Minimal Surfaces Bounded by Convex Curves in Parallel Planes

William H. Meeks, III*
Mathematics Department
University of Massachusetts
Amherst, MA 01003

Brian White†
Mathematics Department
Stanford University
Stanford, CA 94305

September 5, 1990

1 Introduction

In 1956 M. Shiffman [17] proved several beautiful theorems concerning the geometry of a minimal annulus A whose boundary consists of two closed convex curves in parallel planes P_1, P_2. The first theorem stated that the intersection of A with any plane P, between P_1 and P_2, is a convex Jordan curve. In particular it follows that A is embedded. He then used this convexity theorem to prove that every symmetry of the boundary of A extended to a symmetry of A. In the case that ∂A consists of two circles Shiffman proved that A was foliated by circles in parallel planes. Earlier B. Riemann [15] described, in terms of elliptic functions, all minimal annuli in ^{3} that can be expressed as the union of circles in parallel planes (also see [3] for a

*The research described in this paper was supported by research grant DE-FG02-86ER250125 of the Applied Mathematical Science subprogram of Office of Energy Research, U.S. Department of Energy, and National Science Foundation grant DMS-8900285.

†Funded by National Science Foundation grants DMS-8553231 (PYI) and DMS-8703537.

description of these surfaces as well as a computer graphics image of one of them). Together these results yield a classification of all minimal annuli with boundary consisting of circles in parallel planes.

We shall call a compact minimal surface M *stable* if, with respect to any nontrivial normal variation fixing the boundary, the second derivative of area is positive. If the second derivative of area is negative for some variation, then M is called *unstable*. If M is neither stable nor unstable, we will call it *almost-stable*.

Our main theorem, given below, augments Shiffman's theorems.

Theorem 1.1 *If Γ is a pair of smooth convex Jordan curves in distinct parallel planes, then exactly one of the following holds:*

1. *Γ is not the boundary of any connected compact minimal surface, with or without branch points.*

2. *Γ is the boundary of exactly one minimal annulus and this annulus is almost-stable. In this case, Γ bounds no other connected compact branched minimal surfaces.*

3. *Γ is the boundary of exactly two minimal annuli; one stable and one unstable. (Perhpas M is the boundary of a connected minimal surface of positive genus.)*

In certain cases it is known that every connected branched minimal surface with boundary Γ described in Theorem 1.1 is actually an annulus. For example, R. Schoen [16] proved that when Γ is contained in parallel horizontal planes and is invariant under reflection in two vertical planes, then every branched minimal surface with boundary Γ is actually an annulus. Thus, in certain cases, Theorem 1.1 shows that Γ is the boundary of 0, 1, or 2 minimal annuli and no other branched minimal surfaces. In general, Meeks conjectured that every branched minimal surface with boundary consisting of a pair of convex Jordan curves in parallel planes is an annulus (see Conjecture 16 in [10]).

The proof of the main theorem is based on an analysis of the Gauss map of a minimal annulus with boundary Γ and a geometric approach to calculating the index of a minimal annulus with boundary Γ. In the proof of Theorem 1.1 we rely on the description of the space of smooth embedded minimal annuli in $\mathbb{R}^3$ as developed in [22] and techniques from global analysis.

Theorem 1.1 has some interesting applications. Perhaps the most important application occurs in the proof of uniqueness of a natural free boundary value problem, which we now describe. Suppose α is a Jordan curve in a plane P_0 and Σ is a compact branched minimal surface such that $\partial\Sigma$ consists of α together with a nonempty collection of immersed curves on a parallel plane P_1. If Σ is orthogonal to P_1 along $\partial\Sigma \cap P_1$, then Σ is called a *solution of the free boundary value problem for* α *and* P_1. If, with respect to any nontrivial normal variation of Σ that vanishes on α, the second derivative of the area functional is positive, then Σ is called a *stable* solution to the free boundary value problem. Similarly, we can define when Σ is unstable or almost-stable.

Theorem 1.2 *Suppose* Σ *is a solution of the free boundary value problem for a smooth convex plane curve* α *and a plane* P_1 *parallel to the plane containing* α*. Then:*

1. Σ *is embedded;*
2. *There exists a unique stable or almost-stable solution* $\widetilde{\Sigma}$ *to the free boundary value problem for* α *and* P_1*. Furthermore,* $\widetilde{\Sigma}$ *is an annulus that is foliated by convex curves in parallel planes.*

The proof of part 1 of Theorem 1.2 follows immediately from the results of Schoen in [16]. Part 2 of Theorem 1.2 is a simple consequence of the slightly stronger Theorem 3.1 that appears in Section 3 (see Corollary 3.1).

In Section 3 we also show that if Σ is a connected compact stable minimal surface with convex boundary curves Γ in parallel horizontal planes and Γ is invariant under reflection in a vertical plane, then Σ is an annulus. In particular, a compact connected stable minimal surface with boundary two circles in parallel planes is unique and is topologically an annulus.

In Section 4 we make some further applications of Theorem 1.1. There, using Riemann's classification result, we give a simple proof of Shiffman's geometric characterization of minimal annuli having circle boundaries in parallel planes. In Section 5 we give an analytic characterization of the space of smooth minimal annuli whose boundary curves are strictly convex smooth curves in parallel planes.

In [12] we prove some related theorems for the case of minimal annuli bounded by a pair of convex planar curves whose union lies on the boundary of the convex hull of the union.

2 Proof of the main theorem

In this section we shall prove Theorem 1.1, which is stated in the Introduction. Without loss of generality we may assume that $\Gamma = \{\gamma_0, \gamma_1\}$ is a pair of convex Jordan curves where $\gamma_0 \subset P_0 = \{x_3 = 0\}$ and $\gamma_1 \subset P_1 = \{x_3 = 1\}$. In the proof of the Theorem 1.1 we shall apply some techniques of global analysis that are useful in describing spaces of curves and spaces of minimal surfaces. Let $\mathcal{C}$ be the space of pairs $\{\alpha_0, \alpha_1\}$ of smooth simple closed curves where $\alpha_0 \subset P_0$ and $\alpha_1 \subset P_1$. Let $\mathcal{M}$ be the space of embedded minimal annuli with boundary curves in $\mathcal{C}$. It follows from the work of White [22] that the natural projection $p\colon \mathcal{M} \to \mathcal{C}$ is a proper smooth Fredholm map of index 0. (See also earlier work in [20]). In the proof of Theorem 1.1 we shall use these properties of p in conjunction with the Smale Transversality Theorem [18], which holds in this setting. We begin the proof of the theorem with an existence result that is a simple consequence of the results in [14].

Lemma 2.1 *Suppose* $\Delta = \{\alpha, \beta\}$ *is a pair of continuous Jordan curves,* $\alpha \subset P_0$ *and* $\beta \subset P_1$*. Let* D_α *and* D_β *be the compact planar disks with* $\partial D_\alpha = \alpha$ *and* $\partial D_\beta = \beta$*. Suppose there exists a connected nonplanar compact branched minimal surface* Σ *whose boundary is contained in* $\mathcal{D} = D_\alpha \cup D_\beta$*. Then there exists a unique embedded minimal annulus* $\mathcal{A}$ *with* $\partial\mathcal{A} = \Delta$ *and such that the following hold:*

1. *Let* B *be the compact region of* ^{3} *with boundary* $\mathcal{A} \cup \mathcal{D}$*. Then every compact branched minimal surface* M *with* $\partial M \subset \mathcal{D}$ *is contained in* B*.*

2. *If* M *is a nonplanar compact branched minimal surface with* $\partial M \subset \mathcal{D}$ *and* $\mathrm{Int}(M) \cap \partial B \neq \emptyset$*, then* $M = \mathcal{A}$*.*

3. $\mathcal{A}$ *is stable or almost-stable.*

Proof. We first show that Δ is the boundary of some minimal annulus that is stable or almost-stable. Let M be the image of some connected branched minimal surface with $\partial M \subset \mathcal{D}$. The surface M disconnects the slab with boundary $P_0 \cup P_1$ into several components, exactly one whose closure W is noncompact.

Approximate α and β by smooth curves $\alpha_i \subset (P_0 - D_\alpha)$ and $\beta_i \subset (P_1 - D_\beta)$ converging to α and β, respectively. Note that $\alpha \cup \alpha_i$ and $\beta \cup \beta_i$ are each the boundary of annuli whose areas go to zero as i goes to infinity. The curves α_i and β_i are homotopic in W but are not homotopically trivial in W. The boundary of W, although not smooth, is a good barrier for solving least-area problems in W (see Theorem 1 in [14]). Hence the pair of curves $\alpha_i \cup \beta_i$ is the boundary of a least-area annulus in W, *stable or almost-stable in* ^{3}, and these least-area annuli are embedded by Geometric Dehn's Lemma in [14]. After choosing a subsequence, these least-area embedded annuli converge to a least-area (hence, stable or almost-stable) embedded minimal annulus $\widetilde{A}$ with boundary $\alpha \cup \beta$ (see [13] for this type of compactness argument). Furthermore, by the maximum principle, either this annulus is equal to M or $M \cap \widetilde{A} = \Delta$.

Now choose an embedded stable (or almost-stable) minimal annulus $\mathcal{A}$ with the property that the volume of $B_{\mathcal{A}}$ is the greatest. The choice of $\mathcal{A}$ is always possible by the compactness of set of embedded stable minimal annuli with boundary Γ. (See for example [1, 21]). If M is a branched minimal surface with boundary Δ and M is not contained in $B_{\mathcal{A}}$, then using $M \cup \mathcal{A}$ as a barrier we produce from the above procedure a least-area embedded minimal annulus $\mathcal{A}'$ that lies outside $M \cup \mathcal{A}$. Hence, $B_{\mathcal{A}} \subset B_{\mathcal{A}'}$, which contradicts the largest volume property for $B_{\mathcal{A}}$. Thus, $M \subset B_{\mathcal{A}}$ and, by the maximum principle, $\mathrm{Int}(M) \cap \mathcal{A} \neq \emptyset$ implies $\mathcal{A} = M$. $\square$

Remark 2.1 *Notice that Lemma 2.1 gives some partial information on results claimed in Theorem 1.1. Namely, if the convex curves* Γ *are the boundary of some compact branched minimal surface, then* Γ *is the boundary of an embedded minimal annulus that is stable or almost-stable. It remains to prove that if* Γ *is smooth and it is the boundary of a stable or almost-stable minimal annulus* A*, then exactly one of the following holds:*

1. A *is almost-stable and* Γ *bounds no other connected minimal surface.*
2. A *is stable and* Γ *bounds exactly one other minimal annulus, which is unstable.*

The next step in the proof of Theorem 1.1, Lemma 2.2, shows that the interior of a minimal annulus A with continuous convex boundary Γ can be conformally parametrized by the image of its Gauss maps. In order to obtain this result it is convenient to define two Gauss maps for a smooth orientable minimal surface; the first $G\colon M \to S^2$ given by translating the unit normal to the origin and the second $g\colon M \to \ \cup \{\infty\}$ where g is the map G composed with stereographic projection. It follows directly from the definition of a minimal surface that the map g is conformal wherever the derivative of g is nonzero.

Lemma 2.2 *Let $\Gamma = \{\gamma_0, \gamma_1\}$ be a pair of continuous convex curves in the planes P_0, P_1, respectively. If A is a minimal annulus with $\partial A = \Gamma$, then $g\colon \mathrm{Int}(A) \to \ \cup \{\infty\}$ gives rise to a conformal diffeomorphism between* $\mathrm{Int}(A)$ *and* $g(\mathrm{Int}(A))$*.*

Proof. We shall prove the lemma by showing that for every $\varepsilon > 0$, the conformal map g restricted to $A_\varepsilon = x_3^{-1}[\varepsilon, 1-\varepsilon]$ is one-to-one with nonzero derivative. For t, $0 < t < 1$, consider the plane P_t of height t. By Shiffman's first theorem [17], $C_t = P_t \cap A$ is a uniformly convex curve. Since the curve C_t is uniformly convex and smooth, C_t can naturally be parametrized by $\theta \in S^1$ by considering C_t to be parametrized by its outward planar normal. Orient A by the outward pointing normal to the bounded component of $\ ^3 - (P_1 \cup P_2 \cup A)$. With this orientation of A, g has the property that $\arg(g(C_t(\theta)) = \theta$ where $\arg(z)$ is the argument of the complex number z. Hence the derivative of g on $\mathrm{Int}(A)$ is never zero. Since $g\colon \mathrm{Int}(A) \to \ \cup\{\infty\}$ is holomorphic and the derivative of g is never zero, g is a local conformal diffeomorphism.

Suppose that $g(C_{t_1}(\theta_1)) = g(C_{t_2}(\theta_2))$. Then by the above formula $\theta_1 = \theta_2$. If $t_1 \neq t_2$, then let Q be the plane that is tangent to both C_{t_1} and C_{t_2} at θ_1. Let V_Q be the vector, parallel to Q, obtained by orthogonal projection of $(0,0,1)$ onto Q. Clearly the dot products of V_Q with the normals to A at the two points in $Q \cap (C_{t_1}(\theta_1) \cup C_{t_2}(\theta_2))$ are of opposite signs. It follows that $t_1 = t_2$ and hence g is one-to-one. $\square$

We will use Lemma 2.2 in our analysis of the index of a minimal annulus A with boundary Γ. To do this we will use a theorem of Schwarz that states that an eigenfunction (with zero boundary values) of the stability (or Jacobi) operator $\mathbf{S}$ of a compact orientable minimal surface M can be identified with an eigenfunction of the Laplacian $\Delta + 2$ on S^2 for $G(\mathrm{Int}(M))$ when G is one-to-one on $\mathrm{Int}(M)$ (see [2] for a generalized version of Schwarz's theorem). The second eigenvalue of $\Delta + 2$ on S^2 is 0. Thus the second eigenvalue of $\Delta + 2$ on any proper subdomain of S^2 is positive (see corollary 1, page 18 of

[5]), so (equivalently) the second eigenvalue of the stability operator on $\mathbf{S}$ is positive. These remarks together with Lemma 2.2 prove

Lemma 2.3 *A minimal annulus whose boundary consists of two continuous convex Jordan curves in parallel planes has index 0 or 1. Furthermore, if the annulus has index 1, then it does not have a Jacobi vector field.*

Remark 2.2 *If a compact orientable minimal surface has* index + nullity ≥ 2*, then the zero set of the second eigenfunction of* $\mathbf{S}$ *separates the surface into two components, each of which is unstable or almost-stable. Each of these components must have total curvature at least* 2π *[2]. By the Gauss-Bonnet formula any minimal annulus bounded by convex planar curves has total curvature less than* 4π*. Thus, such an annulus satisfies the conclusions of Lemma 2.3, even if the planes are not parallel.*

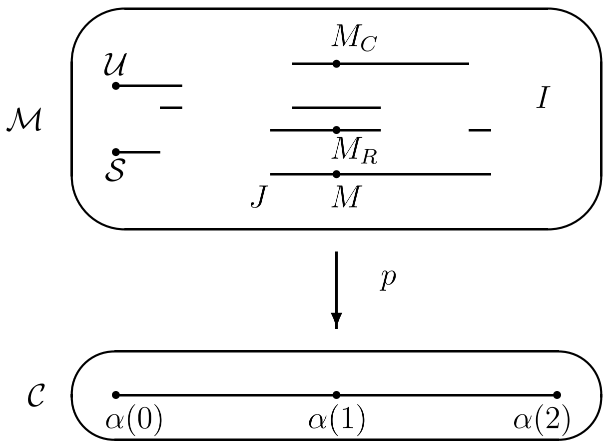
Lemma 2.4 *Suppose* Γ *satisfies the hypotheses of Theorem 1.1. If* Γ *is the boundary of a stable or almost-stable minimal annulus* A*, then every minimal annulus* A' *with* $\partial A' = \Gamma$ *that is contained outside of the ball* B_A *with boundary* $D_\alpha \cup D_\beta \cup A$*, must be stable.*

Proof. To see this first note that if A' lies outside A, then the Hopf boundary maximum principle implies that the boundary curve of $G(A')$ must be contained in the interior of the annulus $G(A)$. Hence, $G(A^{'}) \subset \mathrm{Int}(G(A))$. For compact domains E_1, E_2 with $E_1 \subsetneqq \mathrm{Int}(E_2) \subset S^2$, the first nonzero eigenvalue of $\Delta + 2$ on E_1 is strictly larger than the first eigenvalue on E_2. Hence, the first eigenvalue of the stability operator of A' is greater than 0, which proves A' is stable. $\square$

Proposition 2.1 *Let* W *be the slab with boundary* $P_0 \cup P_1$*. Suppose* M *is a smooth embedded stable compact minimal surface with two smooth boundary curves* $\Gamma = \{\alpha, \beta\}$*, not necessarily convex, contained in the boundary planes of* W *and such that* M *is not the annulus defined in Lemma 2.1. Let* C *be* *the closure of bounded component of $W - M$ and let R be the closure of the unbounded component. Then there exist embedded minimal surfaces $M_C \subset C$ and $M_R \subset R$, diffeomorphic to M, with $\partial M_C = \partial M_R = \partial M$, such that M_C and M_R are not stable.*

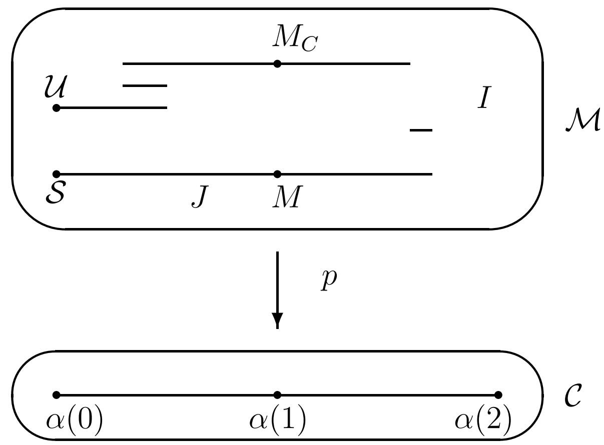
Proof. First assume that $\Gamma \subset P_0 \cup P_1$ is a regular value of the related projection $p\colon \mathcal{M}_g \to \mathcal{C}$ where $\mathcal{C}$ is the space of smooth pairs of curves in $P_0 \cup P_1$ and $\mathcal{M}_g$ is the space of genus-g embedded minimal surfaces with boundary in $\mathcal{C}$ and where g is the genus of M. Consider a path $\alpha\colon [0,2] \to \mathcal{C}$ satisfying the following properties:

1. α is transverse to p;
2. $\alpha(1) = \Gamma$;
3. $\alpha(0)$ consists of a two large concentric circles that enclose Γ;
4. $\alpha[0,2]$ is a union of pairwise disjoint, simple closed curves in $P_0 \cup P_1$.
5. $\alpha(2)$ is a pair of circles "inside" Γ that are so small that they do not bound a connected minimal surface.
6. The variational vector field associated to α is nowhere zero.



The case $M =$ Annulus $\neq \mathcal{A}$

Figure $1a$:



The case $M = \mathcal{A}$

Figure $1b$:

By the results in [1, 22], we know that $\Delta = p^{-1}(\alpha[0,2])$ is a smooth compact one-manifold. By property 5, the boundary of Δ is contained entirely in $p^{-1}(\alpha(0))$. By a theorem of Schoen [16], $p^{-1}(\alpha(0))$ consists of two points, one corresponding to a stable catenoid $\mathcal{S}$ and the other $\mathcal{U}$ corresponding to an unstable catenoid. Let I be the component of $p^{-1}(\alpha[1,2])$ that contains M in its boundary. (See Figure $1a$ and $1b$ for the case M has genus 0; Figure 1, with the interval component of $p^{-1}(\alpha[0,2])$ with boundary $\mathcal{S} \cup \mathcal{U}$ removed, provides a representative picture of the case when M has positive genus.) Note that the other end point of I corresponds to a minimal surface M_C with $\partial M_C = \Gamma$. Since M is transverse to $P_0 \cup P_1$ along Γ and also the variational vector field V for the family $M(t)$, $t \in I$, is nowhere zero along ∂M, the normal projection $V^{\perp}$ restricted to M is a Jacobi vector field for M that is never zero along ∂M. Since M is stable, this Jacobi vector field is in fact never zero on M. This implies that the $M(t)$ in I, close to M, lie in C. By the maximum principle, the entire family of surfaces corresponding to I

must also be contained in C, since their boundary curves are. In particular $M_C \subset C$. Since the sum of the indices of M_C and M is odd by the work of White [22], the index of M_C must be odd, which means M_C is unstable. (In Figure 1, each time Δ turns the index changes by 1.)

Let J denote the component of $p^{-1}(\alpha[0,1])$ that contains M and let M_R denote the surface corresponding to the other end point of J. If the boundary of M_R is Γ, then the argument in the previously considered case shows that M_R is unstable and $M_R \subset R$. Thus we may assume that $\partial M_R = \alpha(0)$ and consists of two large round concentric circles. A theorem of Schoen [16] states that M_R is a catenoid and since the index along Δ changes by 1 at each turn of Δ in Figure 1, M_R has even index and hence is stable (the unstable catenoid has index 1). If $M \neq \mathcal{A}$, then observe that $M(t) \cap \mathcal{A} \neq \emptyset$ for $M(t)$ in J close to M. However, using $\mathcal{A}$ as a barrier, the Geometric Dehn Lemma in [13, 14] shows that $\alpha(0) = \partial M_R$ is the boundary of an embedded least-area annulus outside of $\mathcal{A}$. By uniqueness of the stable catenoid, we conclude that M_R is disjoint from $\mathcal{A}$. Since the boundary of the surfaces in the interior of the family J lie outside of $\mathcal{A}$ and M_R lies outside of $\mathcal{A}$, the maximum principle implies that $M(t)$ must be disjoint from $\mathcal{A}$ for $M(t) \neq M$. This contradicts our earlier observation that if $M \neq \mathcal{A}$, then $M(t) \neq \mathcal{A}$ for $M(t)$ in J close to M. This contradiction implies $M = \mathcal{A}$. This completes the proof of the proposition when Γ is a regular value of $p: \mathcal{M}_g \to \mathcal{C}$.

Suppose now that Γ is not a regular value of p. It is still the case that M is a regular point of p since it is stable. Since M is a regular point of p, there exists a small neighborhood W of M in $\mathcal{M}_g$, such that $p: W \to p(W)$ is a diffeomorphism. By the Smale-Sard Theorem [18], we can approximate M by a sequence $M_i \in W$ of stable minimal surfaces with $\partial M_i \in \mathcal{C}$ that converge smoothly to M as $i \to \infty$ and such that $\Gamma_i = \partial M_i$ is a regular value of p for all integers i. If $M \neq \mathcal{A}$, then Γ_i will bound, by our previous arguments, two unstable minimal surfaces $M_C(i)$ and $M_R(i)$. By well-known compactness theorems [1], there are subsequences of these surfaces that converge to surfaces M_C and M_R that are unstable or almost-stable. Clearly

$M_C \subset C$ and $M_R \subset R$, which completes the proof of the proposition. $\square$

Remark 2.3 *The proof of Proposition 2.1 and Figure 1 show that when* Γ *is a regular value for* $p\colon \mathcal{M}_g \to \mathcal{C}$ *and* M *has genus* g, $g > 0$, *and satisfies the hypotheses of Proposition 2.1, then* Γ *is the boundary of at least 4 embedded minimal surfaces of genus* g.

Corollary 2.1 *Let* Γ *be as in Theorem 1.1. If* Γ *is the boundary of a stable minimal annulus, then this annulus is the annulus* $\mathcal{A}$ *given in Lemma 2.1.*

Proof. Suppose A' is a stable minimal annulus with boundary Γ and that $A' \neq \mathcal{A}$. By Proposition 2.1, Γ is the boundary of a minimal annulus A_R that is outside A' and that is unstable or is almost-stable. This is impossible by Lemma 2.4. $\square$

Lemma 2.5 *Theorem 1.1 is true if* Γ *is a regular value for* $p\colon \mathcal{M} \to \mathcal{C}$.

Proof. Assume that Γ is a regular value for the projection $p\colon \mathcal{M} \to \mathcal{C}$. In this case White [22] proved that the number of odd index minimal annuli in $p^{-1}(\Gamma)$ equals the number of even index annuli. (This is clear from Figure 1. Since for convex planar curves Γ, there is exactly one pair of minimal disks spanning Γ, this result also follows from Morse theory [19].) By Lemma 2.3, the even index annuli are all stable. By Corollary 2.1, Γ is the boundary of only one stable minimal annulus and this annulus is $\mathcal{A}$. Hence Γ is the boundary of one stable minimal annulus and one unstable minimal annulus. $\square$

Lemma 2.6 *Let* D_α *and* D_β *be smooth parallel convex planar disks with* $\partial D_\alpha = \alpha$, $\partial D_\beta = \beta$ *and let* $\Gamma = \{\alpha, \beta\}$. *Suppose there exists a compact connected branched nonplanar minimal surface* Σ *with* $\partial\Sigma \subset \mathrm{Int}(D_\alpha \cup D_\beta)$. *Then every minimal annulus with boundary* Γ *that is disjoint from* Σ *is stable.*

Proof. Suppose A is a minimal annulus with $\partial A = \Gamma$ and $A \cap \Sigma = \emptyset$. Consider the convex curves $\Gamma(\varepsilon)$ of distance ε from Γ inside $D_1 \cup D_2$ where ε is chosen sufficiently small so that $\Gamma(\varepsilon)$ lies outside $\partial\Sigma$ and every point of $\Gamma(\varepsilon)$ has a unique closest point on Γ. Using $\Sigma \cup A$ as a barrier, one produces, as in Lemma 2.1, a least-area minimal annulus $A(\varepsilon)$ with $\partial A(\varepsilon) = \Gamma(\varepsilon)$ and $A(\varepsilon)$ is contained in the region between A and Σ.

We claim that $g(\mathrm{Int}(A)) \subsetneqq g(\mathrm{Int}(A(\varepsilon)))$; this will prove the strict stability of A since $A(\varepsilon)$ is stable or almost-stable. For every $p \in \Gamma(\varepsilon)$, let $\widehat{p} \in \Gamma$ denote the closest point to p. Translate $A(\varepsilon)$ continously in the direction $v = \widehat{p} - p$ until the translated annulus $F = A(\varepsilon) + v$ is obtained. By the maximum principle, F lies inside A. In particular at $\widehat{p}$, F lies on the inside of A. By the Hopf boundary maximum principle there is a positive angle between the conormals of F and A at $\widehat{p}$. It follows that the norm of $g_A(\widehat{p})$ is never equal to norm of $g_F(\widehat{p})$. Since the boundary curves of A and F are tangent at $\widehat{p}$, $\arg(g_A(\widehat{p})) = \arg(g_F(\widehat{p}))$. Since the Gauss map of A and F are one-to-one when Γ is strictly convex, the comparison of norms implies $g(\partial A) \subseteq g(A(\varepsilon))$ and $g(\partial A) \neq g(\partial A(\varepsilon))$. A moments reflection in the weakly convex case also shows $g(\mathrm{Int}(A)) \subsetneqq g(\mathrm{Int}(A(\varepsilon)))$. As we observed earlier, this completes the proof of the lemma. □

Corollary 2.2 *Suppose Γ is as in Theorem 1.1. If the minimal annulus $\mathcal{A}$ given in Lemma 2.1 is almost-stable, then $\mathcal{A}$ is the unique minimal annulus with boundary Γ.*

Proof. Suppose $\mathcal{A}$ is almost-stable. If A is another minimal annulus with boundary Γ, then A is inside $\mathcal{A}$ and so $g(\mathcal{A}) \subsetneqq g(A)$. It follows the first eigenvalue of the stability operator of A is negative and hence A is unstable. Note that A is a regular point of $p\colon \mathcal{M} \to \mathcal{C}$, since A has no Jacobi vector fields by Lemma 2.3. Since A is a regular point for $p\colon \mathcal{M} \to \mathcal{C}$, one can deform Γ slightly to a pair of convex curves $\Gamma(\varepsilon)$ inside the convex planar disks with boundary Γ, so as to obtain a minimal annulus $A(\varepsilon)$ with $\partial A(\varepsilon) = \Gamma(\varepsilon)$. By

Lemma 2.1, $\mathcal{A}$ lies outside $A(\varepsilon)$ and by Lemma 2.6, we conclude that $\mathcal{A}$ is stable. This contradiction proves the corollary. $\square$

Proof of Theorem 1.1. We now complete the proof of Theorem 1.1. By Lemma 2.5 we may assume that Γ is not a regular value of $p\colon \mathcal{M} \to \mathcal{C}$. By the statement and proof of Lemma 2.2, Γ must be the boundary of an almost-stable minimal annulus A. It remains to prove that A is the unique compact branched minimal surface with boundary Γ.

We first prove Γ is the boundary of a unique minimal annulus. If $A = \mathcal{A}$, then uniqueness follows from Corollary 2.2. Hence, we may assume that $\mathcal{A} \neq A$. By Lemma 2.4, $\mathcal{A}$ is stable. Move Γ to the pair of convex curves $\Gamma(\varepsilon)$ of distance ε inside the convex planar disks with boundary Γ. Since $\mathcal{A}$ is stable, for ε sufficiently small, $\Gamma(\varepsilon)$ is the boundary of a stable outermost minimal annulus $\mathcal{A}(\varepsilon)$ and part of $\mathcal{A}(\varepsilon)$ lies outside A (since $\mathcal{A}$ lies outside of A). Recall that A is foliated by convex curves in parallel planes. Since $\partial\mathcal{A}(\varepsilon)$ lies inside A, we can choose planes K_0 and K_1, parallel and close to the planes P_0 and P_1 containing Γ, such that $(K_0 \cup K_1) \cap A$ bounds an annulus $A' \subset A$ with $A \cap \mathcal{A}(\varepsilon) \subset A'$ and $\partial A' \cap \mathcal{A}(\varepsilon) = \emptyset$. Since A is almost-stable and $A' \subsetneq A$, A' is stable. If Δ denotes the slab between K_0 and K_1, then $\Delta \cap \mathcal{A}(\varepsilon)$ is a minimal surface whose boundary is contained in the two planar disks with boundary curves $\partial A'$. Hence, by Lemma 2.1, there is a annulus $\mathcal{A}'$ associated to A' with $\partial\mathcal{A}' = \partial A'$ and $\mathcal{A}'$ lies outside of $\Delta \cap A(\varepsilon)$. Hence, $\mathcal{A}' \neq A'$. Since $\mathcal{A}'$ lies outside A', it is stable. But $\partial A'$ cannot be the boundary of two stable minimal annuli by Corollary 2.1. This contradiction proves that Γ is the boundary of a unique minimal annulus that is $\mathcal{A}$ and $\mathcal{A}$ is almost-stable.

Suppose now that M is a compact branched minimal surface with boundary Γ and M is not an annulus. Let $\mathcal{A}(t)$ denote the subannulus of $\mathcal{A}$ between the planes at heights $\frac{1}{2} - t$ and $\frac{1}{2} + t$ for $t \in (0, \frac{1}{2})$. Since $\mathcal{A}$ is almost-stable, the proper subannuli $\mathcal{A}(t)$ are stable. Let $\widehat{\mathcal{A}}(t)$ denote the unstable minimal annulus with $\partial\widehat{\mathcal{A}}(t) = \partial\mathcal{A}(t)$ whose existence is given by part 3 of Theorem 1.1. Since the minimal annulus $\mathcal{A}$ is unique, the $\widehat{\mathcal{A}}(t)$ converge smoothly to $\mathcal{A}$ as $t \to \frac{1}{2}$. Since M is inside $\mathcal{A}$ (by Lemma 2.1) and M is never tangent to $\mathcal{A}$ along $\partial\mathcal{A}$ (by the boundary maximum principle), the smooth convergence of $\widehat{\mathcal{A}}(t)$ to $\mathcal{A}$ implies that there is a small compact neighborhood $N(\partial M) \subset M$ of ∂M such that for t close to $\frac{1}{2}$, $\widehat{\mathcal{A}}(t) \cap N(\partial M) = \emptyset$. On the other hand, since $\widehat{\mathcal{A}}(t)$ is unstable and $\partial\widehat{\mathcal{A}}(t)$ lies outside of M, Lemma 2.6 implies $\widehat{\mathcal{A}}(t) \cap M \neq \emptyset$. It follows that there exists a sequence $t_i \to \frac{1}{2}$ and a sequence of points $p_i \in \widehat{\mathcal{A}}(t_i) \cap (M - N(\partial M))$ such that $p_i \to p \in \mathcal{A} \cap M$. Since $\mathcal{A} \cap M = \partial\mathcal{A}$, p must be contained in P_0 or P_1. However, the maximum principle applied to the third coordinate function of $M - N(\partial M)$ shows that $X_3(p_i)$ stays a bounded distance from 0 and 1 and hence $X_3(p) \neq 0, 1$. This contradiction shows M cannot exist, which completes the proof of Theorem 1.1. $\square$

Conjecture 2.1 *Theorem 1.1 holds for* continuous *convex* Γ.

Remark 2.4 *In [12] we continue our study of minimal annuli with boundary curves that are planar but not necessarily convex or in parallel planes. We generalize Shiffman's first theorem by showing if* Γ *is a pair of smooth convex extremal planar curves whose union is an extremal set, then every minimal annulus with boundary* Γ *is embedded. With this result in hand we then prove Theorem 1.1 in the case where* Γ *is a pair of extremal convex planar curves, not necessarily in parallel planes.*

Motivated by these results we go on to prove that the space $\mathcal{M}$ *of embedded minimal annuli with boundary curves in parallel planes is a path connected space (in fact, we prove* $\mathcal{M}$ *is contractible). In contrast to this result we prove that the space* $\widetilde{\mathcal{M}}$ *of immersed minimal annuli with the same boundary curves is not path connected (by showing* $\widetilde{\mathcal{M}}$ *contains a nonembedded example). Similar connectedness theorems hold for the space of minimal annuli with extremal boundary.*

3 Uniqueness of the free and partially-free boundary value problems

Theorem 3.1 *Let α be a smooth convex plane curve in $\mathbb{R}^3$, P_0 be a plane parallel to the plane containing α, and $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane P_0. If Σ is a connected stable or almost-stable compact minimal surface with boundary $\alpha \cup R(\alpha)$, then Σ is an embedded annulus.*

Proof. We will assume that Σ is not an annulus and prove that it is unstable. We may assume that P_0 is the (x, y)-plane. By Theorem 2 of [16], there is a nonnegative function u defined on the closure $\overline{\Omega}$ of an open subset Ω of $\mathbb{R}^2$ such that

$$\Sigma = \{(x, y, \pm u(x, y)) \mid (x, y) \in \overline{\Omega}\} .$$

Since Σ is connected, Ω must be the region inside a convex curve C (the projection of α on P_0) and outside several disjoint curves $C_1, C_2, \ldots, C_k$. Note that $k \geq 1$ since Σ is connected. Indeed, $k > 1$ since Σ is not an annulus.

Note that $u|C \equiv h$ where h is the height of α above P_0. Also, $u|C_i \equiv 0$ and $|\nabla u||C_i = \infty$. Since Σ has mean curvature 0 along C_i, C_i is a plane line of curvature and the top half of Σ is a graph, then each C_i must be uniformly convex.

We claim that there must be at least one point in the interior of Σ^+ (the portion of Σ above P_0) at which the curvature vanishes. To see this, consider the Gauss map from Σ^+ to the upper hemisphere H of the unit sphere. Suppose for the moment that α is uniformly convex. By an argument in the proof of Lemma 2.2, the Gauss map takes C diffeomorphically to a simple closed curve $\nu(C)$. The Gauss map takes each C_i homeomorphically to the equator. It follows that the Gauss map has degree k on the region R between $\nu(C)$ and the equator and has degree $k \pm 1$ on the region $H - R$. In particular, the Gauss map covers the region $H - R$ at least once. If there were no zeroes of curvature in Σ^+, then the covering would be unbranched and so Σ^+ would contain a connected component diffeomorphic to a disk. But Σ is connected, so that is impossible. If α is not uniformly convex, then $\nu(C)$ need not be a simple closed curve. However, in this case $S^2 - \nu(C)$ still consists of two disks, which is all we really needed to prove the existence of an interior point of zero curvature.

Let $q \in \mathrm{Int}(\Omega)$ be such that the curvature of Σ at $(q, u(q))$ vanishes. Let v_0 be a unit vector parallel to P_0 and perpendicular to $\nabla u(q)$. (In other words, v_0 is parallel to P_0 and to the tangent plane to Σ at $(q, u(q))$.)

Let $Z \subset \overline{\Omega}$ be the zero set of $\phi\colon (x,y) \mapsto \nu(((x,y), u((x,y))) \cdot v_0$, where $\nu(p)$ is the unit normal to Σ at p. Then $Z = \pi(\nu^{-1}(\Gamma))$ where Γ is a great circle in ∂B and π is orthogonal projection onto P_0. Since ν is a conformal map with branch points, $\nu^{-1}(\Gamma)$ consists of smooth embedded arcs together with isolated points in Ω where an even number of such arcs meet at their end points. In particular, at least four such arcs begin at the point $(q, u(q))$ (because it is a branch point of the Gauss map). Since Z is homeomorphic to $\nu^{-1}(\Gamma)$, it has the same structure. Note that Z meets each C_i and also C exactly twice because those curves are convex.

Now form a topological space $\widehat{\Omega}$ from $\overline{\Omega}$ by identifying each C_i to a point. Then $\widehat{\Omega}$ is topologically a disk. Now the set $\widehat{Z}$ in $\widehat{\Omega}$ is a graph in which each vertex (except for the two on $C = \partial\widehat{\Omega}$) has an even number of edges. Furthermore, the vertex q has at least four edges. It follows that $\widehat{\Omega} - \widehat{Z}$ contains at least one connected component $\widehat{W}$ that does not touch $C = \partial\widehat{\Omega}$. Let W be the corresponding region in $\overline{\Omega}$, and let $W' = \{(x,y,z) \in \Sigma \mid (x,y) \in W\}$. Then the function $p \mapsto v_0 \cdot \nu(p)$ vanishes on $\partial W'$. Since that function is a solution of the Jacobi operator, it follows that zero is an eigenvalue of the Jacobi operator on W'. But W' is a proper subset of Σ, so Σ must be unstable. $\square$

Corollary 3.1 *Let α be a smooth convex plane curve in $\mathbb{R}^3$ and P_0 be a plane parallel to the plane containing α. If Σ is a minimal surface with boundary C and nonempty free boundary in P_0 and if Σ is stable or almost-stable solution* *for the free boundary problem, then* Σ *is an embedded annulus.*

Proof. If R is orthogonal reflection in the plane P_0, then $\Sigma \cup R(\Sigma)$ satisfies the hypotheses of Theorem 3.1. $\square$

Theorem 3.2 *Let* Γ *be a pair of smooth convex curves in parallel planes and let* P_0 *be a plane of reflection symmetry of* Γ *that intersects each component of* Γ*. If* Σ *is a connected stable or almost-stable compact minimal surface with boundary* Γ*, then* Σ *is an embedded minimal annulus.*

Proof. We will assume that Σ is not an annulus and prove it is unstable. We may assume that P_0 is the (x, y)-plane and that each component of Γ is contained in a plane parallel to the (y, z)-plane. By Theorem 2 of [16], there is a nonnegative function u defined on the closure $\overline{\Omega}$ of an open set Ω of . ^{2} such that

$$\Sigma = \{(x, y, \pm u(x, y)) \mid (x, y) \in \overline{\Omega}\}.$$

Since Σ is connected, Ω must be the region inside a curve C and outside several disjoint convex curves $C_1, C_2, \ldots, C_k$. The curve C consists of two parallel line segments corresponding to the projection of Γ onto P_0 together with two concave arcs.

For simplicity we will assume that Γ is uniformly convex. We claim that there must be at least one point in the interior of Σ^+ (the portion above P_0) at which the curvature vanishes. Consider the Gauss map from Σ^+ to the upper hemisphere H of the unit sphere. Note that G restricted to a component C_i of $\partial\Sigma^+$ gives a parametrization of ∂H and that $G|C$ is one-to-one and $G((C - \Gamma)) \subset \partial H$. Let D be the disk in H bounded by $G(C)$ and let D_1 and D_2 denote the two disks that are the closures of the components of $H - D$. Clearly, $\Delta = \overline{\Sigma^+ - G^{-1}(D_1 \cup D_2)}$ is diffeomorphic to Σ^+ and $G|\Delta \colon \Delta \to D$ is a connected covering space. Hence Δ is a disk, which implies that Σ^+ is a disk. Since we are assuming Σ is not an annulus, we have arrived at a contradiction. This contradiction proves the existence of a branch point for G and therefore a zero of Gaussian curvature in $\mathrm{Int}(\Sigma^+)$.

Let $q \in \mathrm{Int}(\Omega)$ be such that the curvature of Σ at $(q, u(q))$ vanishes. Let v_0 be a unit vector parallel to the (y, z)-plane and parallel to a vector in the tangent plane to Σ^+ at q. Let $Z \subset \overline{\Omega}$ be the zero set of $\phi\colon (x, y) \mapsto \nu(x, y, u(x, y)) \cdot v_0$, where $\nu(p)$ is the unit vector normal to Σ at p. Notice that ϕ has exactly two zeros on each boundary component of Ω and on the component C one of these zeros occurs on each component of $C \cap \Gamma$.

From this point on the proof proceeds, with slight modification, as in the proof of Theorem 3.1 to show that Σ is unstable if it is not an annulus. This completes the proof of Theorem 3.2. $\square$

The above theorem together with Shiffman's second theorem proves

Corollary 3.2 *If Γ is a pair of circles in parallel planes and Σ is a stable or almost-stable compact minimal surface with boundary Σ, then Σ is an embedded minimal annulus foliated by circles in parallel planes.*

4 A simple proof of Shiffman's second theorem

Recall that Shiffman's first theorem states that if Γ is a pair of convex Jordan curves in parallel planes, then any minimal annulus A with $\partial A = \Gamma$ is foliated by convex curves in parallel planes and, except for possibly the boundary curves, this foliation is by uniformly convex analytic planar curves. The main geometric argument in the special case Γ is smooth and convex is quite simple, and for completeness we will give it here.

After a rigid motion and a homothety of $\mathbb{R}^3$, we may standardize the minimal annulus A so that it is parametrized conformally by a map $X\colon A(r) \to \mathbb{R}^3$ where $A(r) = \{z \in \mathbb{C} \mid 1 \leq |z| \leq r\}$ for some unique $r > 1$ and such that $X(\partial A(r)) \subset P_0 \cup P_{\ln(r)}$, where $P_t = \{x_3 = t\}$. Since the third coordinate function $X_3 \colon A(r) \to \mathbb{R}$ is harmonic, we may assume that $X_3(z) = \ln|z|$, since it is a harmonic function with the correct boundary values. In this parametrization, the circle $|z| = c$ in $A(r)$ maps by X to an immersed curve $\gamma_c(\theta) = X(c\,e^{i\theta})$ in $P_{\ln(c)}$. (Note that in this parametrization each curve in $X(\partial A)$ is oriented in a clockwise manner and the oriented normal to $X(A)$ is inward pointing along $X(\partial A)$.) Let $g \colon A(r) \to \mathbb{C} \cup \{\infty\}$ denote the Gauss map of $A(r)$. Since g never obtains the values $0, \infty$, the angle $\arg(g(z)) \in S^1 = \mathbb{R}/(2\pi \cdot \mathbb{Z})$ is well-defined. The convexity of the level set curve $\gamma_c(\theta)$ corresponds to $\frac{\partial}{\partial\theta} \arg(\gamma_c'(\theta)) \geq 0$ where we consider $\gamma_c'(\theta) \in \mathbb{C}^*$. Note that $\arg(\gamma_c'(\theta)) = -\frac{\pi}{2} + \arg(g(c\,e^{i\theta}))$. Since $\frac{\partial}{\partial\theta} \arg(L(z))$ is a harmonic function for any nonzero holomorphic function $L(z)$, and $\frac{\partial}{\partial\theta} \arg(g(z)) \geq 0$ on $X|\partial A(r)$, we conclude that $\frac{\partial}{\partial\theta} \arg(g(z)) > 0$ for $z \in \mathrm{Int}(A(r))$ by the maximum principle. Hence, $\frac{\partial}{\partial\theta} \arg(\gamma_c'(\theta)) > 0$ for all c, which proves Shiffman's first theorem in the simplest case of smooth convex boundary. (Note that in the above discussion we have implicitly used Hildebrandt's boundary regularity theorem [7] that implies $X \colon A(r) \to \mathbb{R}^3$ is smooth along $\partial A(r)$.)

As remarked in the Introduction, Shiffman proved a second theorem (and the most difficult one) in the case that the boundary of the annulus consists of circles in parallel planes. In this case he proved that the minimal annulus is foliated by circles in parallel planes. We shall now give a proof of this second theorem of Shiffman; this proof will be a simple consequence of Theorem 1.1 and the classification of minimal surfaces foliated by circles as given by Riemann [15]. Also see [4, 8] for a discussion of Riemann's classification.

Theorem 4.1 (Shiffman's Second Theorem) *Suppose A is a minimal annulus whose boundary consists of a circle in the plane P_0 and another circle in the planes P_1. Then A is foliated by circles in parallel planes.*

Proof. Choose an analytic path $\alpha \colon [0,1] \to \mathcal{C}$ satisfying:

1. $\alpha(1) = \partial A$;

2. $\{\alpha(t) \mid t \in [0,1]\}$ induces a foliation of annuli in P_0, P_1 with $\alpha(0)$ consisting of two circles whose boundary disks contain $\partial A = \alpha(1)$ and the circles $\alpha(0)$ are concentric around the x_3-axis.

By Lemmas 2.1 and 2.6 and Theorem 1.1, $\alpha(t)$ is the boundary of a stable or almost-stable minimal annulus $\mathcal{A}(t)$ and for $t < 1$ this minimal annulus is stable. For $t < 1$, let $\mathcal{U}(t)$ denote the unstable minimal annulus with boundary $\alpha(t)$. Recall that Riemann's one-parameter family of periodic minimal surfaces is foliated by circles and lines in horizontal planes and that the family converges smoothly on compact subsets of $\mathbb{R}^3$ to a catenoid. Since $\mathcal{A}(0)$ and $\mathcal{U}(0)$ are catenoids, Riemann's classification theorem implies that both $\mathcal{A}(t)$ and $\mathcal{U}(t)$ are foliated by circles for t close to 0. Since $\alpha(t)$ is analytic in t, $\mathcal{A}(t)$ and $\mathcal{U}(t)$ must be foliated by circles for all $t < 1$. Theorem 1.1 implies that A must be the limit of the $\mathcal{A}(t)$ or of the $\mathcal{U}(t)$ as $t \to 1$ (or of both if it is almost-stable). Hence, A is foliated by circles in parallel planes, which proves Shiffman's Second Theorem. □

Shiffman's third theorem states that if the minimal annulus A has boundary consisting of two convex curves in parallel planes, then every rigid motion of $\mathbb{R}^3$ that leaves ∂A invariant leaves A invariant. When ∂A is smooth, this symmetry property for A follows immediately from Theorem 1.1 (since there is at most one stable, one unstable and one almost-stable minimal annulus with boundary ∂A). If Theorem 1.1 can be shown to hold for the case where ∂A is continuous, Shiffman's general symmetry theorem can be proved using this alternative method.

5 Analytic parametrizations of minimal annuli bounding uniformly convex Γ

In this section we will describe analytically the examples that arise in Theorem 1.1 in the case that Γ is uniformly convex. Suppose $\widetilde{A}$ is such a minimal annulus. Then $\widetilde{A}$ is conformally parametrized by $A(r) = \{z \in \mathbb{C} \mid 1 \leq |z| \leq$

$r\}$. After a rigid motion and a homothety we may assume that $f\colon A(r) \to \mathbb{R}^3$ is the parametrization, $f(1) = \vec{0}$, and $f_3(z) = 2\ln|z|$. In particular, the boundary curves of $\widetilde{A} = f(A(r))$ now are contained in the planes P_0 and $P_{2\ln(r)}$. By the proof of Lemma 2.2, $g\colon A(r) \to \mathbb{C}^* = \mathbb{C} - \{\vec{0}\}$ parametrizes an annular domain F such that each component of ∂F is star shaped about the origin. From the Weierstrass Representation [9], we have

$$f(z) = \mathrm{Re} \int_1^z [(\frac{1}{g} - g),\, (\frac{1}{g} + g)i,\, 2]\frac{1}{z}\, dz. \tag{5.1}$$

Consider the Laurant expansions for g and $1/g$: $g = \sum_{-\infty}^{\infty} a_n z^n$ and $1/g = \sum_{-\infty}^{\infty} b_n z^n$. Since f is well-defined, the complex valued forms appearing in the integral (5.1) have no real periods. Hence, $\mathrm{Im}(b_0) = \mathrm{Im}(a_0)$ and $\mathrm{Re}(b_0) = -\mathrm{Re}(a_0)$. Since these equations are necessary and sufficient for f to be well defined for a given parametrization $g\colon A(r) \to F$, this process can be reversed. More precisely,

Theorem 5.1 *Suppose $F \subset \mathbb{C}^*$ is a smooth annulus whose boundary curves are star shaped about the origin. Let $g\colon A(r) \to F$ be a conformal parametrization of F. Then F is the image of the Gauss map of a minimal annulus with strictly convex smooth boundary on horizontal planes if and only if the constant term in the Laurant expansion for g is the negative of complex conjugate of the constant term in $1/g$. Suppose F is the image of such a minimal annulus, and it is parametrized by $g\colon A(r) \to F$. Then in this parametrization, the Gauss map can be identified with g and the coordinates of the annulus, after a rigid motion and a homothety, are given by formula (5.1).*

References

[1] M. Anderson. Curvature estimates for minimal surfaces in 3-manifolds. *Ann. Scient. Éc. Norm. Sup.*, 18:89–105, 1985.

[2] J. L. Barbosa and M. do Carmo. On the size of a stable minimal surface in ^{3}. *American Journal of Mathematics*, 19(8):515–528, 1976.

[3] M. Callahan, D. Hoffman, and W. H. Meeks III. Embedded minimal surfaces with an infinite number of ends. *Inventiones Math.*, 96:459–505, 1989.

[4] M. Callahan, D. Hoffman, and W. H. Meeks III. The structure of singly-periodic minimal surfaces. *Inventiones Math.*, 99:455–481, 1990.

[5] Chavel. *Eigenvalues in Riemannian Geometry.* Academic Press, 1984.

[6] A. Enneper. Die cyklischen flachen. *Z. Math. und Phys.*, 14:393–421, 1869.

[7] S. Hildebrandt. Boundary behavior of minimal surfaces. *Archive Rational Mech. Anal.*, 35:47–81, 1969.

[8] D. Hoffman and W. H. Meeks III. Minimal surfaces based on the catenoid. to appear in the Amer. Math. Monthly, special Geometry Issue.

[9] H. B. Lawson, Jr. *Lectures on Minimal Submanifolds.* Publish or Perish Press, Berkeley, 1971.

[10] W. H. Meeks III. *Lectures on Plateau's Problem.* Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, 1978.

[11] W. H. Meeks III. Uniqueness theorems for minimal surfaces. *Illinois Journal of Math.*, 25:318–336, 1981.

[12] W. H. Meeks III and B. White. The space of minimal annuli bounded by an extremal pair of planar curves. preprint.

[13] W. H. Meeks III and S. T. Yau. The classical Plateau problem and the topology of three-dimensional manifolds. *Topology*, 21:409–442, 1982.

[14] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179:151–168, 1982.

[15] B. Riemann. *Ouevres Mathématiques de Riemann*. Gauthiers-Villars, Paris, 1898.

[16] R. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. *Journal of Differential Geometry*, 18:791–809, 1983.

[17] M. Shiffman. On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. *Annals of Math.*, 63:77–90, 1956.

[18] S. Smale. An infinite dimensional version of Sard's theorem. *American Journal of Math.*, 87:861–866, 1965.

[19] Michael Struwe. A Morse theory for annulus-type minimal surfaces. *Journal für die reine und angewandte Mathematik*, 368:1–27, 1986.

[20] F. Tomi and A. J. Tromba. Extreme curves bound an embedded minimal surface of disk type. *Math. Z.*, 158:137–145, 1978.

[21] B. White. Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals. *Inventiones Math.*, 88(2):243–256, 1987.

[22] B. White. New applications of mapping degrees to minimal surface theory. *Journal of Differential Geometry*, 29:143–162, 1989.