ln^2\theta|W|^4 + 2g^2(\phi^2 - \phi_0^2)|W|^2) \\ & -2g(\frac{g}{2}(\phi^2 - \phi_0^2) + 2g\ln^2\theta|W|^2 + \frac{g}{2}\phi_0^2 + 2g\sin^2\theta|W|^2)|W|^2 \\ & \ + 2g^2|W|^4 + g^2\phi^2|W|^2 + \lambda(\phi_0^2 - \phi^2)^2 \\ & \geq \frac{g^2}{\delta\sin^2\theta}\phi_0^4 \end{array}$$

which means that the total energy $\int_{R^2} \mathcal{E} dx$ must be infinite again.

Define as before the new variables

$$u = \ln |W|^2 , \quad w = \ln \phi^2 .$$

Then equations (3.1) are transformed into the system

$$\text{(3.3)} \qquad \begin{cases} \Delta u = -g^2 e^w - e g^2 e^u + 4\pi \sum_{\ell=1}^m n_\ell \delta(z - z_\ell) , \\ \Delta w = \frac{g^2}{2 \ln^2 \theta} (e^w - \phi_0^2) + 2g^2 e^u . \end{cases}$$

Let

$$\begin{cases} u_0 = \sum_{\ell=1}^m \ln |z - z_\ell|^{2n_\ell} , \\ w_0 = -\frac{g^2 \phi_0^2}{4 \ln^2 \theta} (x_1^2 + x_2^2) ; \end{cases}$$

$$\begin{cases} u_1 = u - u_0 , \\ w_1 = w - w_0 . \end{cases}$$

Then u_1, w_1 satisfy

$$\text{(3.4)} \qquad \begin{cases} \Delta u_1 = -g^2 e^{w_0 + w_1} - 4g^2 e^{u_0 + u_1} \\ \Delta w_1 = \frac{g^2}{2 \ln^2 \theta} e^{w_0 + w_1} + 2g^2 e^{u_0 + u_1} \end{cases}$$

The term $e^{u_0 + u_1}$ is a bad term while $e^{w_0 + w_1}$ is a good term, because e^{w_0} decays exponentially fast.

As in the periodic case, we introduce the change of variables as follows:

$$\begin{cases} u_2 = u_1 + 2w_1 , \\ w_2 = u_1 . \end{cases}$$

Then equations (3.4) becomes

$$\text{(3.5)} \qquad \begin{cases} \Delta w_2 = -g^2 e^{w_0} e^{\frac{1}{2}(u_2 - w_2)} - 4g^2 e^{u_0 + w_2} , \\ \Delta u_2 = g^2 \tan^2 \theta e^{w_0} e^{\frac{1}{2}(u_2 - w_2)} \end{cases}$$

As in §2, we make suitable translations:

$$\begin{cases} u_2 = u_3 + \xi , \\ w_2 = w_3 + \zeta , \end{cases}$$

where u_3, w_3 are smooth functions so that

$$\begin{cases} u_3 = \alpha \ln r , \\ & r \geq 1 \\ w_3 = -\beta \ln r , \end{cases}$$

with $\alpha, \beta > 0$. Hence $\Delta u_3, \Delta w_3$ have compact supports and equations (3.5) become

$$\text{(3.6)} \qquad \begin{cases} \Delta \zeta = -g^2 e^{w_0} e^{\frac{1}{2}(u_3 - w_3)} e^{\frac{1}{2}(\xi - \zeta)} - 4g^2 e^{u_0 + w_3} e^{\zeta} + g , \\ \Delta \xi = g^2 \tan^2 \theta e^{w_0} e^{\frac{1}{2}(u_3 - w_3)} e^{\frac{1}{2}(\xi - \zeta)} + f , \end{cases}$$

where

$$f \equiv -\Delta u_3 \qquad g \equiv -\Delta w_3 \,.$$

As before (see (2.9)), we have

$$\int_{R^2} f dx = -2\pi\alpha \,, \qquad \int_{R^2} g dx = 2\pi\beta \,.$$

Let us now impose the constraints:

$$g^2 \tan^2\theta \int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi-\zeta)} dx = 2\pi\alpha \,, \tag{3.7}$$

and

$$g^2 \int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi-\zeta)} dx + 4g^2 \int_{R^2} e^{u_0+w_3} e^{\zeta} dx = 2\pi\beta \,,$$

or

$$4g^2 \int_{R^2} e^{u_0+w_3} e^{\zeta} dx = 2\pi \left(\beta - \frac{\alpha}{\tan^2\theta}\right) . \tag{3.8}$$

In order to make sense out of (3.8), we require:

$$\beta > \frac{\alpha}{\tan^2\theta} \,. \tag{3.9}$$

Recall that

$$e^{u_0+w_3} = O(r^{2N-\beta}) \;\; \text{for large} \;\; r > 0$$

where $N = n_1 + \cdots + n_\ell$. Hence, if

$$\beta \geq 2N + 4 \,, \tag{3.10}$$

then

$$e^{u_0+w_3} = O(r^{-4}) \;\; \text{for large} \;\; r > 0 \,. \tag{3.11}$$

This property is important in our discussions.

On the other hand, since

$$e^{w_0} = O\left(e^{-\frac{g^2\phi_0^2}{4\ln^2\theta} r^2}\right) ,$$

we have

$$e^{w_0} e^{\frac{1}{2}(u_3-w_3)} = O(e^{-r}) \;\; \text{for large} \;\; r > 0 \,. \tag{3.12}$$

Let us consider the optimization problem as in the periodic case:

$$\min \left\{ I(\xi,\zeta) \mid \xi,\zeta \in \mathcal{H},\ (\xi,\zeta) \;\; \text{satisfy the constraints (3.7)–(3.8)} \right\} . \tag{3.13}$$

where

$$I(\xi,\zeta) = \int_{R^2} dx \left[\frac{1}{2}|\nabla\xi|^2 + \frac{1}{2}\sigma|\nabla\zeta|^2 + f\xi + \sigma g\zeta \right] .$$

From (3.11)–(3.12) and Lemma 2.1, it is easily seen that (3.7), (3.8) are well-defined over $\mathcal{H}$.

Lemma 3.1. *If $\sigma = \tan^2\theta$, then a solution (ξ,ζ) of (3.13) is a solution (3.6).*

Proof. For $\sigma > 0$, let (ξ,ζ) be a solution of (3.13). Since the Frechet derivative of the constraint functionals are linearly independent, the Lagrange multipler rule implies there are constants $\lambda, \mu \in$ so that

$$\int_{R^2} (\nabla\xi\cdot\nabla\phi + f\phi)dx = \frac{1}{2}\lambda g^2 \tan^2\theta \int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi-\zeta)}\phi dx\,, \tag{3.14}$$

$$\begin{aligned}\int_{R^2}(\sigma\nabla\zeta\cdot\nabla\psi + \sigma g\psi)dx = &-\frac{1}{2}\lambda g^2\tan^2\theta \int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi-\zeta)}\psi dx\\ &+\mu 4g^2 \int_{R^2} e^{u_0+w_3} e^{\zeta}\psi dx\,, \qquad \phi,\psi\in\mathcal{H}\,.\end{aligned} \tag{3.15}$$

In (3.14), put $\phi \equiv 1$. We obtain $-2\pi\alpha = \frac{1}{2}\lambda 2\pi\alpha$. Hence $\lambda = -2$ and (3.6b) is recovered. Let $\psi \equiv 1$ in (3.15). We get $2\pi\beta\sigma = 2\pi\alpha + \mu\cdot 2\pi\left(\beta - \frac{\alpha}{\tan^2\theta}\right)$. To recover (3.6a), we need $\sigma = \tan^2\theta$. Hence

$$\mu = (\beta\tan^2\theta - \alpha)/(\beta - \alpha/\tan^2\theta) = \tan^2\theta\,.$$

Therefore (3.6a) is recovered as well. This proves the lemma.

Thus we see that it is sufficient to solve the constrained optimization problem (3.13). As in §2, we make the decomposition

$$\xi = \overline{\xi} + \xi'\,, \zeta = \overline{\zeta} + \zeta'\,,$$

where $\xi', \zeta' \in \widetilde{\mathcal{H}}$. Equation (3.8) says that

$$e^{\overline{\zeta}} \int_{R^2} e^{u_0+w_3} e^{\zeta'} dx = \frac{\pi}{2g^2}\left(\beta - \frac{\alpha}{\tan^2\theta}\right),$$

or

$$\overline{\zeta} = \ln\left[\frac{\pi}{2g^2}(\beta - \frac{\alpha}{\tan^2\theta})\right] - \ln\left[\int_{R^2} e^{u_0+w_3} e^{\zeta'} dx\right]. \tag{3.16}$$

From (3.7), we get

$$e^{\frac{1}{2}(\overline{\xi}-\overline{\zeta})} \int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi'-\zeta')} dx = \frac{2\pi\alpha}{g^2\tan^2\theta}\,,$$

or

$$\overline{\xi} = \overline{\zeta} + 2\ln\left(\frac{2\pi\alpha}{g^2\tan^2\theta}\right) - 2\ln\left[\int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi'-\zeta')} dx\right]. \tag{3.17}$$

As a consequence, the objective functional $I(\xi,\zeta)$ takes the form

$$\begin{aligned} I(\xi,\zeta) \;&= \int_{R^2}\left[\frac{1}{2}|\nabla\xi'|^2 + \frac{1}{2}\tan^2\theta\,|\nabla\zeta'|^2\right] dx\\ &+\int_{R^2}(f\xi' + \tan^2\theta\, g\zeta') - 2\pi\alpha\,\overline{\xi} + 2\pi\beta\tan^2\theta\,\overline{\zeta}\,.\end{aligned} \tag{3.18}$$

Let us first try to estimate the term

$$\Lambda = -2\pi\alpha\,\overline{\xi} + 2\pi\beta\tan^2\theta\,\overline{\zeta}\,.$$

We have from (3.16)–(3.17), that

$$\begin{aligned}\Lambda &= -2\pi\alpha\left[\overline{\zeta} - 2\ln(\int_{R^2} e^{w_0}e^{\frac{1}{2}(u_3-w_3)}e^{\frac{1}{2}(\xi'-\zeta')}dx)\right] + 2\pi\beta\tan^2\theta\,\overline{\zeta} + C_1\\ &= -2\pi\tan^2\theta(\beta-\alpha\tan^2\theta)\overline{\zeta} + 4\pi\alpha\ln\left[\int_{R^2} e^{w_0}e^{\frac{1}{2}(u_3-w_3)}e^{\frac{1}{2}(\xi'-\zeta')}dx\right] + C_1\,.\end{aligned} \tag{3.19}$$

Let us find the lower bound for $\int_{R^2} e^{w_0}e^{\frac{1}{2}(u_3-w_3)}e^{\frac{1}{2}(\xi'-\zeta')}dx$. We have

$$\begin{aligned}&\int_{R^2} e^{w_0}e^{\frac{1}{2}(u_3-w_3)}e^{\frac{1}{2}(\xi'-\zeta')}dx = \int_{R^2} h^{-1}e^{w_0}e^{\frac{1}{2}(u_3-w_3)}e^{\frac{1}{2}(\xi'-\zeta')}d\mu\\ &\geq \varepsilon_0\int_{R^2} e^{w_0}e^{\frac{1}{2}(u_3-w_3)}e^{\frac{1}{2}(\xi'-\zeta')}d\mu\\ &\geq \varepsilon_0 C_2\exp\left[\int_{R^2}\left[w_0+\frac{1}{2}(u_3-w_3)\right]d\mu + \int_{R^2}\frac{1}{2}(\xi'-\zeta')d\mu\right]\,.\end{aligned}$$

Here we have used $h^{-1}\geq\varepsilon_0$, $h = O(r^{-4})$ for large $r>0$, $w_0 = O(r^2)$, $u_3, w_3 = O(\ln r)$, so that $w_0+\frac{1}{2}(u_3-w_3)\in L(d\mu)$; then the final inequality above follows from Jensen's inequality.

Thus (3.19) implies:

$$\Lambda \geq 2\pi\tan^2\theta(\beta - \frac{\alpha}{\tan^2\theta})\overline{\zeta} - C_3\,. \tag{3.20}$$

We next analyze (3.16). From (3.11) we see that $e^{u_0+w_3}h^{-1} = O(1)$. Hence,

$$\begin{aligned}&\textstyle\int_{R^2} e^{u_0+w_3}e^{\zeta'}dx = \int_{R^2} e^{u_0+w_3}h^{-1}e^{\zeta'}d\mu \leq C_4\displaystyle\int_{R^2} e^{\zeta'}d\mu\\ &\leq C_5(\varepsilon)\exp\left[\tfrac{1}{4(4\pi-\varepsilon)}\parallel\nabla\zeta'\parallel^2_{L^2(dx)}\right]\quad\text{(using Lemma 2.1)}\quad.\end{aligned} \tag{3.21}$$

Therefore (3.16),(3.21) yields the lower bound

$$\overline{\zeta} \geq C_6 - \frac{1}{4(4\pi-\varepsilon)}\parallel\nabla\zeta'\parallel^2_{L^2(dx)}\,.$$

Thus, from (3.20), there holds

$$\Lambda \geq -\pi\tan^2\theta\left(\beta - \frac{\alpha}{\tan^2\theta}\right)\cdot\frac{1}{2(4\pi-\varepsilon)}\parallel\nabla\zeta'\parallel^2_{L^2(dx)} - C_7\,. \tag{3.22}$$

Also, since f, g have compact supports, we easily obtain using Lemma 2.2, the inequalities:

$$\left\{\begin{aligned}\int_{R^2}|f\xi'|dx &\leq \varepsilon^{-1}C_8 + \varepsilon\int_{R^2}|\xi'|^2d\mu \leq \varepsilon^{-1}C_8 + \varepsilon C\parallel\nabla\xi'\parallel^2_{L^2(dx)}\,,\\ \int_{R^2}|g\zeta'|dx &\leq \varepsilon^{-1}C_9 + \varepsilon C\parallel\nabla\zeta'\parallel^2_{L^2(dx)}\,.\end{aligned}\right. \tag{3.23}$$

Substituting (3.21)–(3.22) into (3.18) we get

$$
\begin{aligned}
I(\xi,\zeta) &\geq \tfrac{1}{2}(1-\varepsilon C') \parallel \nabla\xi' \parallel^2_{L^2(dx)} \\
&+\tfrac{1}{2}\tan^2\theta\left(1-\tfrac{\pi}{4\pi-\varepsilon}(\beta-\tfrac{\alpha}{\tan^2\theta})-\varepsilon C''\right) \parallel \nabla\zeta' \parallel^2_{L^2(dx)} - C_{10} \\
&\equiv \delta_1 \parallel \nabla\xi' \parallel^2_{L^2(dx)} + \delta_2 \parallel \nabla\zeta' \parallel^2_{L^2(dx)} - C_{10}\,.
\end{aligned}
\tag{3.24}
$$

where δ_1, δ_2 are independent of $\varepsilon, \alpha, \beta > 0$. Impose now the conditions

$$
\beta - \frac{\alpha}{\tan^2\theta} < 4\,. \tag{3.25}
$$

Then, if $\varepsilon > 0$ is sufficiently small, we get $\delta_1, \delta_2 > 0$. In particular, I is bounded from below on the admissible set

$$
\mathcal{S} = \{\xi, \zeta \in \mathcal{H} \mid \xi, \zeta \ \text{ satisfy (3.7)–(3.8)}\}\,.
$$

Let $\{(\xi_j, \zeta_j)\}$ be a minimizing sequence of (3.13). Using (3.24) we see that $\{(\xi_j', \zeta_j')\}$ is bounded in $\widetilde{\mathcal{H}}$ (see also Lemma 2.2). From (3.16), (3.21), we see that $\{\overline{\zeta_j}\}$ is a bounded sequence in . as well. Using (3.17), we can show that $\{\overline{\xi_j}\}$ is also a bounded sequence in . . For simplicity, we assume there are $\xi, \zeta \in \mathcal{H}$ so that

$$
\xi_j' \overset{w}{\to} \xi',\ \zeta_j' \overset{w}{\to} \zeta',\ \overline{\xi}_j \to \overline{\xi},\ \overline{\zeta}_j \to \overline{\zeta}\,.
$$

In other words, $\xi_j \to \xi$, $\zeta_j \to \zeta$ weakly in $\mathcal{H}$.

An obvious extension of lemma 2.3 is:

Lemma 3.2. *The injection $\mathcal{H} \to L^2(d\mu)$ is a compact embedding.*

□

Hence, (3.21) says that

$$
\begin{aligned}
&\left|\int_{R^2} e^{u_0+w_3} e^{\zeta_j} dx - \int_{R^2} e^{u_0+w_3} e^{\zeta} dx\right| \\
&\quad \leq C \int_{R^2} e^{|\zeta_j|+|\zeta|} |\zeta_j - \zeta| d\mu \\
&\quad \leq C' \left(\int_{R^2} e^{2(|\zeta'_,|+|\zeta'|)} d\mu\right)^{\frac{1}{2}} \parallel \zeta_j - \zeta \parallel_{L^2(d\mu)} \\
&\quad \leq C'' \exp\left[\frac{1}{4\pi-\varepsilon}(\parallel \nabla\zeta_j' \parallel^2_{L^2(dx)} + \parallel \nabla\zeta \parallel^2_{L^2(dx)})\right] \parallel \zeta_j - \zeta \parallel_{L^2(d\mu)} \\
&\quad\ \to 0 \ \text{ as } \ j \to \infty\,.
\end{aligned}
$$

Similarly, we can show that

$$
\int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi_j-\zeta_j)} dx - \int_{R^2} e^{w_0} e^{\frac{1}{2}(u_3-w_3)} e^{\frac{1}{2}(\xi-\zeta)} dx \ \to 0 \ \text{ as } \ j \to \infty\,.
$$

Therefore (ξ, ζ) satisfies the constraints (3.7)–(3.8). Finally the comparison $I(\xi, \zeta) \leq \liminf I(\xi_j, \zeta_j)$ is trivial. Hence (ξ, ζ) solves (3.13).

For convenience, let us summarize the conditions imposed on $\alpha, \beta > 0$ as follows:

$$
\left\{ \begin{array}{ll} \frac{\alpha}{\tan^2\theta} < \beta < \frac{\alpha}{\tan^2\theta} + 4 & \text{(see (3.9) and (3.25))}\,, \\ \beta \geq 2N + 4 & \text{(see (3.10))}\,. \end{array} \right. \tag{3.26}
$$

So we have obtained a two parameter family of solutions to equation (3.6). We can observe that these solutions give rise to gauge-distinct solutions of the Bogomol'nyi system (3.1).

In fact, we have

$$
\left\{ \begin{array}{l} u = u_0 + u_1 = u_0 + w_2 = u_0 + w_3 + \zeta\,, \\ w = w_0 + w_1 = w_0 + \frac{1}{2}(u_2 - w_2) = w_0 + \frac{1}{2}(u_3 - w_3) + \frac{1}{2}(\xi - \zeta) \end{array} \right. \tag{3.27}
$$

Recall that $|W|^2 = e^u, \phi^2 = e^w$. Hence (3.27), (3.7)–(3.8) imply the relations:

$$
\int_{R^2} \phi^2 dx = \frac{2\pi\alpha}{g^2 \tan^2\theta}\,, \tag{3.28}
$$

$$
\int_{R^2} |W|^2 dx = \frac{\pi}{2g^2}\left(\beta - \frac{\alpha}{\tan^2\theta}\right)\,. \tag{3.29}
$$

Since the left-hand-side, of (3.28)–(3.29) are gauge-invariant, different values of $\alpha, \beta > 0$ give rise to gauge-distinct solutions of (3.1). We can summarize our results as follows.

Theorem 3.3. *Let $\{z_1, \dots, z_m\} \subset {}^2 = \;, n_1, \dots, n_m \in {}_+$. For any $\alpha, \beta > 0$ satisfying (3.26), the Bogomol'nyi equations (3.1) has a solution $(W, \phi, A, P)_{(\alpha,\beta)}$ so that $Z(W) = \{z_1, \dots, z_m\}$, the multiplicity of the zero $z = z_\ell$ of W is n_ℓ, the integral averages of ϕ^2 and $|W|^2$ satisfy (3.28)–(3.29). The solution family $\{(W, \phi, A, P)_{(\alpha,\beta)}\}$ is a family of gauge-distinct solutions.*

In particular, we have nonuniqueness of solutions for each distribution of vortex locations. There is again no restriction to the number of vortices in ^{2}.

These infinite energy vortex solutions are "natural" in the sense that (2.3) or (3.1) do not allow any finite energy solutions .

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