The Structure of Constant Mean Curvature Embeddings in Euclidean Three Space

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Introduction

In these notes we will sketch the known necessary conditions for properly embedded surfaces $\Sigma \subset \mathrm{R}^3$ which have non-zero constant mean curvature and finite topology. Many examples of constant mean curvature immersions and embeddings have been constructed in recent years through the work of H. Wente [13], N. Kapouleas [4, 5], H. Karcher [6], U. Pinkall and I. Sterling [12]. Immersions of tori and cylinders can already have extremely complicated behavior [12, 13], so general *immersed* constant mean curvature surfaces must be quite varied in structure. If Σ is required to be properly *embedded*, it turns out that its structure can be characterized fairly well. Not only is this structure interesting in its own right, but it could also aid in the analysis of physical systems involving several materials or phases, such as those described by the Cahn-Hilliard equation [2, 3], where it is believed that slowly evolving interfaces may have time-varying, approximately constant mean curvature.

Let Σ have mean curvature vector $\mathbf{H}$ with $|\mathbf{H}| \equiv H > 0$, define the "exterior" normal ν to be $-\mathbf{H}/H$, and call such a Σ an MCH surface. The two regions bounded by Σ are called the exterior and interior accordingly. We may scale space so that $H = 1$. The sphere of radius 2, with ν the usual exterior normal, is an MC1 surface, whose interior is the ball. The one-parameter family of axially symmetric MC1 surfaces are known as Delaunay surfaces. They are periodic along the axis and are obtained by integrating a first order O.D.E. for the profile curve (say with initial radius $\rho \leq 2$ and initial derivative zero) which in turn is the first integral of the second order O.D.E. for constant mean curvature. The embedded Delaunay surfaces interpolate between a cylinder of radius 1 and a chain of radius 2 spheres.

Below we list the known necessary conditions on finite topology MC1 surfaces Σ

properly embedded in R^3. (Recall that any surface of finite topology is homeomorphic to a compact surface of genus g with a finite number e of punctures; a neighborhood of each puncture is an annular end.) These results have evolved, except for the famous sphere theorem of A.D. Alexandrov [1], from work by W. Meeks, N. Korevaar, R. Kusner, and B. Solomon: [7, 8, 9, 11]. After the theorems we indicate key ideas and techniques, with references for further details.

1 Theorems

Let Σ be an MC1 surface properly embedded in R^3 with genus g and e annular ends. Then

(i) *The only compact* Σ *is a sphere of radius* 2 [1].

(ii) *There are* no *one-ended* Σ [11].

(iii) *The only two-ended* Σ *are Delaunay surfaces* [9].

(iv) *Each end of* Σ *converges exponentially to a Delaunay surface at infinity.* That is, there is a Delaunay surface in R^3 so that near ∞ the annular end is a normal graph above this Delaunay surface, with parameterizing function that goes to zero exponentially as the distance from the origin approaches infinity [9].

(v) Σ *is contained in a regular neighborhood of a piecewise linear graph in* R^3*.* More explicitly, Σ is localized in space as follows: there exist e solid, half-infinite cylinders of radius 6, at most $3g-3+e$ solid finite-length cylinders of radius 6, and at most $(2g+e-2)^3$ solid cubes of edge-length 20, so that Σ is contained in their union. The solid, half-infinite cylinders contain the asymptotically-Delaunay annular ends discussed in (iv). The solid, finite-length cylinders contain annular subsets of Σ which are Delaunay-like if the cylinders are sufficiently long [7].

(vi) *Any cylindrical ray or segment in* (v) *is canonical in the following sense: the (homology class of the) curve which generates the enclosed annulus has a "balancing line" segment or ray contained in the cylindrical ray or segment.* This balancing line is determined by certain force and torque invariants associated to the given homology class. [7, 10]

(vii) *Solution space pre-compactness.* The area of the intersection of Σ with a ball of radius R is bounded by a constant $C = C(g,e)R^2$. It is possible to bound the second fundamental form on Σ in terms of g, e and the virtual force invariants mentioned in (vi). (So, for example, if g and e are fixed, the norm of the second fundamental form is bounded above by a lower bound on the shortest closed geodesic on Σ.) Furthermore, in case Σ has 3 ends (or more generally, lies in a half-space) the solution space is a finite dimesional real analytic variety, parametrized by a subset of the virtual force and torque invariants, and possibly by an additional compact variety. (We suspect this is true without the half-space assumption, and that the compact variety is just a finite set of points.) [7]

2 Forces and torques

Because an MC1 surface must be in equilibrium with respect to surface tension and pressure forces there are natural (cohomology) invariants associated to a given surface's curve-homology classes [8, 9, 10]. They are natural quantities with which to describe Σ, and we believe the space of MC1 surfaces having given topological type is probably a finite dimensional variety on which appropriate subsets of these invariants are local coordinates. Such a conjecture is nowhere near proven in general, either in its existence or uniqueness aspects, but seems plausible based on the constructions of Kapouleas [4, 5] and our partial results (1vii) [7].

Let Γ be a 1-cycle on Σ, let K be a 2-chain in R^3, with $\partial\mathrm{K} = \Gamma$. Pick normal ν to K, so that the orientations (K, ν) and $(\Gamma, \eta, \nu_\Sigma)$ are right-handed, and conormal η to Γ on Σ. Fix a Killing vector field Y generating isometries of R^3. Then the expression

$$\int_\Gamma \eta \cdot Y - \int_{\mathrm{K}} \nu \cdot Y$$

depends only on the homology class of Γ on Σ. Equivalently, whenever Γ bounds a chain on Σ, this expression is zero. The homology invariance follows from two applications of the divergence theorem to the equation $\Delta x^i = -\nu^i$: once from Γ to the chain on Σ which it bounds, and once from this chain to K, assuming their sum bounds a chain of R^3. (See [8, 9].)

Letting Y generate translations ($Y = (1,0,0), (0,1,0), (0,0,1)$), and rotations ($Y = (0, -x^3, x^2), (x^3, 0, -x^1), (-x^2, x^1, 0)$), and viewing the corresponding homology invariants as a pair of 3-vectors we obtain the "force" and "torque" associated to the homology class of Γ:

$$w(\Gamma) = \int_\Gamma \eta - \int_{\mathrm{K}} \nu \qquad \text{and} \qquad r(\Gamma) = \int_\Gamma \eta \times x - \int_K \nu \times x\,.$$

These expressions represent the virtual force and torque which arise by cutting a piece of surface along Γ, with a "cap" K. The curve and cap contributions to the vectors correspond to surface tension and pressure, respectively. Saying w and r are homology invariants is, physically speaking, to say that if a collection of caps and corresponding compact pieces of Σ bound a region of R^3, then all surface tension along the cuts and pressure through the caps must balance.

If the force $w[\Gamma]$ is non-zero, then it follows that there is a natural "balancing" line in R^3 associated to $[\Gamma]$, namely the set of origin choices for which the torque $r[\Gamma]$ is parallel to $w[\Gamma]$ (equivalently, origins for which $|r[\Gamma]|$ is minimized). Since w and r transform exactly as linear and angular momentum for a moving body, the balancing line conclusion is analogous to the fact that an object of non-zero linear momentum traveling through R^3 has a canonical "world" line associated with it, namely the line along which the center of mass travels.

For a Delaunay surface (with homology generated by a loop around the axis of rotation) the balancing line is simply the axis itself. In general one would like to characterize the sense in which a balancing line and homology class representatives are close to each other, in analogy with the fact that the center of mass of an object lies within its convex hull. In fact, a result of this type does hold for "planar" homology classes (it follows from (5) below), and allows for the naturalness conclusion (1vi).

3 Cylindrical boundedness of subannuli

Subannuli of an MC1 surface Σ tend to be contained in cylindrical regions. This fact makes crucial use of the two-dimensional topolgical properties of an annulus, and is the reason our theorems are for $\Sigma^2 \subset \mathrm{R}^3$ rather than for $\Sigma^n \subset \mathrm{R}^{n+1}$. In this context [11] Meeks developed techniques to show that:

(3.1) *A properly embedded MC1 annulus ($\approx S^1 \times [0, \infty)$ with one end) is contained in the union of a large solid ball and a half-infinite solid cylinder of radius* 3.

Key n-dimensional lemmas for (3.1) are:

(3.2) *A compact MC1 graph S above a plane π, with $\partial S \subset \pi$, has height at most n above the plane (e.g. a hemisphere of radius n).*

(3.3) *A compact MC1 surface S with $\partial S \subset \pi$ lies within distance $2n$ of π (e.g. a sphere of radius $2n$ touching π at a pole).*

Using these ideas one can also show:

(3.4) Let S be an embedded MC1 annulus with $\partial S = \Gamma_1 \cup \Gamma_2$, where the Γ_i lie in parallel planes π_i. (Assume the π_i are horizontal, with π_1 "above" π_2.) Then S lies in the slab from 4 units above π_1 to 4 units below π_2. Furthermore, in the region from 10 units below π_1 to 10 units above π_2, S is contained in a solid cylinder of radius 6.

Immediate consequences of (3.1) and (3.3) are that no one-ended Σ can exist (1ii), and that any Σ with 2 ends is contained in a solid cylinder [11].

4 Estimates in cylinders

Once an MC1 surface Σ or annulus is localized to a cylindrical region it becomes possible to make further useful *a priori* estimates. For example, if Σ is MC1, properly embedded, and contained in a cylinder, then one can extend Alexandrov's reflection techniques to show that Σ is symmetric about an axis parallel to the cylinder, so in particular from (3) we conclude (1iii) that any two-ended Σ is Delaunay.

By integrating $|\nabla x^i|^2 + x^i \Delta x^i$ (and summing on i) over a cylindrically bounded Σ, using surface and R^3 divergence theorems (much as in the force and torque derivation (2)), one can derive *a priori* area estimates: The area of such a Σ grows linearly in the length of the cylinder. In [9] this estimate was dependent on the magnitude of the force of the "planar" homology class on Σ obtained by intersecting the part of it inside the cylinder with a plane perpendicular to the cylinder (and using caps K which are intersections of the plane with the interior of Σ (inside the cylinder). In fact, one can derive the area estimates independently of this force and work backwards to conclude *a priori* estimates on the magnitude of the force.

Using the area estimates, the Gauss-Bonnet theorem, a blow-up argument, and the homology invariance of the force vector, one can derive the fact that the second fundamental form on an annular end is bounded. The area estimate and second fundamental form bound imply that "slide-back" sequences of an annular end must converge to a cylindrically bounded Σ, i.e. to a Delaunay surface, and with fixed force and torque, i.e. unique up to translation along its axis. With a Jacobi field analysis (depending on symmetry improvement at infinity), one can ultimately conclude (iv) [9].

5 Bubbles

We return to the question of which homology classes on Σ have balancing lines, i.e. non-zero forces. It turns out that this is true for any non-zero "planar" class: $[\Gamma]$ is "interior (exterior) planar" if Γ is the boundary of a bounded component of the intersection of a plane π with the interior (exterior) of Σ (and K is this intersection cap). Because of orientation reasons (the Γ and K contributions point to the same side of π), the force is automatically non-zero for exterior planar classes.

For interior classes (such as one encounters in cylindrically bounded annuli), the result is proven by blowing a graphical MC1 bubble inside Σ, spanning Γ. Ones uses an elliptic P.D.E. method of continuity to go from the zero mean curvature cap K, through a family of graphs with mean curvature $t \leq 1$. Σ and its reflection through π provide barriers. By comparing the force of Γ on the MC1 graph (which is zero since Γ bounds the graph) to that of Γ on Σ, one sees that the K terms are the same and that a strict inequality holds on the dot product of the Γ terms with the normal to π, since the graph lies inside Σ and its reflection. (If the strict inequality fails the strong maximum principle implies that the MC1 graph is actually a subset of Σ or its reflection, in which case Γ is trivial on Σ). Hence the force of $[\Gamma]$ on Σ is non-zero, and actually points into a given half-space bounded by π [8]. This is an interesting physical property of embedded Σ, which fails for immersions.

Besides letting one prove the nearness of cylindrically bounded annuli to their balancing lines (1vi), the bubble-blowing argument and its non-zero virtual force consequence, when proven in the hyperbolic space case, enabled us to show the asymptotic symmetry results for MCH properly embedded annuli (of mean curvature $H > 2$), directly from Alexandrov reflection arguments [8], thereby avoiding the Jacobi field analysis.

6 Surface Morse theory and global estimates

Given Σ of genus g with $e \geq 3$ ends, pick three mutually orthogonal directions for which the height functions on Σ are Morse functions. Because 2-dimensional surfaces Σ are built out of $p = |\chi| = 2g - 2 + e$ pairs of "pants", each height function has exactly p "essential" critical points where the homotopy class of the level sets on Σ changes. In between the corresponding critical heights, Σ is made out of annular pieces. Applying (3.4) to the first height function, outside at most p slabs of thickness 20 it follows that Σ decomposes into cylindrically bounded annuli. Applying the same reasoning in succession to the other two height directions, it is possible to conclude the localization results (1v).

The *a priori* area estimates (1vi) follow from the monotonicity formula (see [9] for an easy proof for MC1 embeddings), and the fact that localization (1v) and annular area estimates (5) allow one to bound $|\Sigma \cap B_r|$ for r of intermediate size. The second fundamental form bound may now be proven using the same type of blow up argument as in [9] for the annular case (5).

7 Compactness for families

In case our surfaces Σ lie in a half-space (which is guaranteed when $e = 3$ [9, 11]), we obtain a fixed set of generators for the first homology which are represented by planar cycles. A uniform second fundamental form bound depending only upon the virtual forces follows as above. If we allow "almost embedded" surfaces (i.e. immersed surfaces which extend to immersions of their interiors), then we conclude the variety of such MC1 surfaces Σ (with fixed force and torque invariants) is compact. We must again use a "slide-back" argument [9] to rule out possible "noncompactness propagating down the ends of the surface". The dimension of the solution space of MC1 surfaces Σ is then bounded by that of this compact variety plus a constant times the rank $(2g + e)$ of the first homology of Σ.

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