Some Numerical Methods for the Hele-Shaw Equations [1]

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Classification index numbers: 35Q30, 65D07, 65T10, 76C05, 76T05

Keywords: interface, fingering, surface tension, boundary-integral method, vortex method

18 pages, 4 figures, 2 tables

[1]This work was partially supported by the National Science Foundation under grant DMA-8913482

Abstract

In ([19, 20]), Tryggvason and Aref use a boundary integral method and the vortex-in-cell method to evolve the interface between two fluids in a Hele-Shaw cell. The method gives excellent results for intermediate values of the nondimensional surface tension parameter. The results are different from the predicted results of McLean and Saffman for small surface tension. For large surface tension, there are some numerical problems. In this paper, we implement the method of Tryggvason and Aref but use the point vortex method instead of the vortex-in-cell method. A parametric spline is used to represent the interface. The finger widths obtained agree well with those predicted by McLean and Saffman. We conclude that the method of Tryggvason and Aref can provide excellent results but that the vortex-in-cell method may not be the method of choice for extreme values of the surface tension parameter.

In a second method, we represent the interface with a Fourier representation. In addition, an alternative way of discretizing the boundary integral is used. Our results are compared to the linearized theory and the results of McLean and Saffman and are shown to be highly accurate.

Abstract

In ([19, 20]), Tryggvason and Aref use a boundary integral method and the vortex-in-cell method to evolve the interface between two fluids in a Hele-Shaw cell. This method gives results different from the predicted results of McLean and Saffman for small surface tension. For large surface tension, there are some numerical problems. In this paper, we implement the method of Tryggvason and Aref but use the point vortex method instead of the vortex-in-cell method. A parametric spline is used to represent the interface in order to calculate arclength, tangents and curvature accurately. The finger widths obtained agree well with those predicted by McLean and Saffman. It is found that the method becomes unstable when too many vortices are used. We conclude that the method of Tryggvason and Aref can provide excellent results but that the vortex-in-cell method may not be the method of choice for extreme values of the

surface tension parameter.

1 Introduction

In this paper, the point vortex method is used to evolve the interface between two fluids of different viscosities in a Hele-Shaw cell (two closely placed parallel plates). These fluids are immiscible in the sense that there is a finite surface tension which stabilizes small-scale disturbances at the interface. The equations describing the flow of a fluid in a Hele-Shaw cell are the equation of motion

$$\vec{u} \;=\; -\frac{b^2}{12\mu}\nabla p \tag{1}$$

and the equation of continuity

$$\nabla \cdot \vec{u} \;=\; 0 \tag{2}$$

where μ is the fluid viscosity, b is the spacing between the plates in the Hele-Shaw cell. The plates are taken as horizontal, the positive y-axis is in the direction of the flow, the x-axis is parallel to the plates and the z-axis is perpendicular to the plates. The velocity vector $\vec{u}$ has two components u and v which are functions of t (time), x and y. These velocities result from taking an average of the three-dimensional velocity field. The pressure p is an averaged pressure and the symbol ∇ represents the vector of partial derivatives. Derivations of the above equations are found in Lamb [12] and Bear [2].

For fluids of different viscosities, we have a set of equations to be solved on each side of the interface. These equations are connected by the following two conditions.

1. The normal component of the velocity is continuous at each point of the interface.

2. There is a prescribed jump in the pressure at each point (x_0, y_0) on the interface given by

$$\frac{\sigma}{R(x_0, y_0)} \tag{3}$$

where σ is the surface tension parameter whose value depends on the two fluids involved and $R(x_0, y_0)$ is the radius of curvature at the point (x_0, y_0) on the interface. We suppose that a fluid 1 is injected in the Hele-Shaw cell expelling some fluid 2 and that the parameters in fluids 1 and 2 are subscripted 1 and 2 respectively. In [19], it is shown that the solution to the above problem depends on a dimensionless surface tension parameter B given by

$$B = \frac{\sigma k}{2AWL^2\bar{\mu}} \tag{4}$$

where k is $b^2/12$, $(0, W)$ denotes the velocity of the fluid ahead or behind the interface, $\bar{\mu}$ is the average of the viscosities in the two fluids, L is the dimension of the cell in the x direction and A is defined by

$$A = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}. \tag{5}$$

In a Hele-Shaw cell, for horizontal flow, an interface becomes unstable when a less viscous fluid displaces a more viscous fluid. Saffman and Taylor [16] performed a linearized stability analysis and for zero surface tension, found an analytic solution for the shape of single fingers. McLean and Saffman [13] found an analytic shape for the fingers for nonzero values of the surface tension parameter. Chuoke et al [8] performed experiments in a Hele-Shaw cell and packed bed models. They also performed a linearized stability analysis.

Several numerical methods have been presented for the evolution of the interface in a Hele-Shaw cell. These include the methods of Tryggvason and Aref in [19, 20], Degregoria and Schwartz in [9, 10], Meiburg and Homsy in [14] and Whitaker [21]. The numerical results of Tryggvason and Aref seem to agree well with the predictions of the linearized theory. Their results also agree, in general, with the McLean and Saffman fingers. However, their method predicts a much narrower finger for small values of the surface tension parameter B. The numerical method also gives fingers slightly wider than the McLean and Saffman fingers for very large values of the surface

tension parameter but it is believed that the finger may have not evolved fully. The method uses a boundary integral formulation along with Christensen's vortex-in-cell method [7].

In this paper, we implement the boundary integral formulation given by Tryggvason and Aref but we use the point vortex method instead of the vortex-in-cell method. A parametric cubic spline is used to represent the interface. The point vortex method is known to be unstable when vortices lie too close to each other. It is shown here that it can be used to high accuracy for the Hele-Shaw equations with a modest number of vortices. The results of McLean and Saffman are compared with those produced by the vortex-in-cell method and the point vortex method. The remainder of this paper is organized as follows. In section 2, the method of Tryggavason and Aref is described along with the vortex-in-cell method. In section 3, it is described how the point vortex method is implemented, and in section 4 numerical results are presented.

2 Method of Tryggvason and Aref

In this method, the interface is represented by a vortex sheet. A vortex sheet in two dimensions is a curve along which vorticity, ω, is concentrated as a delta function. The vorticity is related to the velocity field by

$$\omega = \vec{\nabla} \cdot \vec{u}. \tag{6}$$

Using equation (1), we see that $\omega = 0$ on both sides of the interface. The sheet is characterized by a strength, γ, at each point of the interface which has the dimensions vorticity per unit area. Suppose that the fluids in the Hele-Shaw cell meet at an interface with unit tangent vector $\vec{\tau}$. It can be shown then that the vortex sheet strength is exactly the jump in the tangential component of the velocity, i.e.,

$$\gamma = \vec{u_1} \cdot \vec{\tau} - \vec{u_2} \cdot \vec{\tau}. \tag{7}$$

Equation (2) implies that there exists a stream function,ψ, such that

$$\frac{\partial \psi}{\partial x} = -v, \tag{8}$$

and

$$\frac{\partial \psi}{\partial y} = u. \tag{9}$$

Equations (6), (8) and (9) are then combined to give

$$\triangle \psi = -\omega. \tag{10}$$

The Green's function associated with equation (10) can be used to formally write down the velocity field associated with ψ, for an arbitrary ω. For a singular vorticity concentrated along a curve, the velocity field for the sheet can be written in terms of γ, i.e.,

$$\vec{U}(s,t) = \frac{1}{2\pi} \int K(\vec{x}(s,t) - \vec{x}(\hat{s},t))\gamma(\hat{s},t)d\hat{s}, \tag{11}$$

see Birkhoff([3]). The integral is taken over the sheet. $\vec{x} = (x,y)$ denotes a point on the interface being parametrized by time t and arclength s, and

$$K(x) = \frac{1}{2\pi}\frac{1}{|\vec{x}|^2}(-y,x). \tag{12}$$

Equation (11) could be used as an evolution equation for the interface assuming that $\gamma(s,t)$ is known. This is the idea behind the point vortex method.

We now give the an integral equation for the appropriate γ for the Hele-Shaw equations. An equation for γ is derived by taking the dot product of (1) with $\vec{\tau}$ in fluid 1 and fluid 2, subtracting one from the other. After solving for γ one obtains the equation

$$\gamma = \frac{\Delta\mu}{\bar{\mu}}\left[\frac{1}{2}(\vec{u_1} + \vec{u_2})\right] \cdot \vec{\tau} + \frac{b^2}{12\bar{\mu}}\nabla(\Delta p) \cdot \vec{\tau}, \tag{13}$$

where $\Delta p = p_2 - p_1$, $\Delta\mu = \mu_2 - \mu_1$ and $\bar{\mu} = \frac{1}{2}(\vec{\mu}_1 + \vec{\mu}_2)$. It can be shown that the velocity at the sheet, given by (11), is the average of the limiting values of the velocities approaching from each side, i.e.,

$$\vec{U} = \frac{1}{2}(\vec{u}_1 + \vec{u}_2). \tag{14}$$

This fact is used and surface tension (3) is introduced in order to write (13) in the form

$$\gamma(s,t) = \frac{\Delta\mu}{\bar{\mu}}\vec{U}\cdot\vec{\tau} + \frac{\Delta\mu}{\bar{\mu}}\vec{W}\cdot\vec{\tau} + \frac{\sigma b^2}{12\bar{\mu}}\frac{\partial}{\partial s}\frac{1}{R(s,t)}, \tag{15}$$

where $\vec{W}$ is the velocity far ahead and far behind the interface. Equation (15) is an integral equation for the strength of the vortex sheet since $\vec{U}$ is a function of γ. The interface is discretized into a set of point vortices located at arclengths s_i with coordinates $(x(s_i,t), y(s_i,t))$ denoted (x_i, y_i) where $i = 1, \ldots, N$. Equation (11) is replaced by a numerical integration formula and substituted into a discretized version of (15). γ is then solved for at each point vortex through iteration. The point vortices are then advected using a modified version of the vortex-in-cell method. The vortex-in-cell method is a method proposed by Christensen [7] to find the velocity field resulting from some arbitrary distribution of vorticity. A grid is superposed over the (singular) vorticity distribution. Using an interpolation method presented by Meng and Thompson ([15]), the strength is used to approximate the vorticity at the grid points. The stream function and hence the velocity field is then obtained by solving (10), using a fast Poisson solver. The point vortices are then advected according to this velocity field. The interface is evolved by applying this algorithm iteratively. The interface is assumed periodic in the x-direction.

As the vortex sheet evolves it stretches unevenly which could lead to a large segment of the interface represented by only a few point vortices. This is addressed through a redistribution of the vortices after each time step. In this redistribution, the interface is assumed to be a curve consisting of piecewise linear segments connecting the point vortices. New points are then

redistributed evenly onto this curve.

3 The Point Vortex Method

Equation (11) can be written in the complex form

$$\frac{dz(s,t)}{dt} = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\gamma(\hat{s},t)}{x(s,t) - x(\hat{s},t) - i(y(s,t) - y(\hat{s},t))} d\hat{s} \tag{16}$$

where $z = x + iy$ where $i = \sqrt{(}-1)$. As in the formulation of Tryggvason and Aref, let us assume that the interface is periodic in the x-direction. Without loss of generallity, let us assume that this period is 1. It can then be written as

$$\frac{dz(s,t)}{dt} = -\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\int_{0}^{S} \frac{\gamma(\hat{s},t)}{x(s,t) - x(\hat{s},t) - i(y(s,t) - y(\hat{s},t)) - j} d\hat{s} \tag{17}$$

where S is the total arclength taken over one period of x. We interchange the integral and summation, evaluate the complex sum using the well known formula

$$\pi coth(\pi z) = \sum_{j=-\infty}^{\infty} \frac{1}{z - ij} \tag{18}$$

to obtain

$$\frac{dz(s,t)}{dt} = -\frac{1}{2}\int_{0}^{S} \gamma(\hat{s},t) coth(\pi(\Delta y + i\Delta x)) d\hat{s} \tag{19}$$

where $\Delta x = x(s,t) - x(\hat{s},t)$ and $\Delta y = y(s,t) - y(\hat{s},t)$. After separating real and imaginary parts we have

$$\frac{dx(s,t)}{dt} = -\frac{1}{2}\int_{0}^{S} \frac{sinh(2\pi\Delta y)}{cosh(2\pi\Delta x) - cos(2\pi\Delta x)} \gamma(\hat{s},t) d\hat{s} \tag{20}$$

and

$$\frac{dy(s,t)}{dt} = \frac{1}{2}\int_{0}^{S} \frac{sin(2\pi\Delta y)}{cosh(2\pi\Delta x) - cos(2\pi\Delta x)} \gamma(\hat{s},t) d\hat{s} \tag{21}$$

For $0 \leq s \leq S$, our interface $(x(s,t), y(s,t))$ is discretized into a finite number of points, $(x(s_i,t), y(s_i,t)$, denoted (x_i, y_i) for $i = 1, \ldots, N$. We assume a uniform s-mesh with spacing

Δs. Equations (20) and (21) then become

$$\frac{dx_i}{dt} = -\frac{S}{2N} \sum_{j=1, j\neq i}^{N} \frac{sinh(2\pi(y_i - y_j))}{cosh(2\pi(x_i - x_j)) - cos(2\pi(x_i - x_j))} \gamma_j \Delta s \tag{22}$$

and

$$\frac{dy_i}{dt} = \frac{S}{2N} \sum_{j=1, j\neq i}^{N} \frac{sin(2\pi(y_i - y_j))}{cosh(2\pi(x_i - x_j)) - cos(2\pi(x_i - x_j))} \gamma_j \Delta s. \tag{23}$$

The above discretization is the trapezoidal rule applied to the $2N$ ordinary differential equations (20) and (21). In the above, we use a uniform mesh only for convenience. Equations (22) and (23) are solved using the fourth order Runge-Kutta method. γ_i represents the strength of the vortex sheet at point (x_i, y_i) and is approximated using Tryggvason and Aref's boundary integral method at each time step as described in the previous section. Because of (22) and (23), it would appear that the method for approximating the arclength between vortices would be important. In addition, one needs approximations for the tangent vector and the curvature at each vortex in the discretization of (15). We choose here to represent the interface by a parametric cubic spline to obtain approximations in an accurate way for the arclength, the tangent vector and the curvature. The points along the interface are redistributed equally with respect to arclength at each time step.

4 Numerical Results

In this section we compare results from using the point vortex method with those obtained by Tryggvason and Aref using the vortex-in-cell method and the McLean and Saffman solutions. The results of McLean and Saffman assume that $A = 1$, therefore all our computations shown here are for this value of A. Physically this implies that the viscosity of the injected fluid is negligible.

Our interface is initially a cosine wave of small amplitude. This wave is discretized and the numerical method is applied to evolve this wave which evolves into a long finger. By the linearized

theory, there is a wavelength λ_m which grows faster than all other wavelengths. Tryggvason and Aref plotted their results in terms of λ_m divided by λ_i where λ_i is the wavelength that an initially flat interface is perturbed by, i.e., the wavelength of the cosine wave. For fixed A, B is related to this ratio by

$$\frac{\lambda_m}{\lambda_i} = 2\pi\sqrt{(3B)}. \tag{24}$$

In the remainder of this paper, $\frac{\lambda_m}{\lambda_i}$ will be denoted by p.

Figure 1: A=1, p=.41, 1.3 and 1.68 respectively.

In figure 1, three fingers are shown for the values $p = .41$,1.3 and 1.68. The fingers equilibrate faster for small values of p. This is why the fingers are shorter from left to right. The first finger to the left represents an extremely small value of $p(p = .41)$. The middle finger represents a moderate value of p $(p = 1.3)$. The last finger from the left represents an extremely large value of p. This value of p $(p = 1.68)$ is extremely close to values of p where any initial perturbation would be linearly stable and would therefore not grow. The fingers develop a more pronounced neck as p increases. This is observed by other methods also.

Figure 2: McLean and Saffman solutions(solid curve) versus vortex-in-cell method(+).

In figure 2, we show the predicted finger widths of McLean and Saffman which correspond to the solid curve. The "+" in figure 2 represents finger widths obtained by Tryggvason and Aref using their finest resolution. For intermediate values of p, Tryggvason and Aref's finger widths are in excellent agreement with the results of McLean and Saffman, but for small values of p, the method produces a different finger width. Also for large values of p the fingers are slightly wider.

Figure 3: McLean and Saffman solutions(solid curve) versus results from the point vortex method (x,o).

In figure 3, we compare the results of the point vortex method with the McLean and Saffman fingers. The symbols "x" and "o" in figure 3 correspond to 8 and 16 vortices per most unstable wavelength(λ_m) respectively. This number of vortices, per most unstable wavelength, is maintained as the interface evolves. The results clearly converge to the McLean and Saffman fingers. We do not observe a narrower finger for small values of p as Tryggvason and Aref conjecture. The method also seems to do a much better job for the large values of the surface tension parameter. The method becomes unstable if the number of vortices per most unstable wavelength exceeds about 24 and the instability varies depending on p. The method becomes unstable in the sense that the wave begins to exhibit a chaotic behavior and the integral equation for the strength requires more and more iterations to converge. The maximum number of vortices per most unstable wavelength can be increased as the wave grows.

The number of vortices used to compute the equilibrated fingers for different values of p ranged from 178 to 322, using 16 vortices per most unstable wavelength. The 322 corresponds to the smallest value of p in figure 3 and 178 corresponds to the largest value of the p given in figure 3. The fingers grown from smaller surface tensions initially need more vortices since λ_m is smaller but these interfaces settle into equilibrated fingers earlier. The fingers grown from larger surface tension require less vortices initially and require longer to settle down.

Acknowledgements. I would like to thank Professor Gretar Tryggvason for several helpful discussions. These computations were done at the GANG lab at the University of Massachusetts at Amherst.

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Figure 1. Three evolved fingers for the nondimensional surface tension parameter $p = .41$, 1.3 and 1.68.

Figure 2. Finger widths from the method of Tryggvason and Aref(+) versus predicted widths from McLean and Saffman(solid line).

Figure 3. Finger widths from the point vortex method(Method I) versus the predicted widths from McLean and Saffman(solid line). The symbols "x" and "o" correspond to 8 and 16 vortices per unstable wavelength respectively.

Figure 4. Finger widths from the modified point vortex method(Method II) versus the predicted widths from McLean and Saffman(solid line). The symbols "x" and "o" correspond to 4 and 8 vortices per unstable wavelength respectively.