Constant Mean Curvature Surfaces in Hyperbolic Space

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August 27, 1990

*Supported by the National Science Foundation grant DMS-8808002.

†Supported by the National Science Foundation grant DMS-8908064.

‡The research described in this paper was supported by research grant DE-FG02-86ER250125 of the Applied Mathematical Science subprogram of Office of Energy Research, U.S. Department of Energy, and National Science Foundation grant DMS-8900285.

§Supported by the National Science Foundation grant DMS-8800414.

Contents

1 Introduction

A basic problem in differential geometry is the global structure and classification of complete properly embedded constant mean curvature surfaces in three-manifolds of constant sectional curvature. The models for these manifolds are three-dimensional Euclidean space $\mathbb{R}^3$, the three sphere S^3, and hyperbolic three-space $\mathbb{H}^3$.

In 1986, Meeks [16] proved that each annular end of a properly embedded surface Σ of nonzero constant mean curvature in $\mathbb{R}^3$ remains a bounded distance from a ray, i.e. each annular end of Σ is *cylindrically bounded*. This theorem, together with a simple application of the Alexandrov reflection principle, showed that Σ cannot be homeomorphic to a closed surface with a single point removed. He also used this cylindrical boundedness theorem to show that when Σ is topologically a closed surface minus two points, the entire surface stays a bounded distance from a line.

In the same year, many new examples of complete properly embedded surfaces in $\mathbb{R}^3$ of finite topology and nonzero constant mean curvature were constructed by Kapouleas [10]. The ends of his examples converge exponentially to the constant mean curvature surfaces of revolution known as *Delaunay surfaces.*

Korevaar, Kusner and Solomon [11] then developed a theory to deal with the global structure and asymptotic geometry of complete, properly embedded constant mean curvature surfaces. In particular, they proved that *each annular end of* any *properly embedded surface* Σ *in* $\mathbb{R}^3$ *with nonzero constant mean curvature converges exponentially to a Delaunay surface.* The first step in their proof was to show that if Σ stays within bounded distance of a line, it is Delaunay. Together with Meeks' result, this established that *every properly embedded constant mean curvature surface homeomorphic to a closed surface minus two points is Delaunay.*

Lately there has been renewed interest in the geometry of surfaces in hyperbolic manifolds. (See for example, [1, 4, 5, 7, 15, 22].) Let $\mathbb{H}_c^{n+1}$ denote the complete simply-connected spaceform of constant negative sectional curvature c. As in Euclidean space, we will call constant mean curvature surfaces of revolution in $\mathbb{H}_c^3$ *Delaunay surfaces.* We shall prove the following theorems on the geometry of constant mean curvature surfaces in $\mathbb{H}_c^3$.

Theorem 1.1 *Suppose* $\Sigma \subset \mathbb{H}_c^3$ *is a complete properly embedded surface with constant mean curvature greater than that of a horosphere. Then*

1. Σ *is not homeomorphic to a closed surface punctured in one point.*

2. *If* Σ *is homeomorphic to a closed surface punctured in two points,* Σ *is Delaunay. In particular,* Σ *is topologically a cylinder.*

3. *If* Σ *is homeomorphic to a closed surface punctured in three points, then* Σ *remains a bounded distance from a geodesic plane of reflective symmetry. Furthermore, each half of* Σ *determined by the plane of re-*

flective symmetry is a graph over this plane with respect to the distance function to the plane.

See § 2.5 for the definition of *horosphere* and a brief explanation of its role here. The proof of the above theorem is similar to that of the corresponding theorem for constant mean curvature surfaces in $\mathbb{R}^3$ whose proof we outlined earlier. Namely, we first prove that *each annular end of* Σ *is cylindrically bounded.* This boundedness property has several immediate consequences when Σ has finite topology:

1. Σ has more than one end.

2. When Σ has two ends, it is cylindrically bounded.

3. When Σ has three ends, it stays a bounded distance from a hyperplane.

An application of the Alexandrov reflection argument, due to Hsiang [9], together with these three properties for Σ proves Theorem 1.1. This proof is carried out in detail in § 3.

As in the Euclidean case, the annular ends of the surface Σ described in Theorem 1.1 must converge to Delaunay surfaces. More precisely,

Theorem 1.2 *If* $A \subset \mathbb{H}^3_c$ *is a properly embedded annulus with constant mean curvature greater than that of a horosphere, then* A *converges exponentially to a fixed Delaunay surface. In particular, each end of a complete properly embedded surface* $\Sigma \subset \mathbb{H}^3_c$*, with finite topological type and constant mean curvature exceeding that of a horosphere, is asymptotically Delaunay. It follows that* Σ *is conformally a compact Riemann surface with finitely many punctures, and that the limit set of* $\Sigma \subset \mathbb{H}^3_c$ *is a finite collection of points in the sphere at infinity* S_∞*.*

The above theorem results directly from the cylindrical boundedness property for the annular end A and the following general theorem.

Theorem 1.3 *Suppose $\Sigma \subset \mathbb{H}_c^{n+1}$ is a properly embedded noncompact hypersurface with constant mean curvature and compact boundary $\partial\Sigma$. If Σ stays within bounded distance of a geodesic ray in $\mathbb{H}_c^{n+1}$, then it converges exponentially to one end of a Delaunay hypersurface of revolution. In particular, Σ has one end, which is diffeomorphic to a punctured ball.*

The proof of Theorem 1.3 introduces a useful new tool we call the Positive Flux Lemma (Theorem 5.3). It also depends on some careful applications of the Alexandrov reflection method. We remark that the Positive Flux Lemma holds in Euclidean space too, and combined with a more refined Alexandrov reflection argument may help prove an n-dimensional Euclidean version of Theorem 1.3.

For simplicity we will prove all of our theorems in $\mathbb{H}_{-1}^{n+1} = \mathbb{H}^{n+1}$.

2 Preliminaries

Here we introduce some background material and notation.

We will use three models of hyperbolic space: the *upper halfspace*, the *ball*, and the *Minkowski-space* models.

(2.1) Upper halfspace model. Introduce the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2 + dy^2}{y^2}$$

on the upper halfspace $\{(\mathbf{x}, y) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in \mathbb{R}^n, 0 < y \in \mathbb{R}\}$. Familiar calculations show this is a complete metric with constant sectional curvature -1, hence isometric to $\mathbb{H}^{n+1}$.

Horizontal (Euclidean) translations and rotations (i.e., elements of the Euclidean group) are clearly isometries of this model. There are two other families of transformations associated with each $\mathbf{x}_0 \in \mathbb{R}^n$: *the vertical hyperbolic translations*

$$(\mathbf{x}, y) \longmapsto \left(e^t(\mathbf{x} - \mathbf{x}_0),\, e^t y\right), \qquad (t \in \mathbb{R}),$$

(i.e., Euclidean homotheties about $(\mathbf{x}_0, 0)$ with factor e^t), and the *hyperbolic reflections*

$$(\mathbf{x}, y) \longmapsto (\mathbf{x}_0, 0) + \frac{t^2(\mathbf{x} - \mathbf{x}_0, y)}{|\mathbf{x} - \mathbf{x}_0|^2 + y^2}, \qquad (t \in \mathbb{R}),$$

which are simply Euclidean inversions of the upper halfspace in the radius t sphere centered at $(\mathbf{x}_0, 0)$. (Reflection in a vertical hyperplane should also be regarded as a hyperbolic reflection, with $\mathbf{x}_0 = \infty$, $t = \infty$.) As the fixed-point set of its reflection, any such sphere is immediately seen to be totally geodesic. Taking intersections, one then deduces: *Hemispheres of any dimension and radius meeting the $y \equiv 0$ "hyperplane at infinity" orthogonally are totally geodesic. In particular, circles meeting the $y \equiv 0$ hyperplane at right angles and vertical lines are the geodesics of the upper halfspace model.*

The hyperbolic translations will prove useful later. The ones described above associate to each $\mathbf{x}_0 \in \mathbb{R}^n$ a one-parameter group of isometries obtained by translating along the vertical geodesic through $(\mathbf{x}_0, 0)$ with unit speed. Since *every* geodesic can be mapped to a vertical geodesic by some hyperbolic reflection (reflect in a sphere of appropriately larger radius having a great circle tangent to the given geodesic at $y = 0$ in the 2-plane it determines), one sees: *Every constant speed geodesic $\gamma \in \mathbb{R}^{n+1}$ uniquely determines a one-parameter group of hyperbolic translations, whose Killing field extends the velocity vector field of γ.*

(2.2) Ball model. Introduce a complete, rotationally invariant conformal metric

$$ds^2 = \frac{4\left(dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2\right)}{(1-r^2)^2}$$

on the open ball $\mathbf{B}^{n+1}$ with radius 1 centered at the origin $\mathbf{0}$ in $\mathbb{R}^{n+1}$ (here $r^2 := x_1^2 + x_2^2 + \cdots + x_{n+1}^2$). Completeness of this metric and constant curvature -1 are again easy to check. Euclidean rotations (i.e., elements of the group $SO(n+1)$) and reflections across hyperplanes through $\mathbf{0}$ are obvious isometries, and it is straightforward to verify that the following maps from ball to the upper halfspace and back, (treating both as subsets of $(\mathbf{x}, y)$-space) are mutually inverse isometries:

$$(\mathbf{x}, y) \longmapsto \frac{(2\mathbf{x}, -|\mathbf{x}|^2 - y^2 + 1)}{(1-y)^2 + |\mathbf{x}|^2}$$

and

$$(\mathbf{x}, y) \longmapsto \frac{(2\mathbf{x}, |\mathbf{x}|^2 + y^2 - 1)}{(1+y)^2 + |\mathbf{x}|^2}$$

Since both models are conformally flat, these are conformal maps of $\mathbb{R}^{n+1}$. Thus, in the ball model, hyperbolic reflections are again (Euclidean) hyperplane reflections and inversions in spheres meeting the ($r \equiv 1$) sphere at infinity $S_\infty = \partial \mathbf{B}^{n+1}$ orthogonally. Any point q outside $\mathbf{B}^{n+1}$ centers a unique such reflecting sphere, whose Euclidean radius R, satisfies $R^2 = |q|^2 - 1$. Each radial ray emanating from $\mathbf{B}^{n+1}$ thus determines a one-parameter family of such reflections and these are the families we shall use to implement the *Alexandrov reflection method* in hyperbolic space. (Recall in Euclidean space, this method involves successive reflection across each of a family of parallel hyperplanes; see 6.5). The above also shows that k-dimensional spherical caps and disks meeting S_∞ orthogonally ($k \leq n+1$) are totally geodesic in the ball model; Euclidean circular arcs and line segments meeting S_∞ orthogonally are hyperbolic geodesics.

Explicit expressions for hyperbolic translations in the ball model are not particularly easy (or illuminating) to write down, but their behavior is clear from the upper halfspace model: *given any two points on S_∞, there is a one-parameter group of hyperbolic translations moving with unit speed along their unique "connecting" geodesic.*

(2.3) Minkowski-space model. Here we write ${}^{1,n+1}$ for ${}^{n+2}$ with the *Lorentz metric*

$$Q = -dx_0^2 + dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2 \,.$$

The following spacelike hypersurface $\mathcal{S} \subset {}^{1,n+1}$ with induced Riemannian metric then provides the *Minkowski space model* for ${}^{n+1}$:

$$\mathcal{S} := \{x = (x_0, \mathbf{x}) \in {}^{n+2} \mid \mathbf{x} \in {}^{n+1}, 0 < x_0 \in \ , Q(x) = |\mathbf{x}|^2 - x_0^2 = -1\}$$

In this way, we regard ${}^{n+1}$ as the "unit sphere" in $(n+2)$-dimensional Minkowski space. By definition, the group $O^+(1, n+1)$ preserves this metric, and acts transitively on $\mathcal{S}$. To see the latter, consider that any $(x_0, \mathbf{x}) \in \mathcal{S}$ can be sent to $(1, \mathbf{0})$ by an element of the group. For the matrices

$$\begin{bmatrix} \cosh(t) & \sinh(t) & 0 & . & . & . & 0 \\ \sinh(t) & \cosh(t) & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ & . & . & . & & & . \\ & . & . & & . & & . \\ & . & . & & & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix}$$

form a one-parameter subgroup of $O^+(1, n+1)$, and any $(x_0, \mathbf{x}) \in \mathcal{S}$ can be mapped to $(x_0, |\mathbf{x}|\mathbf{e}_1)$ by a "horizontal" rotation (i.e., element of $O(n+1)$; we regard the x_0-axis as "vertical"). Belonging to $\mathcal{S}$, the latter vector can be written $(\cosh(s)\,,\sinh(s)\mathbf{e}_1)$ for some $s \in \mathbb{R}$, and we obtain an isometry mapping it to $(1, \mathbf{0})$ by applying the matrix above with $t = -s$. In fact, since the isotropy group of $\mathcal{S}$ at $(1, \mathbf{0})$ contains $O(n+1)$, $\mathcal{S}$ must have constant curvature. Computing at $(1, \mathbf{0})$, one finds it to be -1. Completeness is also easy to check, as noted below.

Observe further that the matrices above act transitively on the set of all hyperplanes in $\mathbb{R}^{n+2}$ containing the subspace $x_1 = 0$. We therefore obtain involutive isometries of $\mathcal{S}$ by conjugating reflection across the hyperplane $x_1 = 0$, (which induces an obvious isometry) with any of the above matrices. *These involutions are the hyperbolic reflections encountered previously in the ball and upper halfspace models.*

Moreover, conjugating these reflections with horizontal rotations and taking intersections, we see that *the totally geodesic subspaces of* $\mathcal{S}$ *are precisely the intersections of* $\mathcal{S}$ *with vector subspaces of* $\mathbb{R}^{1,n+1}$*; all geodesics of* $\mathcal{S}$ *are intersections of* $\mathcal{S}$ *with Euclidean planes through* $(0, \mathbf{0}) \in \mathbb{R}^{1,n+1}$. In particular, the curve $\cosh(t)\mathbf{e}_0 + \sinh(t)\mathbf{e}_1$ in $\mathcal{S}$ is a geodesic, and the one-parameter group of matrices above translates it along itself with unit speed. This makes the completeness of $\mathcal{S}$ obvious, as well as the following: *the one-parameter hyperbolic translation subgroups on* $\mathcal{S}$ *are all conjugate to the matrix subgroup of* $O^+(1, n+1)$ *displayed above.*

Note also that in this model, the sphere at infinity S_∞ naturally identifies with the (projectivized) null-cone $\{x = (x_0, \mathbf{x}) \in \mathbb{R}^{1,n+1} \mid Q(x) = |\mathbf{x}|^2 - x_0^2 = 0\}$.

(2.4) Hypersurfaces of constant mean curvature. We define the *mean curvature vector* $\mathbf{h}$ of a hypersurface $\Sigma \in \mathbb{R}^{n+1}$ as the trace of the second fundamental form $\mathbf{A}$ on Σ:

$$\mathbf{h} := \ \mathrm{trace}(\mathbf{A}) = \left(\sum_{i=1}^{n} \nabla_{\tau_i} \tau_i \right)^{\perp} .$$

Here $\tau_1, \tau_2, \ldots, \tau_n$ is any local orthonormal frame field on Σ, "∇" is the covariant derivative operator on $^{n+1}$, and "$\perp$" denotes projection on the normal bundle of Σ. We say Σ *has mean curvature H if each of its points has a neighborhood with a continuous unit normal vector field ν such that* $\mathbf{h} = -H\nu$.

It is well-known that $\mathbf{h}$ can also be characterized in variational terms, according to the *first variation formula* for area (§ 9 of [20]):

$$\frac{d}{dt}\bigg|_{t=0} |\Sigma(t) \cap K| = -\int_{\Sigma} \mathbf{h} \cdot \mathbf{X} .$$

Here "$|\cdot|$" signifies n-area on Σ , and the integral is taken with respect to same. $\Sigma(t)$ is any one-parameter family of hypersurfaces having $\Sigma(0) = \Sigma$, with initial velocity vector field $\mathbf{X}$ supported in a compact set K disjoint from $\partial\Sigma$. This formula simply says that the vector field $\mathbf{h}$ represents the "gradient" of the n-area functional on hypersurfaces of $^{n+1}$.

(2.5) Some examples. Any hypersurface of $^{n+1}$ on which the isometries of $^{n+1}$ act transitively must have constant mean curvature: for example, any sphere centered at $\mathbf{0}$ in the ball model, and any "horizontal" hyperplane $y = y_0$ in the upper halfspace model. The latter are known as *horospheres*; in the ball model they appear as Euclidean spheres tangent to $\partial\mathbf{B}^{n+1}$.

In the ball model, a sphere of hyperbolic radius r centered at $\mathbf{0}$ is quickly seen to have Euclidean radius $\tanh(r/2)$, hence (multiply by the conformal factor and raise to the n^{th} power), n-area proportional to $\sinh^n(r)$. The first variation of area formula above then gives it constant mean curvature $H \equiv n \coth(r)$. In the upper halfspace model, the horospheres are similarly found to have constant mean curvature $H \equiv n$.

The fact that spheres in $^{n+1}$ have mean curvature bounded away from zero as their radii tend to infinity — the lower bound is exactly n, the mean curvature of the horospheres — illustrates a key qualitative difference between the theories of constant mean curvature hypersurfaces of $^{n+1}$, and of $^{n+1}$, where a sphere of radius r has mean curvature n/r. This difference shows up again when we consider *Delaunay* hypersurfaces in $^{n+1}$ (Lemma 6.3). Indeed, cylinders of radius r have mean curvature $H \equiv (n-1)\coth(r) + \tanh(r) > n$, and we will see that *cylindrically bounded* Delaunay hypersurfaces in $^{n+1}$ exist only when $H > n$. This is why the latter inequality appears as an important hypothesis in all our main results (e.g., Theorems 1.1 and 1.2). The theory of constant mean curvature hypersurfaces with $H \leq n$ is more like the theory of *minimal* ($H = 0$) hypersurfaces in Euclidean space than the constant mean curvature theory of [11] and [16] which guides the present work.

(2.6) Two elliptic equations. In the Minkowski space model, a constant mean curvature hypersurface Σ^n in $^{n+1}$ may be regarded as a spacelike codimension-two submanifold of $^{1,n+1}$. Two elliptic PDE's associated with this situation will help us get height estimates for constant mean curvature graphs in Lemma 3.3 and § 5.4. We sketch their derivations here.

Let ν be a unit vector field normal to Σ (and tangent to $\mathcal{S}$). Then the position vector field X on Σ forms, with ν, an orthonormal pair spanning the normal bundle to Σ in $^{1,n+1}$. These complete any local orthonormal *tangent* frame field $\tau_1, \tau_2 \ldots, \tau_n$ on Σ to an orthonormal frame for the pullback to Σ of the $^{1,n+1}$ tangent bundle. It turns out the connection Laplacian on this bundle induces a simple endomorphism of the normal bundle:

$$\Delta_\Sigma X = nX - H\nu$$

(2.7)

$$\Delta_\Sigma\,\nu \;\;=\;\; -HX + |\mathbf{A}|^2\nu \;\;.$$

Here H and $\mathbf{A}$ are respectively the (constant) mean curvature and second fundamental form of Σ. The Laplacian Δ_Σ can be written as

$$\Delta_\Sigma := \; D_iD_i - D_{(D_i\tau_i)^\top} \;,$$

where D_v is the standard derivative along $v \in {}^{n+2}$ (D_i abbreviates the case $v = \tau_i$), and "$\top$" signifies projection onto $T\Sigma$.

To obtain equations (2.7), compute at a single point, choosing a "normal coordinate" frame $\{\tau_i\}$ so that $(D_i\tau_j)^\top$ vanishes there for each $i, j = 1, \ldots, n$. Write

$$h_{ij} := -\nu \cdot D_i\tau_j = \tau_j \cdot D_i\nu = \tau_i \cdot D_j\nu$$

for the components of $\mathbf{A}$, (hence $H = \Sigma h_{ii}$) and observe that $D_iX = \tau_i$. Using the summation convention one then has

$$\Delta_\Sigma X \;=\; D_i\tau_i \;=\; (\nu \cdot D_i\tau_i)\nu - (X \cdot D_i\tau_i)X \;=\; -h_{ii}\nu + (\tau_i \cdot D_iX)X \,,$$

and the first equation in (2.7) follows immediately. (Geometrically, this equation establishes that the mean curvature vector $\Delta_\Sigma X$ of Σ, as a codimension-two submanifold of Lorentz space, is the sum of its mean curvature vector as a hypersurface in $\mathcal{S}$, and that of $\mathcal{S}$ itself in ${}^{1,n+1}$.)

For the second equation, first check that the *normal* components of $D_i\nu$ automatically vanish on Σ, so that

$$D_iD_i\nu \;=\; D_i\left((\tau_j \cdot D_i\nu)\tau_j\right) \;=\; D_i(h_{ij}\tau_j) \;.$$

But recall that the Codazzi equations for hypersurfaces of space forms imply $D_ih_{jk} \;=\; D_jh_{ik}$ for all $i, j, k = 1, \ldots, n$. So,

$$\begin{aligned} D_iD_i\nu \;\; &= (D_jh_{ii})\tau_j + h_{ij}D_i\tau_j \\ &= h_{ij}\left((\nu \cdot D_i\tau_j)\nu - (X \cdot D_i\tau_j)X\right) \\ &= -h_{ij}^2\nu + h_{ii}X \;, \end{aligned}$$

which establishes the desired fact.

3 Cylindrical boundedness and topological obstructions

In this section we generalize Meeks' cylindrical boundedness theorem [16] to constant mean curvature surfaces in ^{3}. Our method of proof, which also applies in ^{3}, further simplifies that given in [11]. Throughout this section A will denote a properly embedded annulus ($\approx S^1 \times [0, \infty)$) in ^{3} with boundary ∂A a simple closed curve.

Lemma 3.1 *Let γ_R be a circle of radius R in a geodesic plane in ^{3}. Let N_ε denote the closed ε-tubular neighborhood of γ_R for some fixed $\varepsilon < R$. Suppose Σ is a compact connected oriented surface of positive mean curvature in N_ε with $\partial\Sigma \subset \partial N_\varepsilon$. If Σ does not separate N_ε, then its mean curvature is somewhere less than or equal to that of a sphere $S_\varepsilon \subset {}^3$ with radius ε.*

Proof. We first show that the map $i_*\colon \pi_1(\Sigma) \to \pi_1(N_\varepsilon)$ induced by the inclusion map is the trivial map. Since Σ does not separate N_ε, there is a loop α in N_ε intersecting Σ transversely in a single point, and homotopic to some nonzero multiple of γ_R. Thus, we may assume, after orienting γ_R, that Σ has positive intersection number with γ_R. If $i_*\colon \pi_1(\Sigma) \to \pi_1(N_\varepsilon)$ is non-trivial, there is a simple oriented loop β on Σ homotopic to a nonzero multiple of γ_R. The homology class of β in $H_1(N_\varepsilon)$ must therefore have nonzero intersection number with Σ. However, since the normal bundle to Σ is trivial, we can push the curve β off Σ to obtain a homotopic curve $\widetilde{\beta}$ that is disjoint from Σ. Since $\widetilde{\beta}$ is homologous to β, this is a contradiction. Hence $i_*\colon \pi_1(\Sigma) \to \pi_1(N_\varepsilon)$ is trivial.

Let $\Pi\colon \widetilde{N_\varepsilon} \to N_\varepsilon$ denote the universal Riemannian covering space of N. Since $i_*\colon \pi_1(\Sigma) \to \pi_1(N_\varepsilon)$ is trivial, Σ lifts to a surface $\widetilde{\Sigma} \subset \widetilde{N_\varepsilon}$. Since

$\widetilde{N}_\varepsilon$ is simply connected and $\partial\widetilde{\Sigma} \subset \partial\widetilde{N}_\varepsilon$, the surface $\widetilde{\Sigma}$ separates $\widetilde{N}_\varepsilon$ into two components. If the closure of either component were compact, then Σ would be the boundary of a compact region of N_ε that is the image of this component under projection by Π. This contradicts our hypothesis that Σ does not separate N_ε and so $\widetilde{\Sigma}$ separates the two ends of $\widetilde{N}_\varepsilon$.

Let $\widetilde{\gamma}_R\colon \; \to \widetilde{N}_\varepsilon$ be a parametization of the proper arc covering γ_R. For each t let $S(t)$ denote the sphere of radius ε in $\widetilde{N}_\varepsilon$ with center $\widetilde{\gamma}_R(t)$. Let $t_0 = \min\{t \mid S(t) \cap \widetilde{\Sigma} \neq \emptyset\}$, $t_1 = \max\{t \mid S(t) \cap \widetilde{\Sigma} \neq \emptyset\}$ and let $p_i \in S(t_i) \cap \widetilde{\Sigma}$ for $i = 1, 2$, respectively. A simple geometric comparison of principal curvatures then shows that the mean curvature of $\widetilde{\Sigma}$ at one of the points, p_1 or p_2, is less than the mean curvature of the sphere S_ε. (In this comparison we use the fact that if $S(t_1)$ is contained in one of the closures of a component of $\widetilde{N}_\varepsilon - \widetilde{\Sigma}$, then $S(t_2)$ is contained in the closure of the other component.) $\square$

Lemma 3.2 *Let $A \subset {}^3$ be a properly embedded annulus with mean curvature function greater than a constant $H > 2$. Let $\varepsilon >$ arctanh$(\frac{2}{H})$, the radius of a sphere in ^{3} with constant mean curvature H. Let P_+ and P_- be planes of distance ε from a geodesic plane P in ^{3} and let Π_+ and Π_- denote the two closed halfspaces determined by these planes that do not contain P. Then $\Pi_+ \cap A$ or $\Pi_- \cap A$ consists entirely of compact components.*

Proof. Without loss of generality, assume P_- and P_+ are transverse to A. We will argue by contradiction. If $\Pi_+ \cap A$ and $\Pi_- \cap A$ each contain non-compact components, then there are proper arcs $\alpha_+ \subset P_+ \cap \mathrm{Int}(A)$ and $\alpha_- \subset P_- \cap \mathrm{Int}(A)$ parametized by a half-open interval. Consider α_+ and α_- to be proper disjoint arcs on A and choose an embedded path β joining the end point of α_+ to the end point of α_-. The arc $\delta = \alpha_+ \cup \alpha_- \cup \beta$ is a proper embedded arc in $\mathrm{Int}(A)$ that separates A into two components, where one component is topologically an annulus that contains ∂A, and the other component is simply connected. Let D denote the closure of the simply connected component of $A - \delta$ (see Figure 1).

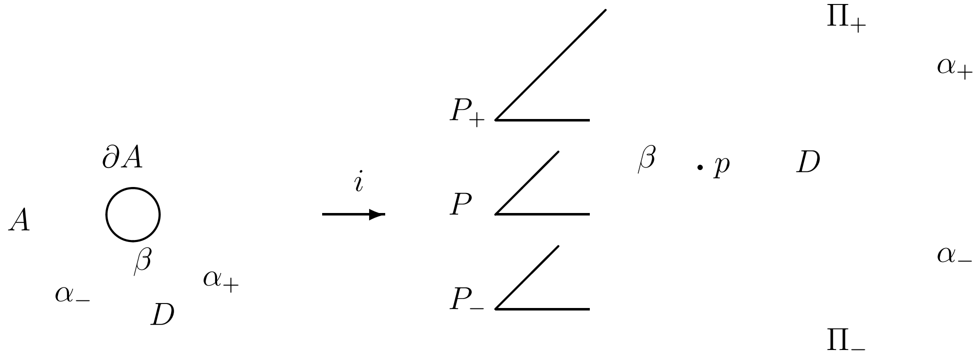

Figure 1.

Choose a ball B centered at some point $p \in P$ and of radius sufficiently large so that the arc β is contained in the interior of B. Since δ intersects P in a compact set contained in the disk $B \cap P$, we can choose a circle $\gamma_R \subset P$ centered at p, whose radius R equals the sum of the radius of B and the positive number ε . Clearly γ_R has linking number 1 with the proper arc δ.

Without loss of generality, assume now that γ_R intersects the simply connected surface D transversely, and that the boundary of the ε-tubular neighborhood N_ε of γ_R is also transverse to D. Since γ_R has linking number 1 with δ, γ_R has odd intersection with D. So one component Σ of $D \cap N_\varepsilon$ has odd intersection with γ_R and Σ does not separate N_ε. But then, Lemma 3.1 implies Σ has some point with mean curvature less than or equal to the mean curvature of a sphere of radius ε, namely, H . This contradiction completes our proof. $\square$

The following estimate is true for constant mean curvature hypersurfaces of $^{n+1}$, and is sharp on geodesic spheres.

Lemma 3.3 *Let $\Sigma \subset {}^{n+1}$ be an embedded compact hypersurface with constant mean curvature $H > n$, and boundary $\partial\Sigma$ contained in a totally geodesic hyperplane P. Then for any $x \in \Sigma$, $dist(x, P) \leq K(H) \equiv 2\,\mathrm{arctanh}(\frac{n}{H})$.*

Proof. Since Σ is compact we can choose a point $p \in \Sigma$ of maximal distance d from the hyperplane P. Let γ be the unit speed geodesic with $\gamma(0) \in P$ and $\gamma(d) = p$. Note that γ is orthogonal to P at $\gamma(0)$. Let $\mathbf{Y}$ be the Killing field that generates the unit speed family of hyperbolic translations h_t along γ with h_0 the identity (see Section 2.1). Let R_t: ${}^{n+1} \to {}^{n+1}$ be hyperbolic reflection in the geodesic hyperplane $P_t = h_t(P)$.

Let $\Sigma_t = \bigcup_{t' \geq t}(P_{t'} \cap \Sigma)$. The Alexandrov reflection principle implies that for $t \geq \frac{d}{2}$, $R_t(\Sigma_t) \cap \Sigma = \partial\Sigma_t$, and that this intersection is transverse. It follows that the surface $S = h_{-\frac{d}{2}}(\Sigma_{\frac{d}{2}})$ is a graph with respect to the Killing coordinate t over a domain $\Omega \subset P$.

We claim that S is also a graph over Ω with respect to the distance function s from P. In fact, it suffices to show that S is a graph with respect to the Killing coordinate associated to any unit speed geodesic γ that meets $\Omega \subset P$ orthogonally, since along γ the Killing coordinate t coincides with s, and in general, $t \leq s$. But this latter claim follows from the Alexandrov reflection argument used previously.

It remains to show the distance function s from P satisfies $s \leq \mathrm{arctanh}(\frac{n}{H})$ on the graph S. To do this, we model ${}^{n+1}$ by the unit sphere $\mathcal{S}$ in Minkowski space ${}^{1,n+1}$ (see discussion in Section 2.3).

We may assume P is the geodesic hyperplane $\{x_1 \equiv 0\}$. In these coordinates, the distance from P is simply $s = \mathrm{arcsinh}(x_1)$. By the above, S is a graph with respect to x_1, and $x_1 = 0$ on the boundary. Also, the first component of the unit normal ν on a graph satisfies $\nu_1 \geq 0$, and in general $\nu_1 \leq \sqrt{1 + x_1^2} = \sqrt{1 + \sinh^2 s} = \cosh s$.

Now by the Cauchy-Schwartz inequality we have $n|\mathbf{A}|^2 \geq H^2$, and so by equations 2.7,

$$\Delta(Hx_1 - n\nu_1) = (n|\mathbf{A}|^2 - H^2)\nu_1 \geq 0.$$

The maximum principle then implies

$$Hx_1 - n\nu_1 \leq (Hx_1 - n\nu_1)\Big|_{\partial S} \leq 0\,;$$

that is

$$\sinh s = x_1 \leq \frac{n}{H}\nu_1 \leq \frac{n}{H}\cosh s\,,$$

which immediately gives the required estimate. $\square$

Corollary 3.4 *Suppose $\Sigma \subset {}^{n+1}$ is an embedded compact hypersurface of constant mean curvature $H > n$. Then the distance from Σ to the convex hull $\mathcal{H}(\partial\Sigma)$ of the boundary of Σ is less than $K(H) \equiv 2\,\mathrm{arctanh}(\frac{n}{H})$.*

Proof. Suppose $p \in (\Sigma - \mathcal{H}(\partial\Sigma))$ and $d = dist(p, \mathcal{H}(\partial\Sigma))$. Let $q \in \partial(\mathcal{H}(\partial\Sigma))$ be a closest point to p, let γ be the geodesic line segment joining p and q, and denote by P_q the geodesic plane orthogonal to γ at q. Let Π denote the halfspace with boundary P_q and that contains the p. Then $\Pi \cap \Sigma$ is a compact hypersurface with boundary contained in P_q. Since $dist(p, P_q) = d$, Lemma 3.3 shows that $d \leq K(H)$. $\square$

Remark 3.5 *The corresponding result holds in Euclidean space, replacing the constant $K(H)$ by $2n/H$.* (This generalizes a result of Serrin [19].)

We now prove the main result of this section.

Theorem 3.6 (Cylindrical Boundedness) *Suppose A is a properly embedded annulus in ^{3} of constant mean curvature $H > 2$. Then for any point $p \in \partial A$ there exists a unique geodesic ray γ_p through p such that A stays a bounded distance C from γ_p. The number C depends only on the diameter of ∂A and on H.*

Proof. Choose a point $p \in \partial A$ and a divergent sequence $\{p_1, \ldots, p_n, \ldots\} \subset A$. Let γ_i denote the unit speed geodesic joining p to p_i. Since the unit sphere in the tangent space to ^{3} at p is compact and the geodesics γ_i are determined by their initial velocities, a subsequence of $\{\gamma_i\}$ converges to a unit speed geodesic ray γ. We now show A stays a bounded distance from γ and that this distance can be estimated from above by a constant depending only on its mean curvature H and the diameter of ∂A. Clearly any geodesic γ satisfying these hypotheses is unique.

Let $C(\gamma, d)$ denote the d-tubular neighborhood of γ where d is the diameter of ∂A. For this choice of d, we have $\partial A \subset C(\gamma, d)$. Suppose P is a totally geodesic plane tangent to $C(\gamma, d)$. Then P intersects $\partial C(\gamma, d)$ at a unique point. Let P_+ and P_- be the planes of constant distance ε from P where $\varepsilon > \operatorname{arctanh}(\frac{2}{H})$. Let Π_+ and Π_- be halfspaces defined as in the similar situation described in Lemma 3.2. Assume that P_+, P_- are indexed so that $(\,^3 - \Pi_+)$ contains γ.

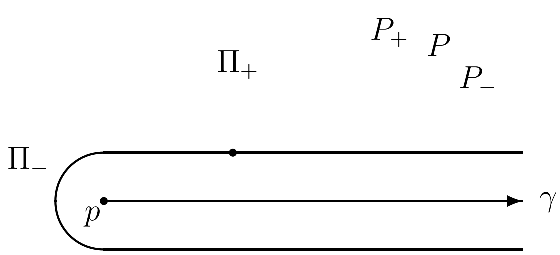

Figure 2a:

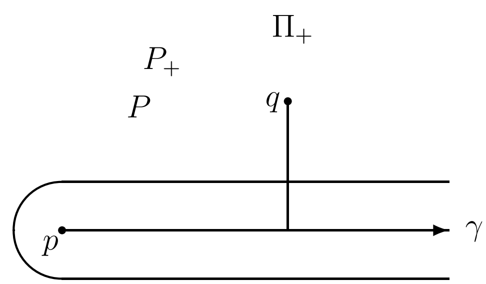

Figure 2b:

We first claim $\Pi_+ \cap A$ consists entirely of compact components. For, if to the contrary $\Pi_+ \cap A$ contains a noncompact component, Lemma 3.2 makes every component of $\Pi_- \cap A$ compact. Since $\partial(\Pi_- \cap A) \subset \partial\Pi_- = P_-$, and P_- is convex toward P (the interior convex hull of P_- is equal to $^3 - \Pi_-$), Corollary 3.1 shows $\Pi_- \cap A$ is within distance $K(H)$ of P_-. So the limit set of A is contained in the limit set S_+ of Π_+. (Here, we view the limit set of a subset of ^{3} as a subset of S_∞ in the ball model.) On the other hand, the sequence $\{p_i\} \subset A$ has the limit point p_∞ of γ in its limit set. But γ and P diverge in ^{3} and hence p_∞ is not contained in S_+. This contradiction proves every component of $\Pi_+ \subset A$ is compact.

Let $q \in A$ be a point in the complement of $C(\gamma, d)$ and let α be the geodesic from q to γ with length $dist(q, \gamma)$. For this q we pick a particular geodesic plane P to apply the above estimates. Specifically, choose P to be the geodesic plane that is tangent to $C(\gamma, d)$ at $\alpha \cap \partial C(\gamma, d)$. Since P_+ is convex toward P, Corollary 3.1 again implies $dist(q, P) \leq K(H)$. Therefore,

$$dist(q, \gamma) \leq dist(q, P_+) + dist(P_+, P) + dist(P, \gamma) \leq K(H) + \varepsilon + d\,,$$

and this estimate proves the theorem. $\square$

We can now prove Theorem 1.1 as stated in the Introduction, assuming without loss of generality that the ambient curvature c equals -1.

Proof of Theorem 1.1. Suppose $\Sigma \subset {}^3$ is homeomorphic to a closed surface punctured in one point. By Theorem 3.1, the annular end of Σ stays a bounded distance C from some geodesic ray γ with $\gamma(0) \in \Sigma$. Hence the entire surface stays a bounded distance from γ.

Let P be a geodesic plane orthogonal to γ at $\gamma(d)$ where d is greater than the constant $K(H)$ of Lemma 3.3. Let Π be the halfspace determined by P and containing the point $\gamma(0)$. Then $\Pi \cap \Sigma$ is a compact surface with boundary in P, and $dist(\gamma(0), P) > K(H)$ contradicting Lemma 3.3. This proves part 1 of the theorem.

For the remainder of the proof we recall a theorem of Hsiang [9], which is proved using the Alexandrov reflection principle (cf. also § 5).

Hsiang's Theorem [9]. *Suppose that* $\Sigma \subset {}^{n+1}$ *is a complete properly embedded hypersurface of constant mean curvature that is a bounded distance from a geodesic hyperplane. Then* Σ *is invariant under geodesic reflection in the hyperplane. If* Σ *is a bounded distance from a geodesic, then* Σ *is a Delaunay hypersurface of revolution.*

Suppose now that Σ is homeomorphic to a closed surface with two points removed. Theorem 3.1 says the annular ends A_1 and A_2 of Σ stay a bounded distance from geodesic rays γ_1 and γ_2, respectively. Given two geodesic rays in ^{3} there is a unique geodesic γ to which γ_1 and γ_2 converge at infinity. It follows that Σ stays a bounded distance from γ. Hsiang's theorem implies that Σ is a surface of revolution, which proves part 2 of the theorem.

Finally, assume Σ is homeomorphic to a closed surface with three points removed. Let A_1, A_2, A_3 be the annular ends of Σ and let $\gamma_1, \gamma_2, \gamma_3$ be the geodesic rays given in Theorem 3.1 such that A_i is a bounded distance from γ_i for $i = 1, 2, 3$. Note that the limit points at infinity of these geodesic rays must be distinct. Otherwise, the rays stay a bounded distance from some fixed geodesic, and Hsiang's Theorem would make Σ a surface of revolution with only two ends. So there exists a unique geodesic plane P in ^{3} to which the geodesic rays $\gamma_1, \gamma_2, \gamma_3$ are asymptotic, and Σ stays a bounded distance from P. Hsiang's Theorem now makes Σ invariant under reflection in the plane P. Furthermore, the argument in the proof of Lemma 3.3 shows, if s is the distance function from P, then the halves of Σ above and below P are graphs over P relative to s. This completes the proof of Theorem 1.1. $\square$

4 Conservation laws and momenta

Using the first variation formula and the symmetry group $O^+(1, n+1)$ of hyperbolic space, we derive *conservation laws* for a constant mean curvature hypersurface $\Sigma \subset {}^{n+1}$. These laws permit, for example, the assignment to each end of Σ a *momentum* in the Lie algebra $o(1,\, n+1)$, and force the sum of these momenta (over all ends) to vanish. We also interpret these momenta geometrically, showing, for instance, how they assign geodesic "worldlines" to certain cycles on Σ (cf. [14] for a discussion of the Euclidean case).

Given a hypersurface $\Sigma \subset {}^{n+1}$ with constant mean curvature H, consider an $(n-1)$-cycle $\Gamma \subset \Sigma$ and any "cap" n-chain $K \subset {}^{n+1}$ with boundary $\partial K = \Gamma$. Let η and ν denote the "outer" unit conormal and normal for Γ and K, respectively. In other words, if ν_Σ is the outer normal to Σ (oriented oppositely to the mean curvature vector), then the orientations $(\Sigma\,,\, \nu_\Sigma)$, $(K\,,\, \nu)$ and $(\Gamma\,,\, \eta\,,\, \nu_\Sigma)$ all agree with the righthanded orientation on ${}^{n+1}$.

Each Lie algebra element $Y \in o(1,\, n+1)$ generates a one-parameter subgroup $\exp(tY)$ in $O^+(1,\, n+1)$. The velocity of the corresponding isometric flow on ${}^{n+1}$ is a Killing vector field $\mathbf{Y}$. This process inverts, yielding an isomorphism between $o(1,\, n+1)$ and the Lie algebra of Killing vector fields on ${}^{n+1}$.

Theorem 4.1 (Conservation Laws) *On any hypersurface* $\Sigma \subset {}^{n+1}$ *with constant mean curvature* H*, there is a natural cohomology* momentum class *with coefficients in the dual Lie algebra,*

$$\mu \in H^{n-1}(\Sigma) \otimes o(1,\, n+1)^*$$

defined by the area $-H\cdot$ volume flux *induced by* $\mathbf{Y}$ *through* (Γ, K)*:*

$$\langle \mu([\Gamma]),\, Y\rangle = \int_\Gamma \eta \cdot \mathbf{Y} - H \int_K \nu \cdot \mathbf{Y}\,.$$

Furthermore, as Σ *is translated in* ${}^{n+1}$ *by the action of* $g \in O^+(1,\, n+1)$*, the momentum class transforms via the co-Adjoint representation*

$$g^*\mu = \mathrm{Ad}^*(g)\cdot \mu\,.$$

Proof. To prove μ is well-defined in $H^{n-1}(\Sigma)\otimes o(1,\ n{+}1)^*$, it suffices to show, for fixed $\mathbf{Y}$, that the flux integral depends solely upon the homology class $[\Gamma] \in H_{n-1}(\Sigma)$. Since the flux integral is linear in its domains of integration, we only need to show it vanishes when (Γ, K) is null homologous, i.e. when there exists a relative $(n, n+1)$-chain $(S, U) \subset (\Sigma, {}^{n+1})$ such that

$$\Gamma = \partial S \ \text{ and } \ K = \partial U - S.$$

In this case the *area* $- H \cdot$ *volume* flux induced by $\mathbf{Y}$ can be rewritten

$$\begin{aligned} \langle \mu([\Gamma]), Y \rangle &= \int_{\partial S} \eta \cdot \mathbf{Y} - H \int_K \nu \cdot \mathbf{Y} \\ &= \int_{\partial S} \eta \cdot \mathbf{Y} \,+\, H \int_S \nu_\Sigma \cdot \mathbf{Y} \,-\, H \left(\int_S \nu_\Sigma \cdot \mathbf{Y} + \int_K \nu \cdot \mathbf{Y} \right) \end{aligned}$$

But, combining the *first variation* formulas (cf. [20]) as in [11] for *area*

$$\delta_{\mathbf{Y}} |S| = \int_S \text{div } \mathbf{Y} = \int_{\partial S} \eta \cdot \mathbf{Y} + H \int_S \nu_\Sigma \cdot \mathbf{Y}$$

and for *volume*

$$\delta_{\mathbf{Y}} |U| = \int_U \text{ DIV } \mathbf{Y} = \int_S \nu_\Sigma \cdot \mathbf{Y} + \int_K \nu \cdot \mathbf{Y}\,,$$

we have

$$\begin{aligned} \langle \mu([\Gamma]), Y \rangle &= \int_S \text{ div } \mathbf{Y} - H \int_U \text{ DIV } \mathbf{Y} \\ &= \delta_{\mathbf{Y}}(|S| - H|U|) \end{aligned}$$

as the pair (S, U) is deformed along $\mathbf{Y}$; but the latter vanishes, since $\mathbf{Y}$ is assumed to be a Killing vector field, preserving area and volume.

The transformation property follows from a change of variables in the flux integral

$$\begin{aligned}
\langle \mu(g_\#[\Gamma]), \mathrm{Ad}(g)\cdot Y\rangle &= \int_{g(\Gamma)} g_*\eta\cdot g_*\mathbf{Y} - H\int_{g(K)} g_*\nu\cdot g_*\mathbf{Y}\\
&= \int_\Gamma \eta\cdot\mathbf{Y} - H\int_K \nu\cdot\mathbf{Y}\\
&= \langle \mu([\Gamma]), Y\rangle
\end{aligned}$$

using the fact that the Lie algebra element $\mathrm{Ad}(g)\cdot Y$ extends to the Killing vector field $g_*\mathbf{Y}$ on $\;^{n+1}$, which is the push-forward of $\mathbf{Y}$ by the derivative of g. □

As a direct consequence, we observe that the ends of a constant mean curvature hypersurface $\Sigma \subset \;^{n+1}$ must "balance".

Corollary 4.2 *On a complete, properly embedded hypersurface* $\Sigma \subset \;^{n+1}$ *with constant mean curvature and finitely many ends, any end* E *has a well-defined momentum* $\mu_E \in o(1, n+1)^*$*, and the sum of the momenta corresponding to the ends is zero.*

Proof. Define $\mu_E = \mu([\Gamma])$, where Γ is an appropriately oriented cycle separating E from the "compact part" of Σ. To prove the second statement, consider cycles Γ_i of this type, where i indexes the ends of Σ. Defining $\mu_i = \mu([\Gamma_i])$, we now simply observe that

$$\mu_1 + \cdots + \mu_k = \mu\left(\sum [\Gamma_i]\right) = 0$$

since the latter sum bounds a compact domain in Σ. □

It is customary to identify the Lie algebra $o(1,\, n+1)$ and its dual $o(1,\, n+1)^*$ with the Lie algebra $\wedge^2 \;^{1,\,n+1}$ of antisymmetric (with respect to the Lorentz form $Q = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2$) linear maps on $\;^{1,\,n+1}$

$$o(1, n+1)^* \cong o(1, n+1) \cong \wedge^2 \;^{1,n+1} \cong \{A \in M_{n+2}(\;) \mid A^t Q + QA = 0\}.$$

The dual pairing is then given by the Killing form $\langle A, Z\rangle = \frac{1}{2}Trace(AZ)$. With these identifications, a little computation shows

$$\mu([\Gamma]) = \int_\Gamma \eta \wedge X - H\int_K \nu \wedge X$$

where X is the position vector on $\ ^{1,n+1}$, and η and ν are regarded as vectors in $\ ^{1,n+1}$ via the "Minkowski-space model" embedding $\ ^{n+1} \subset\ ^{1,n+1}$ discussed in § 2.3. Furthermore, the co-Adjoint action can be identified with the usual action of

$$O^+(1, n+1) \cong \{U \in M_{n+2}(\ ) \mid U^t BUB = I\}$$

on $o(1,\, n+1)$ by conjugation. Thus the geometric invariants of $\mu([\Gamma])$ are simply its conjugacy invariants, namely its (generalized) eigenspaces and eigenvalues.

Before we work these out explicitly (we do so later for constant mean curvature surfaces in $\ ^3$), recall the decomposition of $O^+(1, n+1)$ and its Lie algebra $o(1,\, n+1)$ into *elliptic, parabolic,* and *hyperbolic* elements.

Proposition 4.3 *Consider a nonzero element $\mu \in o(1,\, n+1)$. We say μ is* elliptic $(\mathcal{E})$, parabolic $(\mathcal{P})$, *or* hyperbolic $(\mathcal{H})$, *respectively, if one of the following four correspondingly lettered equivalent conditions holds:*

1. $\exp(\mu)$, *viewed as a conformal automorphism of $\overline{\mathbf{B}^{n+1}}$, has:*

   $\mathcal{E}^{(1)}$: *a fixed point on the interior,*

   $\mathcal{P}^{(1)}$: a unique *fixed point on the boundary sphere S_∞,*

   $\mathcal{H}^{(1)}$: two *fixed points on S_∞.*

2. μ *, extended to a vector field on $\overline{\mathbf{B}^{n+1}}$, has:*

   $\mathcal{E}^{(2)}$: *a zero on the interior,*

$\mathcal{P}^{(2)}$: *a unique zero on* S_∞,

$\mathcal{H}^{(2)}$: *two zeros (a source and a sink) on* S_∞.

3. μ, *viewed as an antisymmetric endomorphism of* $\mathbb{R}^{1,n+1}$, *has:*

   $\mathcal{E}^{(3)}$: *a ray of eigenvectors in the interior* $\{v \mid Q(v) < 0\}$ *of the null cone,*

   $\mathcal{P}^{(3)}$: *one ray of eigenvectors on the null cone* $\{v \mid Q(v) = 0\}$,

   $\mathcal{H}^{(3)}$: *two linearly independent rays of null eigenvectors (with real eigenvalues* $\pm m \neq 0$*).*

4. μ, *as an* $(n+2) \times (n+2)$ *matrix has the property that:*

   $\mathcal{E}^{(4)}$: *each column* (μ_{*j}) $(j \geq 1)$ *satisfies* $(\mu_{0j})^2 < \sum_{i=1}^{n+1} (\mu_{ij})^2$,

   $\mathcal{P}^{(4)}$: *neither* $\mathcal{E}^{(4)}$ *nor* $\mathcal{H}^{(4)}$,

   $\mathcal{H}^{(4)}$: *there is a column* (μ_{*j}) $(j \geq 1)$ *such that* $(\mu_{0j})^2 > \sum_{i=1}^{n+1} (\mu_{ij})^2$.

Proof. The equivalence of (1) and (2) is clear since a zero of μ is a fixed point of $\exp(\mu)$; (1) gives the usual notion of ellipticity, etc. for group elements. Items (1) and (2) are equivalent to (3) by the usual identification of $\overline{\mathbf{B}^{n+1}} = S_\infty \cup \mathbf{B}^{n+1}$ with the projectivization of the future cone $\{v \mid Q(v) \leq 0\}$. These are then equivalent to (4) by the following type of argument (we do case $\mathcal{H}^{(4)}$).

Without loss of generality suppose

$$(\mu_{0,n+1})^2 > \sum_{i=1}^{n+1} (\mu_{i,n+1})^2 .$$

Now consider the hyperplane $(\mathbf{e}_{n+1})^\perp \subset \mathbb{R}^{1,n+1}$, which defines a totally geodesic $\overline{\mathbb{H}^n} \subset \overline{\mathbb{H}^{n+1}}$. Using the relationship between the Minkowski space and ball models (§ 2.2), one readily finds this inequality equivalent to the fact that the vector field

$$\sum_{i,j=0}^{n+1} \mu_{ij} x_j \mathbf{e}_i$$

has a source and a sink on S_∞ (one on each side of $(\mathbf{e}_{n+1})^\perp$) corresponding to the $\pm m$ null eigenvectors of μ.

We leave the other cases to the reader. $\square$

Remark 4.4 *If $\mu \in o(1,\, n+1)$ is hyperbolic, it defines a unique (oriented) geodesic $L \subset \;^{n+1}$ joining the source and sink mentioned above. L is the intersection of the oriented $\pm m$ eigenvector plane with the Lorentzian unit sphere $\mathcal{S} \subset \;^{1,n+1}$* (see § 2.3). *In fact, μ can be put into canonical form:*

$$\begin{bmatrix}
0 & m & 0 & . & . & . & & 0 & 0 \\
m & 0 & & & & & & & 0 \\
0 & & 0 & s_1 & & & & & \\
. & & -s_1 & 0 & & & & & . \\
. & & & & 0 & s_2 & & & . \\
. & & & & -s_2 & 0 & & & . \\
 & & & & & & . & & \\
0 & & & & & & & . & 0 \\
 & & & & & & & 0 & s_{[n/2]} \\
0 & 0 & 0 & . & . & . & 0 & -s_{[n/2]} & 0
\end{bmatrix}$$

with respect to an "orthonormal" basis $\{u_0, u_1, \ldots, u_{n+1}\}$ for $\;^{1,n+1}$, i.e., $Q(u_0,u_0) = -1$, $Q(u_i,u_i) = 1$ $(i > 0)$, $Q(u_\alpha, u_\beta) = 0\,,(\alpha \neq \beta)$.

We call L the worldline*, m the* mass*, and the invariants $s_1, s_2, \ldots, s_{[n/2]}$ the* spins *of μ.*

To justify this remark, note that from $\mathcal{H}^{(3)}$ of Proposition 4.3 we have

$$\begin{bmatrix} m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & \mathbf{R} \end{bmatrix}$$

where $\mathbf{R} \in o(n)$ is some antisymmetric $n \times n$ matrix. The existence of $\{u_2, u_3, \ldots, u_n\}$ putting $\mathbf{R}$ into a diagonal matrix of 2×2 antisymmetric blocks

$$\begin{bmatrix} 0 & s_i \\ -s_i & 0 \end{bmatrix}$$

is then standard linear algebra. Now choose null eigenvectors v_+ and v_- corresponding to $\pm m$ such that the remaining vectors u_0, u_1 defined by $u_0 + u_1 = v_+$ and $u_0 - u_1 = v_-$ are orthonormal.

Remark 4.5 *In the case $n = 2$ (corresponding to $\Sigma \subset {}^{3}$), we have simply*

$$\mu \cong \begin{bmatrix} 0 & m & 0 & 0 \\ m & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & -s & 0 \end{bmatrix},$$

which can be viewed as the initial velocity of a screw-motion along the geodesic axis L with translation and rotation "speeds" m and s respectively.

To be completely explicit with regard to this last remark, if

$$\mu = \begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ w_1 & 0 & r_3 & -r_2 \\ w_2 & -r_3 & 0 & r_1 \\ w_3 & r_2 & -r_1 & 0 \end{bmatrix},$$

is hyperbolic, then

$$m^2 = \frac{1}{2}\left(B + \sqrt{B^2 + 4C}\right), \tag{4.1}$$

$$s^2 = -\frac{1}{2}\left(B - \sqrt{B^2 + 4C}\right), \tag{4.2}$$

where, regarding $\mathbf{w} = (w_1, w_2, w_3)$ and $\mathbf{R} = (r_1, r_2, r_3)$ as vectors in $\mathbb{R}^3$ with the standard dot product, we have

$$B = \mathbf{w} \cdot \mathbf{w} - \mathbf{R} \cdot \mathbf{R} = \langle \mu, \mu \rangle, \quad C = (w_1^2, w_2^2, w_3^2) \cdot (r_1^2, r_2^2, r_3^2)\,.$$

We let the reader work out the analogous formulae when $n > 2$.

Remark 4.6 *On the most important examples, the Delaunay surfaces of revolution, where* $[\Gamma]$ *is represented by a cross-section, one finds that the spins all vanish, the world line* L *is the axis of revolution, and the mass* m *is computed by the* $\text{area} - H \cdot \text{volume}$ *flux generated by a unit speed dilation along the axis (cf. 6.3).*

Finally, we determine which $(n-1)$-cycles $\Gamma \subset \Sigma \subset \mathbb{R}^{n+1}$ give rise to hyperbolic momenta $\mu([\Gamma])$. (It seems natural to state the following result here, although the heart of its proof is really Theorem 5.3. For the definitions of "planar", "interior" and "exterior" homology classes, the reader should refer to § 5.1.)

Theorem 4.7 (Geometric Positive Flux Lemma) *Let* $[\Gamma]$ *be a planar homology class on a complete constant mean curvature* $\Sigma \subset \mathbb{R}^{n+1}$*. Then* $\mu([\Gamma])$ *is either hyperbolic or zero. In the former case,* μ *assigns* $[\Gamma]$ *a geodesic worldline* L *and a nonzero mass* m *(positive if* $[\Gamma]$ *is an interior class, negative if* $[\Gamma]$ *is an exterior class). If* $\mu([\Gamma]) = 0$*, then* $[\Gamma]$ *is an interior class, and bounds a subdomain of* Σ *that is a graph over the corresponding plane domain.*

Proof. We will use the "scalar" Positive Flux Lemma (5.3) to establish the inequality $\mathcal{H}^{(4)}$ from Proposition 4.2:

$$(\mu_{0j})^2 > \sum_{i=1}^{n+1} (\mu_{ij})^2\,, \quad (\text{some} \;\; j \geq 1)\,.$$

Actually, for purposes of proof, we may assume $j = n{+}1$ by relabelling. Thus, let $\Pi = (\mathbf{e}_{n+1})^{\perp} \cap \mathcal{S}$ in the Minkowski space model and rotate coordinates in Π so that $\mu_{i,n+1} = 0$ for each $i > 2$. We must show

$$(\mu_{0,n+1})^2 > (\mu_{1,n+1})^2 \ .$$

Consider the following two parabolic elements of $o(1, n+1)$

$$P_{\pm} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & . & & . & \pm 1 \\ . & . & & . & 0 \\ 0 & . & & . & 0 \\ 1 & \mp 1 & \cdots & 0 & 0 \end{bmatrix} .$$

These extend to Killing vector fields $\mathbf{P}_{\pm}$ on $^{n+1}$ which are perpendicular to Π and satisy $\mathbf{P}_{\pm} \cdot \nu_{\Pi} > 0$. Thus, the pairings

$$\begin{aligned} \langle \mu\,, P_+ \rangle &= \mu_{0,n+1} + \mu_{1,n+1} \\ \langle \mu\,, P_- \rangle &= \mu_{0,n+1} - \mu_{1,n+1} \end{aligned}$$

are both positive from Theorem 5.3. So their product $(\mu_{0,n+1})^2 - (\mu_{1,n+1})^2$ is positive, as needed for $\mathcal{H}^{(4)}$. $\square$

5 Positive flux lemma

Here we prove our "Positive Flux Lemma" (Theorem 5.3) for a properly embedded constant mean curvature surface Σ with compact boundary $\partial\Sigma$.

Roughly, it says that "planar" homology classes (defined in § 5.1) which are far enough from $\partial\Sigma$ must either be trivial, or have non-zero $area - H \cdot volume$ flux relative to certain Killing vector fields. We first establish the analogous fact in Euclidean space. We do so because the Euclidean case is interesting on its own, and because it displays the main ideas of the hyperbolic case without as many technical details.

In § 6, we will apply the Positive Flux Lemma to *cylindrically bounded* Σ to deduce rotational symmetry "at infinity." We hope this lemma will also prove useful in studying the "compact part" of constant mean curvature surfaces. As already observed (see Theorem 4.7, and [14] for the Euclidean case), the fact that a homology class has non-zero mass gives it a well-defined "worldline" which for a Delaunay end is just the axis of revolution. It should be true more generally that constant mean curvature surfaces stay close to their worldlines. For instance, one might hope to prove (at least for the case $n = 2$) that every properly embedded constant mean curvature surface stays near some "balanced diagram" like the examples in $\mathbb{R}^3$ constructed by N. Kapouleas [10].

(5.1) Configuration. We assume Σ comprises part of the boundary of some open set, into which the mean curvature vector of Σ points. Specifically, we assume that there exists an open, connected domain Ω with $\partial\Omega = \Sigma \cup \Sigma'$ and $\partial\Sigma = \Sigma \cap \Sigma'$, where Σ' is a smooth "cap". (We assume nothing about the geometry of Σ'.)

The homology class $[\Gamma]$ of Σ is called *planar* if the cycle Γ is a compact, transverse intersection between part of Σ and some (totally geodesic) hyperplane Π, and Γ is the (possibly disconnected) boundary of a component K of either $\Omega \cap \Pi$ or of $(\mathbb{H}^{n+1} - \Omega) \cap \Pi$. In the first case, or more generally whenever Γ is trivial in Ω, we will call $[\Gamma]$ an *interior* homology class, and in the second case (or whenever Γ is trivial in $(\ ^{n+1}-\Omega))$ we call $[\Gamma]$ an *exterior* class. (Note that the Mayer-Vietoris exact sequence implies that interior and exterior classes *span* the homology of $\Sigma \cup \Sigma'$, but there is no guarantee that *planar* classes do so.)

(5.2) Orientations. We retain the convention that on Σ the normal ν_Σ is the exterior normal to Ω, and require that on a planar cap K, the conormal η along Γ has positive inner product with the plane normal ν. The flux formulas (Theorem 4.1) for interior and exterior homology classes then differ by a sign in the cycle term:

$$\langle \mu([\Gamma]), Y\rangle = \int_\Gamma \eta\cdot \mathbf{Y} - H\int_K \nu\cdot\mathbf{Y} \qquad [\Gamma] \text{ interior planar}$$

$$\langle \mu([\Gamma]), Y\rangle = -\int_\Gamma \eta\cdot \mathbf{Y} - H\int_K \nu\cdot\mathbf{Y} \qquad [\Gamma] \text{ exterior planar}\,.$$

Our main objective for the remainder of this section is now to prove:

Theorem 5.3 (Positive Flux Lemma) *Let $[\Gamma]$ be a planar homology class. Let $\mathbf{Y}$ be an Killing vector field perpendicular to Π , with $\mathbf{Y}\cdot\nu_\Pi$ strictly positive on Π. If $[\Gamma]$ is an exterior planar homology class, then the flux $\langle\mu([\Gamma]),\mathbf{Y}\rangle$ is strictly negative, and thus Γ is nontrivial on Σ.*

If $[\Gamma]$ is an interior planar homology class, and if $dist(K,\Sigma')$ is sufficiently large (depending on H and the diameter of Γ), then either Γ is trivial on Σ or the flux $\langle\mu([\Gamma]),\mathbf{Y}\rangle$ is strictly positive. In the trivial case, Γ actually bounds a subset of Σ that is a graph (with respect to distance) above K.

In particular, only the planar homology classes near $\partial\Sigma$ and the trivial class can have momentum zero. If $\partial\Sigma$ is empty, every *non-trivial planar class has hyperbolic momentum and nonzero mass (cf. Theorem 4.7).*

Proof. For exterior planar classes, the definition of flux immediately yields the result, since $\eta\cdot\mathbf{Y}$ and $\nu\cdot\mathbf{Y}$ are non-negative and positive, respectively, on Γ and K.

The main content of our theorem therefore concerns *interior* planar classes. Our plan is to "blow a bubble" K_H inside Ω with mean curvature H which is bounded by Γ, and then to compare the flux through Γ relative to Σ with the flux relative to K_H (which must be zero since Γ bounds K_H).

Specifically, we will construct a one-parameter family of graphs K_λ over K with constant mean curvature $\lambda \in [0, H]$ and with $\partial K_\lambda = \Gamma$. Let R denote geodesic reflection across Π. Since the K_λ form a continuous family of constant mean curvature surfaces, the strong maximum principle implies that:

1. For $\lambda < H$, the graph K_λ intersects Σ and its reflected image $\widetilde{\Sigma} = R(\Sigma)$ transversely along Γ and $K_\lambda \cap (\Sigma \cup \widetilde{\Sigma}) = \Gamma$.

2. Either $K_H \subset \Sigma \cup \widetilde{\Sigma}$, or $K_H \cap (\Sigma \cup \widetilde{\Sigma}) = \Gamma$ and K_H is transverse to Σ and $\widetilde{\Sigma}$ along Γ.

In case 2 above when $K_H \subset \widetilde{\Sigma}$, the reflected image $R(K_H)$ is contained in Σ. Hence when $K_H \subset \Sigma \cup \widetilde{\Sigma}$, Γ is the boundary of a compact graph in Σ. On the other hand, if Γ does not bound a graph in Σ, the angle of Σ with the plane Π along Γ exceeds the angle of K_H (see Figure 3). Since $\mathbf{Y} \perp \Pi$, this means $\eta_{K_H} \cdot \mathbf{Y} < \eta_\Sigma \cdot \mathbf{Y}$ along Γ. (Here subscripts distinguish between the conormals to Σ and K_H.) Hence

$$
\begin{aligned}
(5.1) \qquad \langle \mu([\Gamma]), Y \rangle &= \int_\Gamma \eta_\Sigma \cdot \mathbf{Y} \, - \, H \int_{K_H} \nu \cdot \mathbf{Y} \\
&> \int_\Gamma \eta_{K_H} \cdot \mathbf{Y} - H \int_{K_H} \nu \cdot \mathbf{Y} = \langle \mu_{K_H}([\Gamma]), Y \rangle = 0.
\end{aligned}
$$

The inequality here is precisely what we sought; it remains to construct the graphs K_λ to complete the proof of the Positive Flux Lemma.

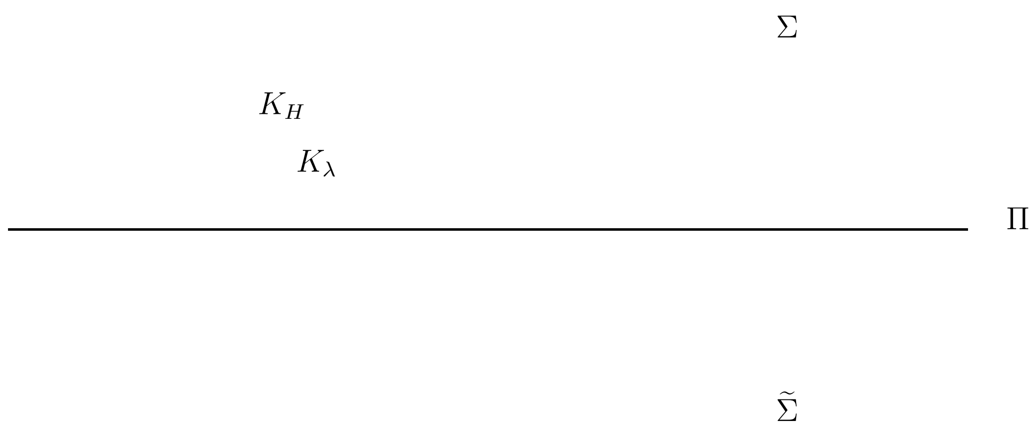

Figure 3.

We first carry out the construction in Euclidean space. In this setting, $\mathbf{Y}$ generates a translation perpendicular to the plane Π. We may assume Π is the $\{x_1 = 0\}$ plane and that its normal $\nu_\Pi = -\mathbf{e}_1 = \mathbf{Y}$. The family of graphs over K will be $K_\lambda = \{x_1 = u_\lambda(x_2, \ldots, x_{n+1})\}$, were the non-negative function u_λ solves the nonparametric constant mean curvature λ equation:

$$\begin{cases} \operatorname{div}\left(\frac{Du_\lambda(y)}{\sqrt{1+|Du_\lambda(y)|^2}}\right) = -\lambda & y \in K \\ \\ u_\lambda(y) = 0 & y \in \partial K = \Gamma\,. \end{cases} \tag{5.2}$$

for $0 \leq \lambda \leq H$.

Note that $u_0 \equiv 0$, so that $K_0 = K$. It follows from the implicit function theorem for elliptic PDE's that we can solve (5.2) for *small* $\lambda > 0$. The following sequence of estimates will enable us to continue this solution-family all the way through $\lambda = H$.

(5.4) Height estimates. Choose an origin in Π so that Γ lies in the ball of radius $R = diam(\Gamma)$. Consider the spherical graph $x_1 = \sqrt{R^2 - |y|^2}$, with non-parametric mean curvature $-n/R$, and non-negative boundary values on Γ. For $\lambda \le n/R$, it provides a barrier above the graph K_λ of u_λ. That is,

$$u_\lambda(y) \le R, \qquad 0 \le \lambda \le n/R. \tag{5.3}$$

Recall the estimate

$$\lambda x_1 \le n\nu_1\,, \tag{5.4}$$

which holds for a graph $x_1 = u(y)$ having mean curvature λ and upper normal ν. (This estimate is proven in [11] by the same method used in § 2.6 above for the analogous hyperbolic estimate: $\Delta(\lambda x_1 - n\nu_1) = (-\lambda^2 + n|\mathbf{A}|^2)\nu_1 \ge 0$ holds for Euclidean graphs as well.) Applying (5.4) to estimate x_1, we conclude that

$$u_\lambda(y) \le n/\lambda \le R, \qquad n/R \le \lambda \le H\,. \tag{5.5}$$

(5.5) Gradient estimates and barriers at the boundary. Note that the $u_\lambda(y)$ increase monotonically in λ: if $\lambda_1 < \lambda_2$ then $u_{\lambda_1}(y) \le u_{\lambda_2}(y)$, with strict (interior) inequality, since the maximum principle applies to equation (5.2). Estimate (5.4) combines with this monotonicity in λ to bound $\nu_1(y)$ below by a positive number depending on y in (the interior of) K, but not on λ. *The gradient $|Du_\lambda(y)|$ is therefore bounded uniformly in λ on compact subsets of K, as long as the graphs K_λ exist.*

Observe next that for $\lambda \le H$, our original surface $\Sigma \cap \{x_1 \ge 0\}$, *and* the "positive" half of its reflection $\tilde{\Sigma}$ through $\{x_1 = 0\}$, namely $\tilde{\Sigma} \cap \{x_1 \ge 0\}$, are barriers "above" the graphs of u_λ, provided that Σ' is more than R units away from K. (This is why Theorem 5.3 requires the distance between Σ' and K to be sufficiently large: we can thereby guarantee that any first point of tangential contact between $\partial\Omega$ (or the reflection $\partial\tilde{\Omega}$) and u_λ occurs on Σ

(or the reflection $\widetilde{\Sigma}$). The maximum principle then controls contact between Σ (and $\widetilde{\Sigma}$) and the graph K_λ of u_λ.)

At the boundary of K, since $\Sigma \cap \{x_1 \geq 0\}$ and $\widetilde{\Sigma} \cap \{x_1 \geq 0\}$ are barriers, we immediately get gradient estimates for the u_λ, independent of λ, wherever the conormal η of Σ is *not* perpendicular to Π along Γ. Of course, η may indeed be perpendicular to Π at many points. The following curvature estimates cover this contingency.

(5.6) Curvature estimates. One could apply geometric measure theory arguments to produce curvature estimates independently of gradient estimates: constant mean curvature graphs minimize (parametric) elliptic variational problems, hence have Holder continuous normals with norm depending only on the smoothness of $\partial K = \Gamma$ [3]. Such an estimate implies that one has functions parameterizing uniformly-sized neighborhoods $U(p)$ of points $p \in K_\lambda$ as Lipschitz graphs above the tangent planes to K_λ at p. Elliptic regularity theory then bounds the curvatures.

However, a certain amount of machinery is required to obtain Holder continuity of the normal, so we use direct PDE techniques below to find tilted planes near Γ, above which the K_λ are (locally) uniform Lipschitz graphs. (These techniques relate to methods used in [13]; more technical versions of "tilting" apply to a variety of Dirichlet and contact angle problems [12].) We then combine the resulting curvature estimates near Γ with the interior estimates of § 5.5 to derive global curvature (and higher derivative) estimates independent of λ and $|Du|$.

To begin, let $\mathbf{n}$ be the exterior normal vector field in a tubular neighborhood of $\Gamma \subset \Pi$. By a mild abuse of notation, we also use $\mathbf{n}$ for the normal vector field $(0, \mathbf{n})$ near the cylinder $\times \Gamma$ lying "above" Γ, and then extend $\mathbf{n}$ smoothly (again without changing notation) to the entire cylinder $\times K$.

Now consider the functional

$$F = \theta e^{kx_1} - \nu \cdot \mathbf{n} - M\nu_1 \tag{5.6}$$

on K_λ. It suffices to show that we can find positive constants θ, k and M so that $F \leq 0$ on K_λ. For, in a neighborhood of $p \in K_\lambda$, $F \leq 0$ can be rewritten as

$$\nu \cdot (M\mathbf{e}_1 + \mathbf{n}(p)) \geq \theta e^{kx_1} + \nu \cdot (\mathbf{n}(p) - \mathbf{n}),$$

and one has a uniformly Lipschitz graph above a plane perpendicular to $M\mathbf{e}_1 + \mathbf{n}(p)$ as long as the quantity on the right is uniformly positive. Elliptic regularity theory for solutions to uniformly elliptic equations having locally smooth boundary data (i.e., the projections of Γ onto such planes when p is near $\times \Gamma$) then produces uniform estimates on the curvature (and all its derivatives) on K_λ.

To find θ, k, M, recall first (paragraph after equation (5.2)), that for λ near 0, say $0 \leq \lambda \leq \lambda_0$, the implicit function theorem gives us solutions u_λ with uniformly bounded gradient. For this range of λ, and for given θ, k, we can therefore ensure $F \leq 0$ by making M large enough. It then suffices to find θ, k, M for the range $\lambda \geq \lambda_0$.

For this purpose, we again use the strong maximum principle, which implies u_{λ_0} has strictly positive normal derivative, *i.e.,* $\nu \cdot \mathbf{n} > \theta$ along Γ for some positive number θ. Since the u_λ increase monotonically in λ, this normal derivative estimate continues to hold for all $\lambda \geq \lambda_0$. Using this θ in our definition of F we see that $F < 0$ on $\partial K_\lambda = \Gamma$, for all $\lambda \geq \lambda_0$.

Now, if for a given k, we already have $F \leq 0$ with $M = 0$, we are done. If not, increase M from 0 until $F \leq 0$ on K_λ and there exists an interior point $p \in K_\lambda$ where $F = 0$. (This is possible because $F < 0$ on Γ and $\nu_1 > 0$ on the interior of K_λ.) Then at p, $\nabla F = 0$ and $\Delta F \leq 0$, where ∇ and Δ are the surface gradient and Laplacian, respectively.) Using the identities

$\Delta x_1 = -\lambda\nu_1$, $\Delta\nu = -|\mathbf{A}|^2\nu$, we compute $\Delta F(p)$ explicitly, and obtain an estimate at p:

$$\begin{aligned} &\theta e^{kx_1}(-k\lambda\nu_1 + k^2|\nabla x_1|^2) \\ &+|\mathbf{A}|^2(\nu\cdot\mathbf{n} + M\nu_1) - C|\mathbf{A}| - C \quad \leq 0, \end{aligned} \tag{5.7}$$

The error terms $-C|\mathbf{A}| - C$ here come from cross derivatives of $\Delta(\nu\cdot\mathbf{n})$. The constants C depend on the domain K, but not on λ. Since $F(p) = 0$ we can substitute $\nu\cdot\mathbf{n} + M\nu_1 = \theta e^{kx_1}$ into equation (5.7), obtaining

$$\theta e^{kx_1}(k\lambda\nu_1 + k^2|\nabla x_1|^2 + |\mathbf{A}|^2) - C|\mathbf{A}| - C \leq 0. \tag{5.8}$$

But $|\nabla x_1|^2 = 1 - (\nu_1)^2 > \frac{1}{2}$ if $(\nu_1)^2 < \frac{1}{2}$, the latter holding for any $M \geq M_0(\theta, k)$ because x_1 is bounded from above (equation (5.4)), and $F(p) = 0$ implies

$$\nu_1 = \frac{\theta e^{kx_1} - \nu\cdot\mathbf{n}}{M}\ . \tag{5.9}$$

For $M \geq M_0$ equation (5.7) therefore implies at p:

$$\theta e^{kx_1}\left(\frac{k^2}{2} + |\mathbf{A}|^2 - Ck - C|\mathbf{A}| - C\right) \leq 0. \tag{5.10}$$

But the arithmetic-geometric mean inequality contradicts this for k sufficiently large. Take one such as our value of k. It follows that our value of M, gotten by increasing from zero until the inequality $F \leq 0$ is first obtained, must satisfy $M \leq M_0(\theta, k)$. This gives the uniform curvature (and higher regularity) estimates we seek in the Euclidean case: we have shown that $F \leq 0$ on K_λ for certain values of θ, k, and $M = M_0(\theta, k)$.

(5.7) Final step. Using the estimates from § 5.4, § 5.5, § 5.6 we deduce that the set of $\lambda \leq H$, for which u_σ exists for all $\sigma \leq \lambda$, is closed. For the limit of u_σ as $\sigma \to \lambda$ from below must exist, and must be a smooth graph by our apriori estimates. We now claim that this closed interval must actually be $[0, H]$. If not, denote it by $[0, \lambda]$ for $\lambda < H$. Then $|Du_\lambda|$ must be infinite at some boundary point $p \in \Gamma$; otherwise the equation (5.2) for u_λ is uniformly elliptic and we may again use the implicit function theorem to obtain solutions with larger mean curvature. Since K_λ lies "beneath" both $\Sigma \cap \{x_1 \geq 0\}$ and the reflection $\widetilde{\Sigma} \cap \{x_1 \geq 0\}$, it follows that at p the conormal η to Σ is perpendicular to Π, hence K_λ and $\Sigma \cap \{x_1 \geq 0\}$ make one-sided tangential contact there. This is a contradiction: it violates the maximum principle for $\lambda < H$. Hence the path of graphs includes all values $\lambda \in [0, H]$.

The same basic strategy proves the theorem in hyperbolic space. We indicate the necessary technical modifications below. We work in the Minkowski space model (§ 2.3). Assume $\Pi = \{x_1 = 0\} \cap \ ^{n+1}$, and that along Π, ν_Π and the Killing vector field $\mathbf{Y}$ are positive (function) multiples of the projection of $-\mathbf{e}_1$ onto $\ ^{n+1}$ (i.e. of $-(\mathbf{e}_1 - (\mathbf{e}_1 \cdot x)x)$, where $x = (x_0, \mathbf{x})$ can be viewed as both position and normal vector to $\ ^{n+1} \subset \ ^{1,n+1}$).

Definition 5.8 *Let $\gamma(y)$ denote the unit speed geodesic in $\ ^{n+1}$ perpendicular to Π, passing through $y \in \Pi$, with tangent vector at y pointing in the positive x_1-direction. Let $\gamma(y, s)$ be its parametrization, where s is the arclength, $\gamma(y, 0) = y$. A surface S is a graph (with respect to distance) above Π if there exists a function v defined on a subset K of Π such that $S = \{\gamma(y, v(y)) \mid y \in K\}$.*

Now we will construct a one-parameter family of graphs K_λ of functions $u_\lambda \geq 0$ above $K \subset \Pi$ having mean curvature $\lambda \in [0, H]$, with $K_0 = K$, as in the Euclidean case. Recall from Section 3 that the geodesic distance s from Π relates to x_1 by $\sinh s = x_1$. Consider the functions $(x_\lambda)_1 \equiv v_\lambda \equiv \sinh u_\lambda$. If estimates corresponding to those in § 5.4, § 5.5, and § 5.6 can be established for the v_λ, then the equivalent estimates will hold for the "hyperbolic" u_λ. The equation corresponding to (5.2) is

$$
(5.11) \qquad \begin{cases} \Delta v_\lambda = n v_\lambda - \lambda(\nu_\lambda)_1 & y \in K \\ v_\lambda(y) = 0 & y \in \partial K = \Gamma \,. \end{cases}
$$

This equation restates the identity $\Delta x_1 = n x_1 - \lambda \nu_1$ from § 2.3, so the Laplacian in (5.11) is that associated to the pull-back of the metric on K_λ onto $K \subset \Pi$. It is uniformly elliptic for $y \in K$ as long as u and $|Du|$ (equivalently v and $|Dv|$) are uniformly bounded above. By considering the (upper) normal ν to K_λ one also finds this equivalent to the existence of a uniform upper bound on u and lower bound $\nu \cdot \mathbf{Y} \geq \delta > 0$. But for bounded u, $\mathbf{Y}$ is a uniform multiple of the projection of $\mathbf{e}_1$ onto $^{n+1}$, so we may replace $\nu \cdot \mathbf{Y} \geq \delta > 0$ by $\nu_1 = \nu \cdot \mathbf{e}_1 \geq \delta > 0$ to guarantee uniformity.

As long as (5.11) remains uniformly elliptic, i.e., as long as one has a solution u_λ for which $u_{\lambda\_}$ and $|Du_\lambda|$ are bounded, the implicit function theorem guarantees an interval of solutions about u_λ. In particular, solutions exist in a neighborhood of $u_0 \equiv 0$.

(5.9) Height estimates. As in Euclidean space, make a choice of Π so that the origin $(0, 0, \ldots, 0, 1)$ of $^{n+1}$ is in K, and so that a geodesic ball of radius $R = diam(\Gamma)$ about $(0, 0, \ldots, 0, 1)$ contains $\Gamma \cup K$.

Consider the equidistant hemisphere obtained by intersecting the plane $\{x_0 = C\}$ with $^{n+1}$. As $C \to \infty$ this hemisphere has mean curvature approaching n. Choose $C = C(R)$ so large that the plane intersects Π outside

Γ. The resulting hemisphere has mean curvature $H_0 > n$ hence provides a barrier above for K_λ when $0 \le \lambda \le H_0$. But for $H_0 \le \lambda \le H$, one can use Lemma 3.3 (a consequence of $\lambda x_1 \le n\nu_1$) to derive uniform estimates above for $x_1 = v_\lambda$.

(5.10) Gradient estimates and barriers at the boundary. The monotonicity (strict in the interior K) of v_λ with respect to λ follows from the maximum principle, just as in the Euclidean case. The estimate $\lambda x_1 \le n\nu_1$ and the monotonicity in λ yield interior gradient estimates. Again, the original surface $\Sigma \cap \{x_1 \ge 0\}$ and the reflection $\widetilde{\Sigma} \cap \{x_1 \ge 0\}$ of Σ through $\{x_1 = 0\}$ are barriers "above" the K_λ, provided that Σ' is sufficiently far from Π, depending on the height estimates of § 5.9.

(5.11) Curvature estimates. Follow the reasoning used in § 5.6, considering again the function F from equation (5.6) on the surface K_λ. Extend the normal $\mathbf{n}$ as before; first from a tubular neighborhood of Γ to all of K, then from K to the solid cylinder above it by parallel transport along the geodesics $\gamma(y)$. By the arguments after equation (5.11) it again suffices to find θ, k, M for which $F \le 0$ on K_λ. Find θ as before. For a given k, we let M vary as before and find an interior point p at which $F(p) = 0$, knowing that $F \le 0$ elsewhere. The analog of estimate (5.7) (computing $\Delta F(p)$, using the formulas for Δx_1 and $\Delta\nu$ in § 2, and noting that $\Delta F(p)$ is non-positive) is:

$$
\begin{aligned}
&\theta e^{kx_1}(k(-\lambda\nu_1 + nx_1) + k^2|\nabla x_1|^2 \\
&+ |\mathbf{A}|^2(\nu \cdot \mathbf{n} - \lambda x \cdot \mathbf{n} + M\nu_1) - C|\mathbf{A}| - C \quad \le 0.
\end{aligned}
\tag{5.12}
$$

But $x \cdot \mathbf{n} = 0$ since x is the normal to ${}^{n+1}$ at x. Using $F(p) = 0$ we now estimate as in equation (5.8):

$$\theta e^{kx_1}(k^2|\nabla x_1|^2 + |\mathbf{A}|^2 - Ck - C|\mathbf{A}| - C) \leq 0. \tag{5.13}$$

In the present setting we have $\nabla x_1 = \mathbf{e}_1 - (x_1)\, x - \nu_1 \nu$, and, because $x \cdot x = -1$, we have $|\nabla x_1|^2 = 1 + (x_1)^2 - (\nu_1)^2$. Estimate (5.13) lets us again find an M_0 such that $M \geq M_0$ implies $|\nabla x_1|^2 \geq \frac{1}{2}$. Therefore we can complete the argument as in § 5.6, and conclude uniform curvature and higher regularity estimates for the K_λ, as in the Euclidean case. $\square$

6 Asymptotic symmetry of cylindrically bounded ends

We now show that any noncompact properly embedded constant mean curvature hypersurface $\Sigma \subset \ ^{n+1}$, which is cylindrically bounded and has compact boundary $\partial\Sigma$, must approach a Delaunay surface exponentially at infinity (Theorem 6.9). The proof is conceptually simpler than that [11] in the Euclidean case $\Sigma^2 \subset \ ^3$, and holds in all dimensions. This happens mainly because Alexandrov reflection is “stronger” in hyperbolic space, since there are “more” totally geodesic hyperplanes parallel to a given hyperplane.

(6.1) Configuration. Let Σ be a properly embedded noncompact constant mean curvature hypersurface in $\ ^{n+1}$, with compact boundary $\partial\Sigma$ and the property that Σ stays in a solid half cylinder $C(\gamma, R)$, *i.e.,* the geodesic radius R tubular neighborhood about a geodesic ray γ.

Let $\mathbf{Y}$ be the Killing vector field generating unit-speed translation along γ, and write $\mathbf{Y} = \partial_t$, where arclength t varies from 0 to ∞ along $\gamma = \gamma(t)$. For each fixed t, let D_t be the cross-section of $C(\gamma, R)$ at t, *i.e.,* the totally geodesic n-disk of radius R perpendicular to γ and containing $\gamma(t)$. Use ρ to denote distance from γ *i.e.,* $\rho(p)$ is the distance from $p \in D_t$ to $\gamma(t)$.

We may assume, after translating along $\mathbf{Y}$ and discarding some compact pieces of Σ, that $\partial\Sigma$ is contained in D_0, and that Σ bounds a possibly disconnected domain $\Omega \subset C(\gamma, R)$. We allow the possibility that Σ is now a finite union of disconnected ends, each one bounding a connected domain. Our analysis will show *a posteriori* that there is really only one end.

Next, perform an isometry of $\,^{n+1}$ so that in the upper half-space model (§ 2.1) γ is the y-axis, $\{0 < y < 1\}$. Then $C(\gamma, R)$ is part of a solid Euclidean cone vertex at the origin; its intersection with each Euclidean n-sphere centered at $\mathbf{0}$ is a totally geodesic hyperbolic disk D_t of hyperbolic radius R. We then have $\mathbf{Y} = \partial_t = -r\partial_r$, for $t = -\ln r$, and $r =$ Euclidean distance to $\mathbf{0}$.

Lemma 6.2 *Let Σ and $\mathbf{Y}$ be as in Configuration 6.1. Then the flux $\mu = \langle \mu([\partial\Sigma]), Y\rangle$ is positive. (Here we orient the conormal to $\partial\Sigma$ to be inward on Σ, i.e. it points toward infinity).*

Proof. Note that $\Gamma = \partial(\Omega \cap D_t)$ is an interior planar cycle homologous to $\partial\Sigma$. Hence by Theorem 5.3, for t large, Γ must have positive flux with respect to $\mathbf{Y}$. This proves $\partial\Sigma$ has positive flux. $\square$

Lemma 6.3 *For fixed $H > n$ there exists a unique (up to translations generated by $\mathbf{Y} = \partial_t$) embedded axially symmetric Delaunay surface $\mathcal{D}(m)$ about γ with mean curvature H and mass $m = \langle \mu([\mathcal{D}(m) \cap D_t]), Y\rangle$. The mass is positive, is maximized for the cylinder of mean curvature H, and approaches zero as the family evolves into a chain of spheres.*

Proof. Fix a Delaunay surface $\mathcal{D}$ and express it as a graph in cylindrical coordinates, $\{\rho = \rho_{\mathcal{D}}(t)\}$. Let Ω be the solid region of $\,^{n+1}$ "inside" $\mathcal{D}$.

Taking rotational symmetry into account, the flux through the homology class $[\mathcal{D} \cap D_t]$ generated by $\mathbf{Y} = \partial_t$ is actually the mass:

$$m = \int_{\mathcal{D}\cap D_t} \eta \cdot \partial_t \;-\; H \int_{\Omega\cap D_t} \nu \cdot \partial_t\,. \tag{6.1}$$

We calculate m explicitly now in terms of ρ and ρ_t, using elementary hyperbolic geometry. Noting first that geodesic distance ρ is related to polar angle φ ($r\cos\varphi = y$) by $d\rho = \sec\varphi\, d\varphi$ we integrate to get

$$\tan\varphi = \sinh\rho. \tag{6.2}$$

Letting ω_n be the area of the unit Euclidean n-sphere S^n, we next evaluate (6.1) with the aid of (6.2):

$$m = \omega_{n-1}(\tan\varphi)^{n-1}\sec\varphi\frac{1}{\sqrt{1+\varphi_t^2}} - \frac{H}{n}\omega_{n-1}(\tan\varphi)^n. \tag{6.3}$$

Thus $\rho_{\mathcal{D}}$ satisfies a first order ODE whose derivative is the usual second order ODE for mean curvature. Inspection of this first order equation immediately reveals that *$\rho_{\mathcal{D}}$ must be periodic in t.*

Writing $z = \tan\varphi$, we find that when $\varphi_t = 0$, (6.3) becomes

$$m = \omega_{n-1}z^{n-1}\left[\sqrt{1+z^2} - \frac{H}{n}\,z\right]. \tag{6.4}$$

The function $m(z)$ in (6.4) is zero at $z = 0$ (corresponding to a chain of spheres), positive for small $z > 0$, and has a second zero at $Hz = n(1+z^2)^{1/2}$. It is straightforward to check that $m(z)$ has one positive critical point, so m is monotone increasing until that value and monotone decreasing afterwords. Thus, except at the maximum, each non-negative value is attained exactly twice. So each embedded Delaunay surface is then uniquely determined (up to translation) by its minimum (or maximum) ρ-value, since $\rho_{\mathcal{D}}$ solves the second order initial value problem $\rho = \rho_{\min}$, (or $\rho = \rho_{\max}$), $\rho_t = 0$. But (6.4) determines these two values of ρ (and the corresponding z-values) uniquely in terms of the mass m. Since $\rho_{\max} = \rho_{\min}$ on the cylinder, that surface maximizes the mass. $\square$

Lemma 6.4 (Earp-Rosenberg [6]) *Given Configuration 6.1, the mean curvature of Σ is at least as big as the mean curvature $H_1 > n$ of the cylinder containing it. Furthermore, there is a sequence $\{p_i\} \subset \Sigma$ with $t(p_i) \to \infty$ and with $\rho(p_i) \geq \rho_0$, where ρ_0 is the geodesic radius of a cylinder having mean curvature H.*

Proof. (For the reader's convenience, we sketch the Earp-Rosenberg argument.) Fix a T so that $p \in \partial\Sigma$ implies $t(p) < T$. Let the mean curvature of $C(\gamma, R)$ be H_1. Deform the cylinder along the one-parameter family of Delaunay surfaces with constant mean curvature H_1 that have maximum "bulge" at $t = T$. The periods of these Delaunay surfaces are bounded from above by the diameter of the corresponding sphere, and since the family converges to a chain of spheres, the "necks" pinch to zero radius. The bulge radius increases, so the bulge region of these surfaces stays "outside" of Σ. But since Σ extends beyond the maximum period of the Delaunay surfaces, there must be a Delaunay surface in the family that first makes one-sided tangential contact with Σ at a point within one period of $t = T$. Hence the mean curvature H of Σ is at least that of $C(\gamma, R)$.

To prove the second part of the lemma, assume the sequence does not exist. Repeat the above argument, using the cylinder with mean curvature H and with T large enough so that $p \in \Sigma$, $t(p) \geq T$ implies that p is inside the cylinder. At the first point of one-sided tangential contact the strong maximum principle would be violated. $\square$

(6.5) Alexandrov Reflection. Recall that in the ball model of $^{n+1}$ (Section 2.2) the totally geodesic hypersurfaces are Euclidean n-spheres meeting $S_\infty = \partial \mathbf{B}^{n+1}$ at right angles. "Reflection," corresponding to inversion through these spheres, is an isometry of $^{n+1}$. Alexandrov reflection applied to a constant mean curvature hypersurface $\Sigma = \partial\Omega$ is the process of inverting through a continuous family of these spheres $\{S(q)\}$, and studying the possibility of one-sided interior reflection contact between the part of Σ exterior to the sphere and its reflection from inside the sphere. The strong maximum principle implies reflection symmetry if they contact.

(6.6) New Configuration. We adapt Configuration 6.1 to the ball model. Let $(1, 0, \ldots, 0)$ be the vertex v at infinity of our cylinder $C(\gamma, R)$, which we may assume is axially symmetric about the x_1-axis. Given t as in Configuration 6.1, make a hyperbolic translation along the x_1-axis so that D_t crosses it at the origin. We will use hyperbolic translations and a particular set of reflection families to study the asymptotic symmetry of $D_t \cap \Sigma$. Denote the translated $(\Sigma, \partial\Sigma)$ by $(\Sigma_t, \partial\Sigma_t)$. Note that as $t \to \infty$, $\partial\Sigma_t$ moves uniformly towards the point $(-1, 0, \ldots, 0)$. In fact, the translation moves $D_0 = \{x_1 = 0\} \cap C(\gamma, R)$ to the Euclidean reflection of D_t through $\{x_1 = 0\}$. Hence $\partial\Sigma_t$ is within Euclidean distance $O(e^{-t})$ of $(-1, 0, \ldots, 0)$.

Lemma 6.5 (Weak gradient estimate) *For every $\delta > 0$ there is a $T = T(\delta)$ and an $\varepsilon = \varepsilon(\delta)$ so that for any $p \in \Sigma$ satisfying $t(p) > T$ and $dist(p, \gamma) = \rho(p) > \delta$, we have the estimate $\nu(p) \cdot \partial_\rho > \varepsilon$. (Here $\nu(p)$ is the exterior normal to Σ at p and ∂_ρ is the unit vector field in D_t which is the derivative of geodesic distance ρ from $\gamma(t)$.)*

Proof. Consider the translations Σ_t in Configuration 6.6. There are various possibilities for Alexandrov reflection from "above" (say, after Euclidean rotation about the x_1-axis, from the positive x_2 direction). The simplest one begins with a small Euclidean sphere $S(q)$ orthogonal to S_∞, centered at q on the x_2-axis just above $(0,1,0,\ldots,0)$, with radius chosen to satisfy the orthogonality condition. As q approaches infinity along the x_2-axis, the sphere $S(q)$ approaches the $\{x_2 = 0\}$ plane.

Because $\partial\Sigma_t \to (-1,0,\ldots,0)$ as $t \to \infty$, one can approach arbitrarily close to this limiting plane as $t \to \infty$ without attaining boundary reflection contact for Σ_t, nor interior contact, by the strong maximum principle.

If one only wants the reflection spheres to reach $(0,\delta,0,\ldots 0)$ before first reflection contact, nearby reflection families are permitted: Consider families of spheres whose centers are Euclidean distance r from the origin, $1 < r < \infty$, and which lie on rays making small angle with the x_2-axis. The geometry of these spheres as $r \to \infty$ implies that for t sufficiently large and angle sufficiently small ($O(\delta)$), first reflection contact only occurs after the family passes through $(0,\delta,0,\ldots,0)$. But by the definition of first reflection contact, the angle between the reflection sphere's interior normal and the exterior normal ν to Σ_t is strictly less than $\pi/2$ at any point on the intersection of such a sphere and Σ_t. We see that the set of all such reflection families for a point $p = (0,\rho(p),0,\ldots,0) \in \Sigma_t$ with $\rho(p) \geq \delta$, includes a whole cone of directions about the positive x_2 direction, and this cone has aperture greater than or equal to $O(\delta)$. Hence we conclude that $\nu(p)\cdot\partial_\rho > O(\delta)$, for t sufficiently large. This proves Lemma 6.5, with $\varepsilon \geq O(\delta)$ as $\delta \to 0$. $\square$

Lemma 6.6 (Weak asymptotic axial symmetry) *For every $\delta > 0$ there is a $T = T(\delta)$ so that whenever $t > T$ and $D_t \cap \Sigma$ contains a point of distance $\rho > \delta$ from $\gamma(t)$, then $D_t \cap \Sigma$ is a radial graph with respect to ρ over $S^{n-1}(\gamma(t))$, the unit sphere in D_t with center $\gamma(t)$. Furthermore the function $\rho(\theta)$ parametrizing $D_t \cap \Sigma$ above $S^{n-1}(\gamma(t))$ is smooth and satisfies $|\rho_\theta| < Ce^{-t}$ for some $C = C(\delta,T)$.*

Proof. We study the translates Σ_t, and reflection from above as in the beginning of Lemma 6.5 (*i.e.,* the admissible reflection spheres are now those symmetric with respect to the $\{x_1 = 0\}$ plane and centered on a ray perpendicular to the x_1-axis). Because $\partial\Sigma_t$ is within $O(e^{-t})$ of $(-1, 0, \ldots, 0)$, such a reflection family of spheres can be followed until the reflecting spheres approach to distance $O(e^{-t})$ of the origin, before any reflection contact occurs.

Let $\delta > 0$ be given. Consider t for which the maximum ρ value on $\Sigma \cap D_t$ is at least δ. Pick any $p \in \Sigma \cap D_t$ with $\rho(p) = \overline{\rho} > \delta/2$. Then after translation, the corresponding point $p_t \in D_0$ also satisfies $\rho(p_t) = \overline{\rho}$. The reflecting spheres considered above map D_0 to itself. Write ε $(= O(e^{-t}))$ for a distance to which any such sphere may approach the origin before reflection contact, and assume t has been chosen so that $\varepsilon \ll \delta$.

Now study the images of p_t in D_0, under all these reflection spheres, until the reflection sphere of distance ε from the origin is attained. They all lie in the interior of $\Omega \cap D_t$, since there has been no reflection contact. Call the image set U_p and note that its boundary ∂U_p is the set of reflection images of p by the final (distance $= \varepsilon$) sphere in each family. We will prove:

Claim 6.7 *U_p is a teardrop-shaped region, symmetric about the axis $\overline{\mathbf{O}p_t}$. Furthermore, ∂U_p lies within $O(\varepsilon)$ of the sphere of radius $\overline{\rho}$, the "vertex" of the teardrop is at p_t, and its tangent cone is almost flat, with aperture half-angle within $O(\varepsilon)$ of $\pi/2$.*

Assuming the claim, Lemma 6.6 follows easily: Apply the claim to the point $p \in \Sigma \cap D_t$ that has maximum $\rho = \rho(p) > \delta$. It follows that all of $\Sigma \cap D_t$ lies within an $O(\varepsilon)$ distance of $\rho \cdot S^{n-1}(\gamma(t))$. From the weak gradient estimate in Lemma 6.5, we conclude that since $\nu \cdot \partial_\rho$ is positive (for t large enough), $\Sigma \cap D_t$ is a transverse intersection and is given as the graph of a smooth function $\rho = \rho(\theta)$. Applying the claim to each $q \in \Sigma \cap D_t$, the statement about aperture angle implies that $|\rho_\theta|$ is at most $O(e^{-t})$.

So it remains to verify the claim. By a rotation about the x_1-axis, we assume $p_t = (0, \overline{\rho}, 0, \ldots, 0)$. Because our reflection spheres are symmetric about the x_1-axis and preserve D_0, it suffices to consider the two-dimensional picture: families of reflection circles in the x_2x_3-plane. For simplicity write $y = x_2, z = x_3$. Then p_t corresponds to $(0, \overline{\rho})$ and all reflection families of circles in the yz-plane continue up to the reflection circle of distance ε from the origin.

It is possible to show (and geometrically clear, see Figure 4), that if one replaces the reflecting circles with reflecting lines (perpendicular to the reflection family ray and tangent to the given circle at the point on the ray nearest the origin), the corresponding image set V_p of p_t is contained in U_p. Hence it suffices to establish the claim for V_p. Let θ be the angle between the y-axis and the lines in an admissible family able to reach p_t. (So $\sin^{-1}(\overline{\rho}) < \theta \leq \pi/2$). The image of p_t under this family will be a line segment extending from p_t to the reflection $\widetilde{p}_t$ of p_t through the final line in the family. One can calculate the coordinates of $\widetilde{p}_t$ explicitly using elementary geometry. Assuming the lines have positive slope we do so:

$$\widetilde{p}_t = \left(0, \frac{\varepsilon}{\sin\theta}\right) + \left(\overline{\rho} - \frac{\varepsilon}{\sin\theta}\right)(\sin 2\theta, \cos 2\theta). \tag{6.5}$$

Taking norms gives

$$|\widetilde{p}_t|^2 = \overline{\rho}^2 + 4\varepsilon^2 - 4\varepsilon\overline{\rho}\sin\theta. \tag{6.6}$$

Clearly $|\widetilde{p}_t|$ is within $O(\varepsilon)$ of $\overline{\rho}$, and this establishes part of the claim.

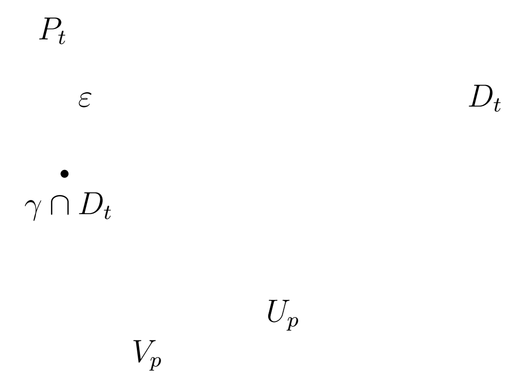

Figure 4.

To estimate the vertex-angle of the teardrop, compute $(\widetilde{p}_t)_\theta$ from (6.5), and evaluate at the limiting $\theta = \sin^{-1}(\varepsilon/\overline{\rho})$:

$$(\widetilde{p}_t)_\theta = (2(\overline{\rho} - \frac{\varepsilon^2}{\overline{\rho}}),\ 2\varepsilon\sqrt{1 - \frac{\varepsilon^2}{\overline{\rho}^2}} - \frac{\varepsilon}{\overline{\rho}^2}). \tag{6.7}$$

Since this tangent vector is within $O(\varepsilon)$ of being horizontal, the angle in question is within $O(\varepsilon)$ of $\pi/2$. $\square$

Lemma 6.8 *There exist T, C, and $\rho_{\min}$ so that $\{p \in \Sigma \mid t(p) \geq T\}$ is the cylindrical graph of a function $\rho(t, \theta)$ with ρ uniformly bounded. Then $R > \rho > \rho_{\min} > 0$, with uniformly bounded gradient $|(\rho_t, \rho_\theta)| < C$, and with $|\rho_\theta| < Ce^{-t}$.*

Proof. Given $\delta > 0$, choose T, ε so that Lemmas 6.5 and 6.6 hold. Use Lemma 6.5 to pick p with $\rho(p) \geq \rho_0$, and $|\rho_\theta| \ll \delta$ for $t \geq t(p)$ and $\rho > \delta$. Then as t increases from $t(p)$, the cross-sections $D_t \cap \Sigma$ remain nearly spherical, provided their maximum radius ρ exceeds δ. But as the radius approaches δ, the flux $\mu = \langle \mu([\partial\Sigma]), \partial_t \rangle$ is of order δ^{n-1}. If δ is chosen sufficiently small (depending on the positive value of μ in Lemma 6.2), this contradicts the homology-invariance of flux. So the radius never approaches

δ, and ρ is uniformly bounded above and below by positive numbers. The estimates for $|D\rho|$ now follow from Lemmas 6.5 and 6.6. $\square$

Theorem 6.9 (Exponential convergence to Delaunay surface) *Given Σ as in Configuration 6.1, there exists an axially-symmetric Delaunay surface $\mathcal{D}$ about γ to which Σ converges exponentially* [1]*. That is, for t sufficiently large, Σ is the graph in geodesic cylindrical coordinates of a function ρ, there is a Delaunay $\mathcal{D}$ surface that is the graph of a function $\rho_{\mathcal{D}}$, and $|\rho - \rho_{\mathcal{D}}| < Ce^{-t}$ as $t \to \infty$. (Convergence holds in any C^k norm.)*

Proof. From Lemma 6.8 we know that for t sufficiently large, Σ is a graph in geodesic cylindrical coordinates of a function $\rho(t, \theta)$ satisfying $R > \rho > \rho_{\min} > 0$, $|(\rho_t, \rho_\theta)| < C$ and $|\rho_\theta| < Ce^{-2t}$.

Since Σ has constant mean curvature, ρ satisfies a quasilinear elliptic equation in (t, θ). The equation is $H = g^{ij}h_{ij} = \mathcal{M}(\rho)$, where $[g^{ij}]$ and $[h_{ij}]$ are the inverse of the first and second fundamental forms, respectively, in the (t, θ) coordinates. The uniformity of ρ ($0 \leq \rho_0 \leq \rho \leq R$) and the bound on D_ρ imply that in the (t, θ) coordinate system the pull-back metric $[g_{ij}]$ is uniformly positive, and hence the operator $\mathcal{M}(\rho) = g^{ij}h_{ij}$ is uniformly elliptic. So ρ is C^∞ , with uniform C^k estimates for sufficiently large t.

Now by differentiating $\mathcal{M}(\rho) = H$ in any θ direction, and interchanging derivatives, we conclude that ρ_θ also satisfies a uniformly elliptic equation. Interior elliptic regularity and the estimate $|\rho_\theta| < Ce^{-t}$ then imply that $|D(\rho_\theta)|$ (and $|(D\rho)_\theta|$) are also bounded by $\widetilde{C}e^{-t}$ for sufficiently large t. That is, all second derivatives involving at least one θ differentiation are uniformly bounded by $\widetilde{C}e^{-t}$.

[1] This Delaunay surface $\mathcal{D}$ is unique if we require its axis to be the world-line of $[\partial\Sigma]$, i.e. if $\mathcal{D}$ and Σ have the same momenta.

Because of this decay in θ derivatives of ρ, we may consider solutions of $\mathcal{M}(\rho) = H$ as perturbations of Delaunay ODE solutions. Define an "approximate bulge" of Σ to be a t value for which the area of the cap $D_t \cap \Omega$ is maximized. At a bulge ρ_t cannot be of one sign, and since $|\rho_{t\theta}|$ is $O(e^{-t})$, $|\rho_t(t,\theta)|$ must be $O(e^{-t})$ as well. It follows from the flux formula, and the fact that the variation of ρ and $|\rho_\theta|$ are also $O(e^{-t})$, that ρ must be within $O(e^{-t})$ of the maximum ρ value on a Delaunay surface $\mathcal{D}$ having momentum equal to that of $\partial\Sigma$. By continuous dependence of parameters for ODE's, it now follows that the time interval until the next approximate bulge (assuming $\mathcal{D}$ is not the cylinder) differs by $O(e^{-t})$ from the period of the Delaunay solution $\rho = \rho_{\mathcal{D}}(t)$. (Compare the PDE and ODE solutions, where the ODE has initial values $\rho = \max(\rho_{\mathcal{D}})$, $\rho_t = 0$.) If $\mathcal{D}$ is the cylinder, our estimates immediately show $|\rho - \rho_0|$ is $O(e^{-t})$. In any case, the total variation of the difference between PDE and ODE "periods" from $t = t_0$ to $t = \infty$ is now seen to be $O(e^{-t_0})$. But then, $\rho(t,\theta)$ converges to a translate $\rho_{\mathcal{D}}(t+T)$, and the total variation of period differences is $O(e^{-t_0})$, which also bounds the rate at which $\rho(t,\theta)$ converges to $\rho_{\mathcal{D}}(t+T)$. Now elliptic regularity, applied to the difference of the two solutions, yields C^k convergence. $\square$

Theorem 1.3 in the introduction follows immediately.

Remark 6.10 *One may want to consider the asymptotic behavior of a properly embedded surface* $\Sigma \subset {}^{n+1}$ *with constant mean curvature* $H \leq H_{\text{horosphere}}$. *In this case the ends of* Σ *are* never *cylindrically bounded. In fact, examples show that the asymptotic boundary behavior is essentially arbitrary when* $H < H_{\text{horosphere}}$. *However, the case* $H = H_{\text{horosphere}}$ *in* ^{3} *is similar to the case of minimal surfaces in* ^{3} *[5]. Perhaps in this case the asymptotic behavior of annular ends can be classified (see [8] for partial results on this problem in the case of minimal surfaces in* ^{3}*).*

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