Structure Theorems for Constant Mean Curvature Surfaces Bounded by a Planar Curve

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1 Introduction

A circle C in $\mathbb{R}^3$ is the boundary of two spherical caps of constant mean curvature H for any positive number H, which is at most the radius of C. It is natural to ask whether spherical caps are the only possible examples. Some examples of constant mean curvature immersed tori by Wente [7] indicate that there are compact genus-one immersed constant mean curvature surfaces with boundary C that are approximated by compact domains in Wente tori; however, this has not been proved. Still one has the conjecture:

Conjecture 1 *A compact constant mean curvature surface bounded by a circle is a spherical cap if either of the following conditions hold:*

1. *The surface has genus 0 and is immersed;*
2. *The surface is embedded.*

If M is a compact embedded constant mean curvature surface in $\mathbb{R}^3$ with boundary C and M is contained in one of the two halfspaces determined by the plane containing C, the Alexander reflection method [1] immediately proves M has the planar reflectional symmetries of C; hence, M is a surface of revolution. Since the only compact constant mean curvature surfaces of revolution are spherical caps (by Delaunay's classification [3] of constant mean curvature surfaces of revolution), Conjecture 1 holds for the subclass of surfaces that are embedded and contained in a halfspace. It is therefore of interest to obtain natural geometric conditions that force a compact embedded constant mean curvature surface to be contained in a halfspace.

One result of our paper is to give the following sufficient condition for a compact constant mean curvature surface to be contained in a halfspace.

Theorem 1 *Let C be a convex curve in a plane P and let M be a compact connected surface with boundary C. Assume M is embedded, of constant mean curvature, and transverse to P along C. Then M is contained in one of the halfspaces of $\mathbb{R}^3$ determined by P.*

Theorem 1 can be generalized to the case where the planar curve C is not necessarily convex.

Theorem 2 *Let M, C and P be as in Theorem 1, except that C is not necessarily convex. Let H^+ and H^- denote the closed halfspaces determined by P and indexed so that there is a connected component M^+ of $M \cap H^+$ with $C \subset \partial M_+$. Let $M^- = M \cap H^-$. Let $D \subset P$ be the planar disk with $\partial D = C$. Then*

1. *$M \cup D$ is an embedded surface;*
2. *$M^+ = M \cap H^+$;*
3. *M^- is a graph over the compact domain in P bounded by $\partial M^+ - C$;*
4. *No component of ∂M^+ lies completely outside of the convex hull of C.*

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2 The proof of Theorem 1.

The proof of Theorem 1 uses the balancing formula [5]:

$$\int_D Y \cdot \eta_D = \frac{1}{2H} \int_{\partial M} Y \cdot \nu \, .$$

Here Y is a constant vector field in $\mathbb{R}^3$, H is the positive mean curvature of M, ν is the inward pointing conormal to M along ∂M, $D \subset P$ is the planar disk with boundary C, and η_D is the unit normal vector field to D. The orientation of η_D is that which orients the cycle $M \cup D$ when M is oriented by its mean curvature vector.

Proof of the balancing formula. The flux of Y across the oriented cycle $M \cup D$ is zero; hence

$$\int_M Y \cdot \eta_M + \int_D Y \cdot \eta_D = 0\,,$$

where the unit normal η_M of M is oriented by the mean curvature vector of M.

Let X denote the position vector field of M, so that $\Delta X = 2H\eta_M$. Then

$$\begin{aligned}\int_M Y \cdot \eta_M &= \frac{1}{2H}\int_M Y \cdot \Delta X = \frac{1}{2H}\int_M \Delta(Y \cdot X)\\ &= -\frac{1}{2H}\int_{\partial M} Y \cdot \frac{\partial X}{\partial \nu} = -\frac{1}{2H}\int_{\partial M} Y \cdot \nu\,,\end{aligned}$$

since $\frac{\partial X}{\partial \nu} = \nu$. $\square$

We remark that the balancing formula still holds when M is not transverse to P along C.

Proof of Theorem 1. Assume the contrary, so that M meets P elsewhere than C. We can assume M meets P transversally by making a small vertical translation (we think of P as horizontal).

Let $C_1, \ldots, C_n$ be the simple closed curves of M in $\mathrm{Int}(D)$ (if there are any). For each j, $1 \leq j \leq n$, let $C_j^+(\varepsilon)$ be the planar curve on M, near C_j, obtained by intersecting M with the horizontal plane $P(\varepsilon)$, at height ε. Similarly, let $C_j^-(\varepsilon)$ be the curve in $M \cap P(-\varepsilon)$ that is near C_j.

We form an embedded surface N by removing from M the annuli bounded by the $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$ and attaching the horizontal planar disks $D_j^+ \cup D_j^-$, bounded by the $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$. Also we attach D to M along C. To ensure that N is embedded, one uses different values of ε when several C_j are concentric; see Figure 1.

Let $\widetilde{M}$ be the connected component of N that contains C. $\widetilde{M}$ separates ^{3} into two components; let A be the closure of the bounded component. The nonsmooth points of $\widetilde{M}$ are on C and the $C_j^{\pm}(\varepsilon)$ and the mean curvature

Figure 1:

vector of $\widetilde{M}$ points into A. Notice that this is possible since $\widetilde{M}$ is obtained by attaching disks to a smooth connected submanifold of M (where the mean curvature vector is never zero) and the mean curvature vector extends across these disks (cf. Figure 1). We orient $\widetilde{M}$ by the mean curvature vector. Denote the set $P - D$ by $\mathrm{Ext}(D)$ and the interior of D by $\mathrm{Int}(D)$. We will see that M is disjoint from $\mathrm{Ext}(D)$ and $\mathrm{Int}(D)$, thus proving the theorem.

We first observe that $M \cap \mathrm{Int}(D) \neq \emptyset$ and $M \cap \mathrm{Ext}(D) = \emptyset$ leads to a contradiction. Assume now that $M \cap \mathrm{Ext}(D) = \emptyset$. Let M_1 be the connected component of $M \cap \mathbb{R}^3_+$ that contains C, where $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \mid x_3 \geq 0\}$. M_1 together with a proper submanifold D_1 of D bound an embedded three-manifold $W_1 \subset \mathbb{R}^3_+$; see Figure 2.

We apply the balancing formula to $M_1 \cup D_1$:

$$\int_{\partial M_1} Y \cdot \nu = 2H \int_{D_1} Y \cdot \eta_{D_1} \,,$$

with $Y = e_3 = (0,0,1)$. Notice that the conormal ν along ∂M_1 points into M_1, hence, $\nu \cdot e_3 > 0$ along ∂M_1 and $\displaystyle\int_{\partial M_1} Y \cdot \nu > \int_C Y \cdot \nu$, when M_1 has other components in $\mathrm{Int}(D)$.

The right side of the balancing formula is $2H(\text{area } D_1) < 2H(\text{area } D)$.

Now apply the balancing formula to $M \cup D$: $\displaystyle\int_C Y \cdot \nu = 2H \int_D Y \cdot \eta_D$, again with $Y = e_3$. Since the right side is $2H(\text{area } D)$, this is incompatible

Figure 2:

with the previous inequalities. Thus we can assume $M \cap \mathrm{Ext}(D) \neq \emptyset$.

We will show that $\widetilde{M} \cap \mathrm{Ext}(D) \neq \emptyset$ leads to a contradiction by using the Alexandrov reflection principle with vertical planes coming from infinity towards $\widetilde{M}$.

Recall that $\widetilde{M}$ separates ^{3} into two connected components (A is the bounded component) and $\widetilde{M}$ is smooth away from C and the disks D_j^+ and D_j^- contained in it. First we observe that $\widetilde{M} \cap \mathrm{Ext}(D)$ has no components that are null homotopic in $\mathrm{Ext}(D)$. To see this, suppose α were such a component. Let ℓ be an infinite line segment, starting at a point of D and intersecting α in at least two points. Consider a family $Q(t)$, $t < \infty$, of parallel vertical planes coming from infinity and orthogonal to ℓ. Suppose the family $Q(t)$ intersects $\widetilde{M} \cap \mathrm{Ext}(D)$ for the first time at $t = t_0$. Continuing the movement of $Q(t_0)$ by parallel translation towards C, would produce for some $t_1 < t_0$ a point of tangential contact of $\widetilde{M}$ with the reflection of $\widetilde{M} \cap (\bigcup_{t_1 \leq t \leq t_0} Q(t) \cap \widetilde{M})$ in $Q(t_1)$, before reaching C. This would yield a plane of symmetry of $\widetilde{M}$ with D on one side of the plane; a contradiction. We remark that the Alexandrov reflection principle applies here because $\widetilde{M}$ bounds the compact region A and the first point of contact of the symmetry with $\widetilde{M}$ occurs before D, hence at a smooth point of $\widetilde{M}$. We used the fact that C is convex here to ensure that a symmetry of α touches α before the

Figure 3:

plane reaches C.

So we can assume $Q(t)$ touches $\widetilde{M} \cap \mathrm{Ext}(D)$ for the first time along a curve α that is homotopic to C in $\mathrm{Ext}(D)$. Let E be the annulus in $\mathrm{Ext}(D)$ bounded by $\alpha \cup C$. Observe first that $\mathrm{Int}(E)$ contains no components α_1 of $\widetilde{M} \cap \mathrm{Ext}(D)$ that are homotopic to C in E. This follows by using the reflection principle with vertical planes $Q(t)$ as above: a symmetry of $\widetilde{M}$ would intersect $\widetilde{M}$ for a first time before arriving at D; see Figure 3.

Hence the mean curvature vector along $\alpha \cup C$ points into E; in particular, along C, it points towards $\mathrm{Ext}(D)$. But this contradicts the balancing formula:

$$\int_C Y \cdot \nu = 2H \int_D Y \cdot \eta_D \,,$$

where $Y = e_3$, so $Y \cdot \nu > 0$ along C. Since $\vec{H}$ points towards $\mathrm{Ext}(D)$ along C, $\eta_D = -e_3$; so the right side is $-2H(\text{area } D)$. $\square$

The proof of Theorem 1 generalizes immediately to constant mean curvature hypersurfaces in $^{n+1}$ to give:

Theorem 3 *Let C be a convex $(n-1)$-sphere in a hyperplane $P \subset {}^{n+1}$ and let M be a compact submanifold with boundary C. Assume M is embedded of constant mean curvature, and transverse to P along C. Then M is contained in one of the halfspaces of ^{3} determined by P.*

We make one further conjecture directly related to Theorem 1.

Conjecture 2 *A compact, embedded, constant mean curvature surface in $\mathbb{R}^3_+$ bounded by a convex planar curve in the x_1x_2-plane must have genus zero. (This conjecture is even unknown for soap bubbles bounding a convex planar curve.)*

3 The proof of Theorem 2.

We now prove Theorem 2. The main idea of the proof is to construct an abstract flat three-manifold that isometrically submerses into $\mathbb{R}^3$ and then to apply Alexandrov reflection in this abstract manifold.

Let M, M^+, M^-, C, D and P be as defined in the statement of Theorem 2. Assume for convenience that P is the (x_1, x_2)-plane, H^+ is the upper halfspace in $\mathbb{R}^3$ and H^- is the lower halfspace. Let $C_1, \ldots, C_n$ be the simple closed curves in $M \cap \mathrm{Int}(D)$ (if there are any). As in the proof of Theorem 1 define the related curves $C_1^+(\varepsilon), C_1^-(\varepsilon), \ldots, C_n^+(\varepsilon), C_n^-(\varepsilon)$ bounded by the planar disks $D_1^+, D_1^-, \ldots, D_n^+, D_n^-$. Similarly, define the embedded surface N by removing from M the annuli bounded by the $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$ and attaching the disks $D_j^+ \cup D_j^-$. Also we attach D to M along C. As in the proof of Theorem 1, to ensure that N is embedded, one uses different values of ε when several of the C_j are concentric. We may choose these values of ε sufficiently small so that the image of the orthogonal projection of $\cup \partial D_j^+$ onto P is an embedded 1-manifold, as is the projection of $\cup \partial D_j^-$.

Note that there is a natural partial ordering of the disks $D_1^+, \ldots, D_n^+$ as well as for $D_1^-, \ldots, D_n^-$. Namely, we say that $D_i^+ \geq D_j^+$ if the third coordinate of D_i^+ is greater than the third coordinate of D_i^+ and if $C_i \subset P$ lies inside $C_j \subset P$. Of course $D_i^+ \geq D_j^+$ geometrically just means that D_i^+ lies over D_j^+. Similarly we can partially order the disks $D_1^-, \ldots, D_n^-$ and it makes sense to make the statement that D_i^- lies over D_j^-.

Let $\widetilde{M}$ denote the component of N that contains D. $\widetilde{M}$ separates $\mathbb{R}^3$ into two components; let A be the closure of the bounded component. Note that we orient $\widetilde{M}$ by its mean curvature vector and this vector points into A.

Assertion 1 $\widetilde{M} \cap (\cup D_i^-) = \emptyset$.

Assume for the moment that Assertion 1 holds and we will show how the theorem follows. First note that $\widetilde{M} \cap (\cup D_i^+) = \emptyset$ by applying the flux formula to the cycle $\widetilde{M}$. (See Figure 2 and the part of the proof of Theorem 1 where it is shown that $\widetilde{M} \cap \mathrm{Ext}(D) = \emptyset$ implies $\widetilde{M} \cap \mathrm{Int}(D) = \emptyset$.) This proves $\widetilde{M} = M \cup D$ and therefore part 1 of Theorem 2 is proved. It follows that A is nonsmooth only along C. Also note by the flux formula applied to $\widetilde{M}$ and $Y = e_3$, the mean curvature vector of $\widetilde{M}$ along C points into the region above D. Therefore A lies above D.

The usual application of the Alexandrov reflection principle, applied using horizontal planes with negative third coordinate, shows that $M^- = \widetilde{M} \cap H^-$ is a graph over its projection onto P. This proves part 3 holds.

We now prove the part of M above P is connected; this will establish part 2 of Theorem 2. Suppose, on the contrary, that the part of M above P is not connected. Let Σ be the union of all components of M above P except the component M^+ that contains C. Σ together with a planar domain in P bounds a compact region R in H^+. We claim that M^+ is disjoint from R. Otherwise M^+ is contained in a connected component R_1 of R. Then take a path Γ in R_1 joining a highest point of ∂R_1 to a highest point of M^+ (this is easy since M^+ is connected). Γ meets $\widetilde{M}$ only at its end points. At the end point on ∂R_1, the mean curvature points down (and into A), hence $\Gamma \subset A$. But at the end point of Γ on M^+ the mean curvature vector is also pointing into A. Since it's also pointing down there, this is impossible. This proves M^+ is disjoint from R.

Note that every path in $\mathrm{Int}(R)$ is contained completely inside A or completely outside A. The reason for this is that such a path is disjoint from M since $M \cap H^+ = M^+ \cup \Sigma$. However every component of R intersects $\mathrm{Int}(A)$ near a highest point of the component. Hence, $R \subset A$.

Now the Alexandrov reflection principle, using horizontal planes above P, shows Σ is a graph over a domain in P. It is possible to apply Alexandrov reflection in this case because M^+, and hence D, is disjoint from R. We know that M^- is also a graph over a domain in P. Let $x \in M^- \cap \Sigma$. Σ is not vertical at x since P would be a plane of symmetry of M if this were so. (Σ is a graph so Alexandrov reflection would show that P is a plane of symmetry of M if Σ were vertical at x.) It follows that $M^- \cup \Sigma$ is a graph of a function over a smooth domain in P with zero boundary values and on some component domain the function changes sign. Hence on this component the function has both local maxima and local minima and the sign of the mean curvature of the graph at these points must be opposite. Thus $M^- \cap \Sigma = \emptyset$ and $\Sigma = \emptyset$ as desired. This completes the proof of parts 1, 2 and 3. The proof of part 4 now proceeds as in the proof of Theorem 1; one does Alexandrov reflection using vertical planes. It remains to prove Assertion 1.

Proof of Assertion 1. Suppose $\widetilde{M} \cap (\cup D_i^-) \neq \emptyset$. After reindexing we may assume that $\widetilde{M} \cap (\bigcup_{i=1}^n D_i^-) = \bigcup_{i=1}^k D_i^-$. Let $M_1, \ldots, M_p$ denote the components of N, other than $\widetilde{M}$, that have at least one of the disks $D_1^+, \ldots, D_k^+$ in their boundary. Each M_i separates ^{3} into two components; let A_i denote the closure of the bounded component. Note that the mean curvature of M_i points into A_i. We will orient all planar disks in ∂A or in ∂A_i by the inward pointing normal.

Choose an $i \leq k$. We know that $D_i^- \subset \partial A$. Let j be such that $D_i^+ \subset \partial A_j$. Since the mean curvature vector of the annulus on M joining ∂D_i^- to ∂D_i^+ is continuous, D_i^- is oppositely oriented from D_i^+. Let Q_i denote the solid cylinder bounded by $D_i^- \cup D_i^+$ together with the annulus on M bounded by $\partial D_i^+ \cup \partial D_i^-$.

Form a space X as follows. First take the disjoint union of $A, A_1, \ldots, A_k$. For each i, $i \leq k$, such that the outward orientation of Q_i agrees with the orientation of $D_i^- \cup D_i^+$, we attach Q_i to the disjoint union of $A, A_1, \ldots, A_k$. Note that the space X is a flat compact three-manifold that submerses iso-

Figure 4:

metrically into ^{3}.

Suppose for the moment that there exists a disk $D_{t_1}^-$ in ∂X such that $D_{t_1}^-$ is also in ∂A, i.e., $1 \leq t_1 \leq k$. In particular $D_{t_1}^-$ is oriented by e_3. Notice that there exists a disk $D_j^- \subset \partial A$ that lies directly above $D_{t_1}^-$. (A lies above D and A lies above $D_{t_1}^-$.)

Claim 1 *Suppose E_1^+ is the component in $\{A, A_1, \ldots, A_p\}$ such that $D_j^+ \subset \partial E_1^+$. Then there does not exist a disk of E_1^+ that lies between D_j^+ and $D_{t_1}^+$ and furthermore $D_{t_1}^+ \subset \partial E_1^+$.*

Claim 1 will allow us to modify X as follows. By Claim 1, for every disk $D_{t_1}^-$ in ∂X such that $D_{t_1}^-$ is also in ∂A, the component E_1^+ in $\{A, A_1, \ldots, A_s\}$ such that $D_j^+ \subset \partial E_1^+$ has the property that $D_{t_1}^+ \subset \partial E_1^+$ and there no disks of E_1^+ between D_j^+ and $D_{t_1}^+$. We next modify X by lifting the cylinder Q_{t_1} into X and then form $\overline{X - Q_{t_1}}$. After removing all such solid cylinders in X, we obtain a new manifold $\widetilde{X}$ with the property that every disk $D_s^- \subset \partial X$ satisfies $D_s^- \cap \partial A = \emptyset$.

Proof of Claim 1. Suppose that there is a disk $D_{t_2}^+ \subset \partial E_1^+$ that lies between D_j^+ and $D_{t_1}^+$. We choose $D_{t_2}^+$ to be the lowest such disk, i.e., there are no disks of ∂E_1^+ between D_j^+ and $D_{t_2}^+$. Then $D_{t_2}^+$ is oriented by $-e_3$. Now the disk $D^-_{t_2}$ is between $D^-_{t_1}$ and D^-_j. $D^-_{t_2}$ is not in ∂A (since D^-_j is the disk of ∂A directly above $D^-_{t_1}$) and $D^-_{t_2} \subset A$, therefore the connected component of N containing $D^-_{t_2}$ is contained in A. Let E^-_2 be the compact region bounded by this connected component; $E^-_2 \subset A$. Since $D^+_{t_2}$ is oriented by $-e_3$, $D^-_{t_2}$ is oriented by e_3. Hence there is a disk $D^-_{t_3}$ of ∂E^-_2 directly above $D^-_{t_2}$. Clearly $D^+_{t_3}$ lies between $D^+_{t_2}$ and D^+_j. Since $D^+_{t_3} \not\subset \partial E^+_1$ and $D^+_{t_3} \subset E^+_1$, the connected component of N that contains $D^+_{t_3}$ is contained in E^+_1. Therefore the compact region of ^{3} bounded by this component is contained in E^+_1. We label this region E^+_3. Since $D^+_{t_3}$ is oriented by e_3, there exists a disk $D^+_{t_4}$ of ∂E^+_3 directly above $D^+_{t_3}$. The connected component of N that contains $D^-_{t_4}$ is contained in E^-_2 so the compact region E^-_4, that it bounds, is also contained in E^-_2. Clearly this process yields an infinite sequence of compact regions E^-_2, E^-_4, etc., which is impossible. This proves that there is no disk of ∂E^+_1 between $D^+_{t_1}$ and D^+_j.

Next we prove $D^+_{t_1} \subset \partial E^+_1$. If not, then the connected component of N that contains $D^+_{t_1}$ is contained in E^+_1, since $D^+_{t_1} \subset E^+_1$. Let E^+_2 be the compact region bounded by this connected component. We know $D^+_{t_1}$ is oriented by $-e_3$ so there is a disk $D^+_{t_2}$ of ∂E^+_2 directly below $D^+_{t_1}$ and above D^+_j. Now $D^-_{t_2}$ is between $D^-_{t_1}$ and D^-_j and $D^-_{t_2}$ is not in ∂A. Hence the connected component of N that contains $D^-_{t_2}$ is contained in A. Let E^-_3 be the compact region bounded by this connected component. Clearly this process (just as in the last paragraph) yields an infinite sequence of compact regions $E^-_3, E^-_5, \ldots$, which is impossible. This proves Claim 1. $\square$

Now consider the disks of $\partial\widetilde{X}$ in H^+. We claim there are no disks of $\partial\widetilde{X}$ that are directly above the D^+_i that came from the D^-_i of ∂A oriented by $-e_3$. One proves this exactly as the proof of Claim 1: just turn ^{3} over ($x_3 \to -x_3$) and call such a disk of $\partial\widetilde{X}$ (if it existed) $D^-_{t_1}$. Then proceed exactly as in the proof of Claim 1.

Now consider the surface $\widetilde{M}^- = \widetilde{M} \cap H^-$. Let $A_{-t} = \{p \in A \mid x_3(p) \leq -t\}$ for any $-t \leq 0$. Let $P_t = P + (0,0,t)$ and let $R_t\colon {}^3 \to {}^3$ be reflection in P_t.

Consider $A_{-t} \subset \widetilde{X}$ and the largest value $-t_0$ such that the induced reflection $\widetilde{R}_t\colon A_{-t} \to \widetilde{X}$ is defined. Since there are no disks in $\partial\widetilde{X}$ directly above the D_i^+ that came from the D_i^- of ∂A oriented by $-e_3$, the Alexandrov reflection principle shows that $t_0 = 0$ and $\widetilde{M}^-$ is a graph over its projection Δ onto P. (It is helpful to think of $\widetilde{M}^-$ as having its boundary on P by letting ε go to zero.) Notice that $\Delta \subset \widetilde{X}$ and $\Delta \cap C = \emptyset$. On the other hand, a component of Δ that has a boundary curve in D must also have another boundary curve in $P - D$ and so such a component must contain the curve C. Since $\Delta \cap C = \emptyset$, we conclude that $\Delta \cap D = \emptyset$. This establishes Assertion 1 and completes the proof of Theorem 2. $\square$

Remark 1 Brito and Sá Earp [2] recently gave a somewhat simpler proof of Theorem 2, again using the balancing formula and a reflection principle related to the Alexandrov reflection principle.

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