Computing Minimal Surfaces With and Without Conformal Representation

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1 Introduction

In this short presentation we will discuss two ways to compute minimal surfaces. The first way takes advantage of the classical conformal representation of a minimal surface using analytic data on the surface. The second way involves solving the Plateau Problem for a specific boundary, usually polygonal. Our interest is in complete and properly embedded minimal surfaces. Since we will be discussing ways to compute, there will have to be some finiteness imposed on the problem. We will assume compactness, not of the surfaces themselves, of course, but of the underlying conformal structure. According to Osserman's Theorem, a complete minimal surface of finite total curvature is conformally a compact Riemann surface that has been punctured in a finite number of points, each point corresponding to an end of the surface. In particular, such a surface has finite topology. The converse is not true as the example of the helicoid shows; the helicoid is singly-periodic and not flat, so it has infinite total curvature. Whether there are more examples is an open question. See [**?**] for more details.

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In addition to these surfaces we want to consider periodic embedded ones. Modulo the orientation-preserving translations, the surfaces should have finite topology. Here in contrast to the above case, finite total curvature is equivalent to finite topology. For triply-periodic surfaces, this is well-known. But for singly- and doubly-periodic surfaces it has been established only recently. [**?**], [**?**], [**?**]. Finite total curvature again means that the quotient is a compact Riemann surface that may have punctures.

2 Computation with conformal representation: Abelian minimal surfaces

We will describe one of the above minimal surfaces as an *Abelian* minimal surface. Let M be a compact Riemann surface of genus k; $\wp = \{p_1, \ldots, p_r\}$ a finite collection of points on M; $g : M \to S^2$ a meromorphic function on M; and dh a holomorphic 1-form on $M - \wp$ with correct behavior at poles of g and at $\wp$. Define

$$\Phi = (g^{-1} - g, i(g^{-1} + g), 2)dh.$$

Then

$$X(p) = \Re e \int_y \Phi. \tag{1}$$

is a conformal minimal immersion of a covering of $M - \wp$ [**?**]. The formula (1) is the famous Weierstrass-Enneper Representation. For a closed curve γ in $M - \wp$, define

$$\text{Period}(\gamma) = \Re e \int_\gamma \Phi. \tag{2}$$

We consider Period(γ) as a vector. We now point out that to get a finite total curvature example we want (2) to hold for every γ in $M - \wp$. For singly-periodic examples we want all the nonzero Periods to be equal to a fixed vector T. For doubly- (or triply-) periodic surfaces we need to have exactly two (or three) linearly independent periods.

One can argue that every topologically finite, complete periodic (via translations) minimal surface or finite total curvature minimal surface has a representation as an Abelian minimal surface. We will try to find them using the conformal representation (1).

Before describing how this is done, we mention that since we are interested in finding embedded examples, namely examples for which the mapping X in (1) is one-to-one on the covering of $M - \wp$ where it is defined, there are additional necessary conditions to be met. The meromorphic function g is actually the stereographic projection of the Gauss map and is well-defined at the puncture points in $\wp$, so there is a limiting normal at each end. If the surface is embedded, the ends must be *parallel*. This means we can assume that, after a rotation of space, the limiting normals are vertical, or

$$g(\wp) = \{0, \infty\}.$$

There is an additional condition concerning dh and the order of g at the punctures which will ensure that the ends are separately embedded. Meeting all the conditions laid out here can be done, but then it remains to prove existence and embeddedness. We will concentrate on the existence problem in this short presentation.

The procedure goes something like this. With the behavior in mind of the surface that you wish to find, it is usually possible to write down all the possibilities for $M, \wp, g$, and dh. In all cases this *behavior* means that there is a great deal of symmetry, reducing the number of periods that need to be killed. One then has a multiparameter family of punctured symmetric Riemann surfaces with appropriate holomorphic data. Then it is necessary to solve the associated period problem. In practice, this is first done on a computer, and an image of the surface is made in order to check embeddedness and to also see that the computations to kill all the desired periods is in fact correct. Then it remains to give a mathematical proof.

The program we use to do the heart of these computations at the Center for Geometry Analysis Numerics and Graphics at the University of Massachusetts is called MESH. It was written by Jim Hoffman. Helping to define and develop it as a mathematical tool has been a job that has been shared by many people, principal among them are Michael Callahan, Eric Boix and Meinhard Wohlgemuth. [?].

In the video, we see several examples of embedded minimal surfaces discovered by this process.

2.1 *Finite Topology:*

Four ended example of genus two (Callahan, Hoffman, and Meeks) with two flat ends and two catenoid ends.

2.2 *Singly-Periodic:*

Singly-periodic minimal surfaces with an infinite number of flat ends (Callahan-Hoffman-Meeks, Hoffman-Wohlgemuth) [?] [?].

2.3 *Doubly-Periodic:*

Karcher family of genus one surfaces with parallel ends [?], [?].

Wei family of genus-two surfaces [?].

3 Minimal surfaces without conformal representation

It is possible to construct complete embedded minimal surfaces by solving the Plateau problem for an appropriately chosen polygonal boundary. The full surface is produced by means of reflection across the boundary line segments. This will produce a surface invariant under a translation or a screw motion. We will describe several recent constructions of new examples using this method. The trick is to choose the boundary so that the resulting extended surface is embedded and periodic. If the symmetries produced by this process are translations, it will necessarily be an Abelian surface in the sense of the previous section. However, this procedure can produce an embedded periodic surface that is invariant under a screw motion. (We will see this in the last sequence of the video.) The quotient surface will still be a punctured compact Riemann surface. However, the Gauss map g does not descend to the quotient. It's differential does however, and it is possible to get a formula that extends (1) to this case. Thus, the data is still defined in a compact setting and we will still call such a surface Abelian.

In the video the Plateau Problem is solved using a program written by James Riordan. The basic computational idea is very simple. An initial surface is given in the form of a tessellation by triangles. Then each node not on the boundary moves to a point in space defined by the condition that the sum of the area of the triangles of which it is a vertex is minimized. This procedure is inherently parallelizable.

We give several examples in the video.

3.1 *Triply-periodic example:*

Fischer-Koch surface with non-cubic symmetry [**?**], [**?**], [**?**], [**?**].

3.2 *Singly-periodic example with translation symmetry:*

Generalization of the idea in the previous example to produce a singly-periodic surface with Scherk ends. The quotient is a genus one surface with six ends.

3.3 *Singly-periodic examples with screw symmetry:*

These surfaces form a one-parameter family of embedded examples with screw motion symmetry [**?**].