The Ordering Theorem for the Ends of Properly Embedded Minimal Surfaces

Charles Frohman*
Mathematics Department
University of Iowa
Iowa City, IA 52242

William H. Meeks III†
Mathematics Department
University of Massachusetts
Amherst, MA 01003

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1 Introduction

A fundamental problem in the classical theory of minimal surfaces is to describe the asymptotic geometry of properly embedded minimal surfaces in R^3. In the special case that the surface has *finite total curvature*[1] its asymptotic behavior is well understood. For, in this case, the surface is conformally diffeomorphic to a finitely punctured closed Riemann surface and each end of the surface, one for each puncture point, is asymptotic to a plane or an end of a catenoid (see [19]). Thus the plane and the catenoid are the models for describing the asymptotic behavior of these minimal surfaces. When the properly embedded minimal surface has infinite total curvature, but still finite topology, the question has been asked whether the surface must be asymptotic to a helicoid.

A first step towards understanding the asymptotic behavior of a surface is to characterize its topological behavior. For example, doubly and triply-periodic minimal surfaces in R^3 that are not flat must have infinite genus and one end [2]. In [7] the authors' proved that any two properly embedded minimal surfaces in R^3 with the

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[1] A surface has finite total curvature if $(\int |K| dA < \infty)$.

same genus and one end are *ambiently isotopic*[2]. Hence, topologically speaking, there is only one configuration for a one-ended minimal surface at infinity, depending on whether the surface has finite or infinite genus.

On the other hand, there exist examples of properly embedded minimal surfaces in $\mathbb{R}^3$ that have an infinite number of ends. The most famous examples of this type were discovered by Riemann who showed that there exists a one-parameter family of singly-periodic genus 0 minimal surfaces $\mathcal{R}_t$ satisfying:

1. Every horizontal plane intersects $\mathcal{R}_t$ in a single component that is a circle or a straight line;
2. $\mathcal{R}_t$ is invariant under reflection in the (x_1, x_3)-plane;
3. $\mathcal{R}_t$ is invariant under translation by $v_t = (0, t, 1)$;
4. Far away from the line passing through the origin and in the direction v_t, $\mathcal{R}_t$ is asymptotic to the family of parallel horizontal planes at integer heights.

The annular ends of $\mathcal{R}_t$ are naturally ordered by their heights above the (x_1, x_2)-plane with the top and bottom limit ends having heights $+\infty$ and $-\infty$, respectively (for a rigorous geometric definition of end see Definition 2.1). Recently other singly-periodic minimal surfaces with an infinite number of ends have been found that have similar asymptotic behavior (see [1] or [11] for computer graphics pictures of some of these new surfaces as well as one of Riemann's examples).

In this paper we will prove that every properly embedded minimal surface with more than one end has asymptotic behavior that mimics the behavior of the Riemann examples. Loosely speaking our main theorem states that, after a rotation of $\mathbb{R}^3$, the ends of the surface can be ordered by their heights over the (x_1, x_2)-plane. In order to make precise the statement of this theorem one needs the concept of a limit tangent plane, which we rigorously define in Section 2. This definition, as well as the proof of existence and uniqueness of *the* limit tangent plane, was first given in [2]. A more precise statement of the following theorem appears in Theorem 2.1 of Section 2.

Theorem 1.1 (Ordering Theorem) *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end and whose limit tangent plane is the (x_1, x_2)-plane. Then the ends of M are naturally ordered by their "height" over the (x_1, x_2)-plane.*

[2]Two properly embedded surfaces in $\mathbb{R}^3$ are ambiently isotopic if one can be deformed to the other by a one-parameter family of diffeomorphisms of $\mathbb{R}^3$.

Meeks and Yau [17] have proven a topological uniqueness result for properly embedded minimal surfaces with more than one end. Their main theorem states that two proper diffeomorphic minimal surfaces in R^3 of finite topology are ambiently isotopic. An important first step in the proof of their theorem is to show that the ends of a minimal surface of finite topology are topologically parallel and hence ordered, a result similar to the statement of the Ordering Theorem. In the case of finite topology the ordering of the ends is obviously a topological ordering. When the surface has infinite genus, this ordering property is not obvious but we can still prove it holds.

Theorem 1.2 *Suppose M_1 and M_2 satisfy the hypotheses of M in Theorem 1.1 and $F\colon \mathrm{R}^3 \to \mathrm{R}^3$ is a diffeomorphism such that $F(M_1) = M_2$. Then F preserves or reverses the natural ordering of the ends of M_1 and M_2. In particular, if M satisfies the hypotheses of Theorem 1.1 and $F\colon \mathrm{R}^3 \to \mathrm{R}^3$ is a diffeomorphism such that $F(M) = M$, then F preserves or reverses the ordering of the ends of M.*

On the basis of all of these results, one might be tempted to conjecture that two properly embedded diffeomorphic minimal surfaces in R^3 are isotopic. We strongly believe this conjecture to be false but a related conjecture to be true (see Conjecture 1.2 below).

The ordering theorem and its proof motivates three conjectures concerning the topology of properly embedded minimal surfaces with more than one end.

Conjecture 1.1 *The ordering of ends given in Theorem 1.1 is almost a well-ordering in the sense that it is equivalent to the ordering on a compact subset $\mathcal{S}$ of the interval $[0,1]$ with $\mathcal{S} \cap (0,1)$ discrete. (Also see the statement of Theorem 2.1.)*

Definition 1.1 Suppose M is as in Theorem 1.1. A nonlimit end $\overline{\alpha}$ of M has even (odd) multiplicity if a one-ended representative (see Definition 2.2) of M has even (odd) intersection number with every sufficiently large horizontal translation of the x_3-axis.

The next conjecture is motivated by the classifications of Heegaard splittings of a ball by Frohman [6].

Conjecture 1.2 *Suppose M_1 and M_2 are two properly embedded minimal surfaces with more than one end. A necessary and sufficient condition for M_1 to be isotopic to M_2 is for there to exist a diffeomorphism $f\colon M_1 \to M_2$ that preserves or reverses the ordering of the ends of these surfaces and such that f preserves the even-odd multiplicity of the nonlimit ends of M_1 and M_2.*

Conjecture 1.3 *Suppose M is a properly embedded minimal surface with more than one end. An end of M fails to have an end representative (see Definition 2.2) with quadratic area growth*[3] *if and only if it is a limit end of M and it is the maximal or minimal element in the induced ordering of ends.*

It is important to note that Conjecture 1.1 implies that a properly embedded minimal surface in $\mathbb{R}^3$ can have at most two limit ends and that the number of ends of the surface is countable. In particular, the validity of Conjecture 1.1 would show that the surface obtained by taking $\mathbb{C} - \{0, 1\}$ and removing a closed discrete subset of points with limit points at $0, 1$, and ∞ can not properly minimally embed in $\mathbb{R}^3$.

Conjecture 1.3 is our most descriptive and important conjecture on the asymptotic behavior of properly embedded minimal surfaces with more than one end. It implies among other things that any end of such a surface M, which is not a highest or lowest end, has an end representative that has a unique limit tangent cone that is an integer multiple of the limit tangent plane of M. When M has two limit ends, this conjecture implies that the nonlimit ends of M are each asymptotically close to a horizontal plane. It also follows from this conjecture that if such an M has finite topology, then it would have finite total curvature. This last consequence of Conjecture 1.3 is closely related to a theorem of Hoffman-Meeks [10] whose statement, reinterpreted in terms of ordering theorem, states that an annular end, of a properly embedded minimal surface with more than one end, that has infinite total curvature must be a highest or lowest end in the ordering given by the Ordering Theorem. (In the case of finite topology having finite total curvature is equivalent to having quadratic area growth.)

The results of this paper were announced in [8].

2 The Ordering Theorem

Intuitively, the ends of a noncompact surface can be thought as the number of different ways to travel to infinity on the surface. More precisely, an end of surface M is an equivalence class of proper arcs on the surface that describes one way to travel to infinity. We now recall the definition of these equivalence classes.

Definition 2.1 Consider two proper arcs $\alpha_1, \alpha_2 \colon [0, \infty) \to M$. Then α_1 is *equivalent* to α_2, written $\alpha_1 \approx \alpha_2$, if there exists an exhaustion $M_1 \subset M_2 \subset \ldots$ of M by

[3] A surface M in $\mathbb{R}^2$ has quadratic area growth if there exist constants K_1, K_2 such that for large balls $B(R)$ of radius R, $K_1 R^2 \leq Area(M \cap B(R)) \leq K_2 R^2$.

smooth compact subdomains, such that for every i the noncompact components of $\alpha_1 - Int(M_i)$ and $\alpha_2 - Int(M_i)$ are contained in the same component of $M - Int(M_i)$. The relation $\approx$ is an equivalence relation and we denote by $\overline{\alpha}$ the equivalence class of α and we call $\overline{\alpha}$ the *end associated to the proper arc* α.

With the above definition of end, it is easy to check that a closed surface punctured in n points has n ends, one corresponding to each removed point. It follows that R^2 has one end and the cylinder has two ends. However, in the general case, the structure of the ends of a noncompact surface can be much more complicated as occurs, for instance, in a surface obtained by removing a Cantor set from a closed surface.

In order to work with the ends of a surface, it is useful to make some further definitions.

Definition 2.2 A smooth proper subdomain Σ of M with $\partial\Sigma$ compact is said to be an *end-representative* for an end $\overline{\alpha}$ of M if $\alpha \cap \Sigma$ is noncompact.

Note that whether or not Σ is an end-representative of $\overline{\alpha}$ does not depend on the choice of representative in $\overline{\alpha}$.

Definition 2.3 A smooth compact exhaustion $M_1 \subset M_2 \subset \cdots$ of M is called *good* if, for all i, each component of $M - Int(M_i)$ is noncompact and has one boundary curve. It is called *excellent* if it is good and for all i, each component of $M - Int(M_i)$ has either one end or an infinite number of ends.

Lemma 2.1 *A noncompact surface M has an excellent exhaustion.*

Proof. First choose a smooth exhaustion $M_1 \subset M_2 \subset \cdots$ of M by connected compact subdomains. By adjoining the compact components of $M - Int(M_i)$ to M_i, we may assume that every component of $M - Int(M_i)$ is noncompact for every i.

If for each integer i every component of $M - Int(M_i)$ has connected boundary, then the exhaustion is good. If not, let M_k denote the first domain such that some component C of $M - Int(M_k)$ has more than one boundary component. In this case choose an embedded arc δ in C with end points on distinct boundary curves of C. Let $N(\delta)$ be a small regular neighborhood of δ, chosen so that $M'_k = M_k \cup N(\delta)$ is smooth. Since $M_1 \subset M_2 \subset \cdots$ exhausts M, there is an integer I such that M'_k is contained in the interior of M_I.

Consider the new exhaustion $M_1 \subset \cdots \subset M_{k-1} \subset M'_k \subset M_I \subset M_{I+1} \subset \cdots$ but reindex to obtain $M'_1 \subset M'_2 \subset \cdots$ where $M'_i = M_i$ for $i < k$ and $M'_i = M_{I+i-k}$ for

$i > k$. The new exhaustion agrees with the previous exhaustion for the first $k-1$ terms. It is better in the k^{th} term in that the difference between number of boundary components in $M - Int(M_k)$ and the number of components of $\partial(M - Int(M'_k))$ is one less than the original exhaustion. This replacement argument can be continued *ad infinitum* to obtain a good exhaustion of M.

If $M_1 \subset M_2 \subset \cdots$ is a good exhaustion and $M - Int(M_i)$ has a component with a finite number of ends greater than one, then, by a variation of the previous argument, we can enlarge M_i by adding on a compact subdomain so that for the new M'_i, $M - M'_i$ has fewer components with a finite number n of ends, $n > 1$. Continued replacements of this type will result in an excellent exhaustion. $\square$

Suppose $\Sigma \subset \mathrm{R}^3$ is a properly embedded noncompact minimal surface of finite curvature and $\partial\Sigma$ compact. In this case Σ has a finite number of ends of planar and catenoid type [19]. Since Σ is embedded, the normal lines to the ends of Σ are asymptotically parallel to the same line at infinity. The plane passing through the origin and perpendicular to this line is called the *limit tangent plane* of Σ. One can extend this concept to an arbitrary properly embedded minimal surface M without boundary and more than one end. When M has more than one end, it is shown in [2] that there exist properly embedded, noncompact, finite total curvature minimal surfaces contained in the closure of one of the components of $\mathrm{R}^3 - M$ and which have compact boundary contained in M. Furthermore, the limit tangent planes of these surfaces coincide (Theorem 5 in [2]). One defines *the* limit tangent plane of M to be the limit tangent plane of any of these finite total minimal surfaces contained in the closure of a complement of M.

The Ordering Theorem, Theorem 1.1 in the introduction, is an interpretation of the following ordering theorem.

Theorem 2.1 *Suppose M is a properly embedded minimal surface in R^3 with more than one end and whose limit tangent plane is the (x_1, x_2)-plane. Then there is natural geometric ordering of the ends of M that is equivalent to the ordering of a compact subset of $[0,1]$.*

Proof. We first give a brief outline of the proof of Theorem 2.1. By Lemma 2.1, we can choose an excellent exhaustion $M_1 \subset M_2 \subset \cdots$ of M. We assume M_1 is chosen large enough so that $M - M_1$ is not connected. Given this exhaustion we will construct a properly embedded minimal surface $\mathcal{M}$ in R^3, each component of which has compact boundary and finite total curvature. Then with respect to this exhaustion and $\mathcal{M}$, we will assign to every end $\overline{\alpha}$ of M a "height" in the interval

$[0,1]$; in this way the ordering on $[0,1]$ induces an ordering of the ends of M. Finally, we shall show that this ordering of the ends of M is independent of the excellent exhaustion, $\mathcal{M}$, and other choices made along the way.

We begin the proof of the theorem by establishing some further notation. For each i we wish to define a subcollection $C(i)$ of boundary components of ∂M_i. Namely, $\alpha \in C(i)$ if α is a component of ∂M_i and α is not homologous in M to a component of ∂M_j for $j < i$. Given an $\alpha \in C(i)$ we let $\Delta(\alpha)$ denote the component of $M - Int(M_i)$ with boundary curve α. Let $\mathcal{C} = \bigcup_i C(i)$. Let N^+ and N^- denote the closures of the the two components of $\mathrm{R}^3 - M$. Since a stable orientable minimal surface in R^3 is a plane [3, 5], M is unstable. We will assume that M_1 is chosen large enough so that M_1 is an unstable minimal surface.

Assertion 2.1 *$\mathcal{C}$ is the boundary of a complete stable orientable properly embedded minimal surface $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) in N^+ (resp. N^-) such that each component Y of $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) satisfies:*

1. *∂Y is a single component in $\mathcal{C}$;*
2. *Either $Y = \Delta(\partial Y)$ or $Y \cap M = \partial Y$;*
3. *If $Y = \Delta(\partial Y)$, then Y is asymptotic to an end-representative of a catenoid or a plane in R^3;*
4. *If $Y \cap M = \partial Y$ and Y is noncompact, then Y is a complete minimal surface of finite total curvature with each end asymptotic to an end-representative of a catenoid or a plane and this end-representative is contained in the interior of N^+.*

Proof of Assertion 2.1. We first show that for each integer i, $C(i)$ is the boundary of a stable orientable properly embedded minimal surface $M^+(i)$ in N^+ such that each component $Y \in M^+(i)$ satisfies Properties 1–4 of the assertion. It will follow by our inductive construction of $M^+(i)$ that $M^+(i) \cap M^+(j) = \emptyset$, if $i \neq j$. After constructing the surfaces $M^+(i)$, we let $\mathcal{M}^+ = \bigcup_i M^+(i)$ and prove that $\mathcal{M}^+$ is proper; this will complete the proof of Assertion 2.1. For notational convenience, we let $\mathcal{M}^+(i) = \bigcup_{j=1}^{i} M(j)$.

The proof of the existence of $M^+(i)$ will be by induction on i. Therefore, suppose $M^+(i-1)$ exists ($M^+(0) = \emptyset$) and we shall construct $M^+(i)$. Arbitrarily choose an $\alpha \in C(i)$ and let $N^+(\alpha)$ denote the closure of the component of $N^+ - \mathcal{M}^+(i-1)$ that contains α. Note that $\partial N^+(\alpha)$ is piecewise smooth with interior angles less than

π and with the smooth portions having zero mean curvature. Since $\partial(N^+(\alpha))$ has nonnegative mean curvature, it is an appropriate barrier for solving Plateau type problems in $N^+(\alpha)$. See Theorem 1 in [16] for a discussion of the barrier property.

Choose an excellent exhaustion $\Delta_1 \subset \Delta_2 \subset \ldots$ of $\Delta(\alpha)$. Replace Δ_1 by a least-area surface $\widetilde{\Delta}_i \subset N^+(\alpha)$ with $\partial\widetilde{\Delta}_i = \Delta_i$. A subsequence of these least-area surfaces converges to a properly embedded stable minimal surface $Y \subset N^+(\alpha)$ with $\partial Y = \alpha$. (See Lemma 3.1 in [7] or Proposition 3.1 in [17] for details on this convergence of a subsequence of the $\widetilde{\Delta}_i$ to Y.) Since either $Y \cup \Delta(\alpha)$ or $Y \cup (M - \Delta(\alpha))$ is a properly embedded surface that separates R^3, Y is orientable. Since Y is stable and orientable, a theorem of Fischer-Colbrie [4] implies that Y has finite total curvature. Hence, each end of Y is asymptotic to a plane or a catenoid in R^3.

We first consider the case when $Y \cap M = \partial Y$ and Y is noncompact. After removing a small open neighborhood of ∂Y from Y we obtain a new surface Y' with $\partial Y'$ compact and $Y' \cap M = \emptyset$. Hence, by the maximum principle at infinity in [14], the distance between Y' and M is positive. (The maximum principle at infinity states the distance between two properly embedded disjoint minimal surfaces with compact boundary is positive.) Hence, the planes or catenoids that are asymptotic to the ends of Y must have end-representatives that are contained in the interior of N^+, which proves that Y satisfies Property 4.

Suppose now that $Y \cap M \neq \partial Y$. In this case the maximum principle implies that $Y \subset \partial N^+(\alpha)$. Since $\partial N^+(\alpha) - \Delta(\alpha)$ is either not smooth or contains M_1 which is unstable, $Y = \Delta(\alpha)$. Since the exhaustion of M is excellent and $\Delta(\alpha)$ has a finite number of ends (since Y has finite total curvature), Y has exactly one end, which completes the proof that Properties 1–4 hold for Y.

If $C(i) = \{\alpha\}$, then let $M^+(i) = \{Y\}$. Otherwise, let $Y_1 = Y$ and choose an $\alpha_2 \in C(i) - \{\alpha\}$. Then using the barrier $\mathcal{M}^+(i-1) \cup Y_1$, instead of $\mathcal{M}^+(i-1)$ as we just did for α, we produce a new surface Y_2 with $\partial Y_2 = \alpha_2$, satisfying Properties 1–4 and such that Y_2 is disjoint from $\mathcal{M}^+(i-1) \cup Y_1$. It is clear that this process can be continued to produce a collection $M^+(i) = \{Y_1, Y_2, \ldots, Y_n\}$ of surfaces satisfying the required properties. By induction, we can construct $M^+(i)$ for all i.

It remains only to prove that $\mathcal{M}^+$ is a properly embedded minimal surface. Since $\mathcal{M}(i)$ is properly embedded for each i, we need only check that $\mathcal{M}^+$ is proper. If $\mathcal{M}^+$ were not proper, then there would exist a sequence of points $p(i_j) \in M^+(i_j)$ for some sequence i_j, $j \to \infty$, such that $p(i_j) \to p$ for some $p \in N^+$. Since the $\partial M^+(i_j)$ diverge to infinity, the distance from $p(i_j)$ to $\partial M^+(i_j)$ goes to infinity as $j \to \infty$. By the curvature estimates of Schoen [18], there exists a $c > 0$, such that the Gaussian curvature of a point $q \in Int(\mathcal{M}^+)$ is bounded from below by $-c/d^2$ where d is the intrinsic distance of q to $\partial\mathcal{M}^+$. Hence, there exist disk neighborhoods $D(i_j)$ of $p(i_j)$ in $M^+(i_j)$ that are giving better and better approximations to larger and larger flat disks in R^3 as $j \to \infty$. It follows that a subsequence of the $D(i_j)$ converges to a flat plane in N^+ passing through the point p. The existence of such a plane would imply M is contained in a halfspace of R^3. By the Halfspace Theorem [12] a properly immersed minimal surface in a halfspace is a plane but M is not a plane. This contradiction completes the proof of the assertion. $\square$

Definition 2.4 $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ and $\mathcal{M}(i) = \mathcal{M}^+(i) \cup \mathcal{M}^-(i)$.

Since the exhaustion $M_1 \subset M_2 \cdots$ of M is excellent, then either $C(i)$ is nonempty for every i or $C(i) = \emptyset$ for $i > 1$. Of course, the first case occurs when M has an infinite number of ends and the second case when M has a finite number of ends. We shall prove Theorem 2.1 in the case that $C(i) \neq \emptyset$ for all i; the proof of the case $C(i) = \emptyset$ for $i > 1$ uses a similar and simpler argument and will be left to the reader.

Assume that $C(i) \neq \emptyset$ for all i. First note that if $\alpha \in C(i)$, then α cannot bound compact components $Y^+ \subset \mathcal{M}^+$ and $Y^- \subset \mathcal{M}^-$. The reason for this is that $Y^+ \cup Y^-$ is a compact surface in R^3 that bounds a compact region of R^3 that contains $\Delta(\alpha)$ or $M - Int(\Delta(\alpha))$, both of which are proper and noncompact, an impossibility.

Let $\mathcal{M}$ be the properly embedded surface whose existence is given by Assertion 2.1 and Definition 2.4. Note that $\mathcal{M}$ is minimal and smooth except along $C(i)$. Also note that a component Y of $\mathcal{M}$ has nonempty boundary only if $Y = \Delta(\partial Y)$.

First note that outside of a sufficiently large cylinder of radius $R(k)$ around the x_3-axis, $\mathcal{M}(k)$ consists of $n(k)$ graphs over the annulus $A(k) \subset (x_1, x_2)$-plane which is the exterior of the disk of radius $R(k)$ centered at the origin. We choose $R(k)$ so that $R(k)$ is an increasing function in k and $R(k) \to \infty$ as $k \to \infty$. Complete these graphs to be pairwise disjoint graphs $G_k(1), \ldots, G_k(n(k))$ over the (x_1, x_2)-plane. Assume that these graphs are ordered by their relative heights; in otherwords, if $G_k(i)$ lies above $G_k(j)$, then $i > j$. These graphs separate R^3 into a lowest open slab $S_k(0)$ and the half open slabs $S_k(1), \ldots, S_k(n(k))$ where $S_k(j) = \{x \in \mathrm{R}^3 \mid x$ lies on or above $G_k(j)$ but below $G_k(j+1)\}$. If α is a proper arc in M representing $\overline{\alpha}$, then for any fixed k eventually α is contained in exactly one of the regions $S_k(j)$ and $S_k(j)$ only depends on $\overline{\alpha}$. Define $\overline{\alpha}(k) = j$. See Figure 1 for a picture.

Suppose that $\overline{\alpha} \neq \overline{\beta}$. Fix an integer k sufficiently large so that the associated end-representatives $M^{\overline{\alpha}}, M^{\overline{\beta}}$ in $M - Int(M_k)$ are disjoint. We will now prove that $\overline{\alpha}(k) \neq \overline{\beta}(k)$. It is straightforward to show that $\overline{\alpha}(k) \neq \overline{\beta}(k)$ when both $M^{\overline{\alpha}}$ and

Figure 1: Note that $\overline{\alpha}(1) = 1$, $\overline{\beta}(1) = 1$, $\overline{\alpha}(2) = 2$, $\overline{\beta}(2) = 1$

$M^{\overline{\beta}}$ have finite total curvature, so we may assume, after a possible change of indices, that $M^{\overline{\alpha}}$ has infinite total curvature. Let Y^+ denote the component of $\mathcal{M}^+$ with $\partial Y^+ = \partial M^{\overline{\alpha}}$ and let Y^- denote the component of $\mathcal{M}^-$ with $\partial Y^- = \partial M^{\overline{\alpha}}$. Since Y^+ and Y^- are each stable and have finite total curvature, $M^{\overline{\alpha}}$ has infinite total curvature, and $M - M^{\overline{\alpha}}$ is unstable, then Assertion 2.1 implies that $M \cap (Y^+ \cup Y^-) = \partial M^{\overline{\alpha}}$. Since $Y^+ \cup Y^-$ is a properly embedded surface in R^3, it separates R^3 into two regions, $R^{\overline{\alpha}}$ and $R^{\overline{\beta}}$, where $M^{\overline{\alpha}} \subset R^{\overline{\alpha}}$ and $M^{\overline{\beta}} \subset R^{\overline{\beta}}$. By Assertion 2.1, we know that $Y^+ \cup Y^-$ has a finite number n of ends. In particular, the intersection of $R^{\overline{\alpha}}$ with the complement of any solid cylinder with axis the x_3-axis and of sufficiently large radius, consists of $n+1$ slab type regions (with a solid cylinder removed from each slab) and each of these regions can contain points of $M^{\overline{\alpha}}$ or $M^{\overline{\beta}}$ but not both surfaces. Choose representatives $\alpha' \in \overline{\alpha}$ and $\beta' \in \overline{\beta}$ such that $\alpha' \subset R^{\overline{\alpha}}$ and $\beta' \subset R^{\overline{\beta}}$. It is clear that α' and β' eventually are contained in different slab type regions determined by the ends of $Y^+ \cup Y^-$ and hence in different slabs determined by $\mathcal{M}(k)$. This completes our proof that $\overline{\alpha}(k) \neq \overline{\beta}(k)$.

With these remarks in mind, we now give a procedure for ordering the end $\mathcal{E}$ of M relative to $\mathcal{M}$. Let L be a linearly ordered set. A *Dedekind cut* of L is a subset $D \subseteq L$ with the property that if $p \in D$ and $q < p$, then $q \in D$. Notice that the set L^* of Dedekind cuts at L is a complete and bounded linearly ordered set, under inclusion.

Note the least element is the empty set and the greatest element is the set L. There is a map $L \to L^*$ given by sending each $x \in L$ to $\{p \mid p < x\}$. Further, if $\phi : L \to S$ is an order preserving map from L into a complete bounded linearly ordered set S, then ϕ extends to $\overline{\phi} : L^* \to S$.

We make the ends $\mathcal{D}$ of $\mathcal{M}$ into a linearly ordered set, as described above. This allows us to define a map $h : \mathcal{E} \to \mathcal{D}^*$ from the ends of M to the set of Dedekind cuts to $\mathcal{D}$. This map sends an end E of M to the set of all ends of $\mathcal{M}$ that are eventually strictly below E. Notice that since for every $\overline{\alpha}$ and $\overline{\beta}$ there exists a k such that $\overline{\alpha}(k) \neq \overline{\beta}(k)$, h is injective. Hence we have ordered $\mathcal{E}$.

Since we are ordering the ends of $\mathcal{M}$ by their topologically parallel circle intersections with large cylinders, it is easy to find an order preserving map of the ends of $\mathcal{M}$ into the interval $[0, 1]$. This map extends to a map of $\mathcal{E}^*$ into $[0, 1]$, hence we have ordered the ends of M as a subset of $[0, 1]$.

We now show that $h(\mathcal{E})$ is a compact subset of $[0, 1]$ by showing that every subsequence of $h(\mathcal{E})$ has a convergence subsequence in $h(\mathcal{E})$. If not, then there exists an increasing or decreasing sequence e_j in $h(\mathcal{E})$ converging to point L in $\mathcal{D}^*$. We need to produce a proper arc α in M such that $h(\overline{\alpha}) = L$. Given an M_i in the exhaustion of M, one of the components $R(i)$ of $M - Int(M_i)$ must be an end representative for an infinite subsequence $e_j(i)$ of $\{e_j\}$. We can of course choose $R(i + 1) \subset R(i)$. Choose an arc in $M_{i+1} - Int(M_i)$ with boundary points in $\partial R(i) \cup \partial R(i + 1)$ such that the union of these arcs is a proper arc α. It is clear from the definition of h that $h(\overline{\alpha}) = L$.

It remains to prove that the ordering of the ends of M induced by the height function h is independent of the choice of $\mathcal{M}$. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two properly embedded minimal surfaces, associated to two excellent exhaustions of M and that satisfy the conclusions of Assertion 2.1. Let h_1 and h_2 be the associated height functions to the interval $[0, 1]$. We will show that $h_1(\overline{\alpha}) < h_1(\overline{\beta})$ implies $h_2(\overline{\alpha}) < h_2(\overline{\beta})$. Suppose to the contrary that for some pair of ends $\overline{\alpha}, \overline{\beta}$, that $h_1(\overline{\alpha}) < h_1(\overline{\beta})$ and $h_2(\overline{\beta}) \leq h_2(\overline{\alpha})$. Notice in this case that $h_2(\overline{\beta})$ is strictly less than $h_2(\overline{\alpha})$ since h_2 is one-to-one.

If $\overline{\alpha}$ has an end-representative with finite total curvature, then, by the definition of excellent exhaustion, for large values of k, the end-representative of $\overline{\alpha}$ in $M - Int(M_k)$ is asymptotic to a plane or catenoid with horizontal limit tangent plane. In this case it is straightforward to prove that any other end $\overline{\beta}$ of M lies "above" or "below" the catenoid end of $\overline{\alpha}$ and hence if $h_1(\overline{\alpha}) < h_1(\overline{\beta})$, then $h_2(\overline{\alpha}) < h_2(\overline{\beta})$. Assume now that every end-representative of $\overline{\alpha}$ and of $\overline{\beta}$ has infinite total curvature.

By part 4 of Assertion 2.1, there exist catenoid or planar-type ends $E_1 \subset \mathcal{M}_1$ and $E_2 \subset \mathcal{M}_2$ and $E_1 \cup E_2 \subset Int(N^+) \cup Int(N^-)$ such that $\overline{\alpha}$ lies "above" E_1, $\overline{\beta}$ lies "below" E_1, $\overline{\alpha}$ lies "below" E_2 and $\overline{\beta}$ lies "above" E_2; the E_1 and E_2 are chosen to be graphs over the (x_1, x_2)-plane. The ends E_1 of $\mathcal{M}_1$ and E_2 of $\mathcal{M}_2$ are asymptotic to the ends C_1, C_2, respectively, of catenoids or planes and the boundary of E_i is disjoint from M. By the weak maximum principle at infinity [14], $dist(E_1, M)$ and $dist(E_2, M)$ are both positive, so we can make the substitution of C_1, C_2 for E_1, E_2 in our discussions of the relative ordering of $\overline{\alpha}$ and $\overline{\beta}$ with respect to h_1 and h_2. Assume that ∂C_1 and ∂C_2 are round circles that are boundary curves of planar disks D_1, D_2, respectively. Let $\mathcal{C}_i = C_i \cup D_i$.

If we can choose C_1 and C_2 to be disjoint, then, after replacing then by subends, we can assume, that $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint. But if $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint, then $\mathcal{C}_1 \cup \mathcal{C}_2$ separates R^3 into three "parallel" slabs, which clearly contradicts our ordering assumptions on $\overline{\alpha}$ and $\overline{\beta}$.

If C_1 and C_2 have different logarithmic growths as graphs over their projections onto the (x_1, x_2)-plane P, then $C_1 \cap C_2$ is compact. Thus, by choosing subends, we may assume that C_1 and C_2 are disjoint. By the discussion in the previous paragraph, we may therefore assume that the logarithmic growths of C_1 and C_2 are the same.

When C_1 and C_2 have zero logarithmic growth, then they are contained in the same horizontal plane, an obvious impossibility because of the different orderings of $\overline{\alpha}$ and $\overline{\beta}$ by h_1 and h_2. Thus, without loss of generality, we may assume, after a rigid motion of M and a replacement of C_1 and C_2 by subends, that $\partial C_1 \cup \partial C_2$ is contained in P and C_1 and C_2 are non-negative graphs over P of the same positive logarithmic growth.

A simple analysis of catenoids, using their analytic definition, shows that whenever K_1 and K_2 are nonnegative catenoidal graphs over P with round circle boundary curves in P, then either $K_1 \cap K_2$ is compact or else K_2 is obtained from K_1 by reflection in a vertical plane. By our previous discussion, we know that $C_1 \cap C_2$ is noncompact, and so we conclude that C_2 is obtained from C_1 by reflection in a vertical plane. See Figure 2 for a picture of the two possible cases.

Let H denote the upper halfspace and note that $C_1 \cup C_2$ separates H into four regions $\mathcal{R}_{\overline{\alpha}}, \mathcal{R}_{\overline{\beta}}, \mathcal{R}_T, \mathcal{R}_B$. Here $\mathcal{R}_T$ is the "Top" region above $C_1 \cup C_2$, R_B is the "Bottom" region below $C_1 \cup C_2$, $\mathcal{R}_{\overline{\alpha}}$ is the region containing an end-representative of $\overline{\alpha}$ of M and $\mathcal{R}_{\overline{\beta}}$ is the region containing an end-representative of $\overline{\beta}$ of M. Notice that reflection in the plane Q interchanges the regions $\mathcal{R}_{\overline{\alpha}}$ and $\mathcal{R}_{\overline{\beta}}$ and Q is disjoint from the $Int(\mathcal{R}_{\overline{\alpha}})$ and $Int(\mathcal{R}_{\overline{\beta}})$. It follows that $\mathcal{R}_{\overline{\alpha}}$ or $\mathcal{R}_{\overline{\beta}}$ is contained in a quarter-space of

Figure 2: Noncompact intersection on right

R^3 determined by $P \cup Q$. Hence M contains an end-representative of $\overline{\alpha}$ (or of $\overline{\beta}$) that is contained in a quarter-space. But Theorem 3 in [12] states that the convex hull of a properly immersed noncompact nonplanar minimal surface with compact boundary in R^3 must contain a slab in its convex hull. This contradiction completes this proof of the theorem. $\square$

3 The Proof of Theorem 1.2

In this section we prove Theorem 1.2 that implies that the natural geometric ordering of the ends of a properly embedded minimal surface is a topological ordering. We break the key steps of the proof of this theorem into four lemmas.

Lemma 3.1 *Suppose Σ is a properly embedded minimal surface in R^3. Suppose P is the image of a proper embedding of a plane. Suppose that $\Gamma = P \cap \Sigma$ is a simple closed curve on Σ that separates Σ into two noncompact surfaces. Let N denote the closed complement of Σ that contains the end of P. Then Γ is the boundary of a properly embedded annulus in N whose end is the end of a flat plane or a catenoid in $Int(N)$.*

Proof. Let $B_1 \subset B_2 \subset \dots$ be an exhaustion of R^3 by round balls centered at the origin such that $\Gamma \subset B_1$ and ∂B_i is transverse to $\Sigma \cup P$. Let $\widetilde{P}_i$ be the component of $P \cap B_i$ with $\Gamma \subset \partial \widetilde{P}_i$. After performing surgery on $\widetilde{P}_i$ in $B_i \cap N$ we obtain an incompressible planar surface and let P_i denote the component of this surface containing Γ. Replace

P_i by a least-area minimal surface D_i in the isotopy class of P_i relative to ∂P_i in $B_i \cap N$.

We claim that the family of surfaces $\{D_i\}$ have bounded area and bounded curvature in any fixed ball B_i. These estimates together with standard regularity and compactness theorems for minimal surfaces imply that a subsequence $\{D_{i_j}\}$ converges smoothly on compact regions of R^3 to a properly embedded orientable minimal surface A bounded by Γ (see the proof of Theorem 3.2 of [7]). The required curvature and area estimates can be found in the proof of Theorem 3.2 of [7]. It is also shown in the proof of Theorem 3.2 of [7] that the resulting limit A is an incompressible stable minimal surface in N and the usual loop lifting argument proves that the genus of A is zero is no greater than the genus of the D_i which is zero. Hence, A also has genus zero.

We now show that A has one end. If A has more than one end, then we can choose two embedded homotopically nontrivial loops $\alpha_1, \alpha_2 \subset A$, based on a point $p_0 \in A$, such that

$$[\alpha_1][\alpha_2] \neq [\alpha_2][\alpha_1] \quad \text{where} \quad [\alpha_i] \in \pi_1(A, p_0)\,.$$

Since the planar surfaces D_{i_j} converge smoothly to A, for i_j large we may assume that we can lift $\alpha_1 \cup \alpha_2$ to D_{i_j}. Since the D_{i_j} are obtained from the annulus $P \cap B_1 \cap N$ by surgery and an isotopy, we can perform a bounded isotopy of $P \cap N$ so that $\alpha_1 \cup \alpha_2$ is contained on $P \cap N$, which is an annulus with cyclic fundamental group. Hence $[\alpha_1][\alpha_2] = [\alpha_2][\alpha_1]$ in $\pi_1(N)$. Since $\pi_1(A)$ injects into $\pi_1(N)$ under inclusion, $[\alpha_1][\alpha_2] \neq [\alpha_2][\alpha_1]$ in $\pi_1(N)$, which is a contradiction. This contradiction proves that A is a stable minimal annulus. The annulus A has finite total curvature by the results of Fischer-Colbrie [4].

Since A is a finite total curvature, it is asymptotic to an end of a catenoid or a plane. The strong maximum principle at infinity [14], shows distance of the end of A to Σ is positive when A is not contained in Σ. In this case, after a small isotopy, one can move the end of A slightly so that it is equal to the end of the plane or catenoid to which it is isotopic. If A is contained in Σ, then the usual perturbation arguments prove that A can be pushed slightly off itself to have the required property. $\square$

Lemma 3.2 (Haken's Lemma) *Suppose Σ is a properly embedded minimal surface in R^3 and P is a properly embedded plane such that $P \cap \Sigma$ is compact. Furthermore, suppose that P separates two ends of Σ. Then after a bounded isotopy of P, the new plane intersects Σ in a single simple closed curve that separates Σ into two noncompact surfaces.*

Proof. Corollary 3.2 in [7] states that the fundamental group of Σ maps onto the fundamental group of each closed complement of Σ in R^3. For any properly embedded surface M in R^3 that satisfies this topological property on fundamental groups and for any properly embedded plane that intersects M in a compact set, the proof of Haken's lemma [9] shows that after an isotopy of P in some compact region of R^3, the new isotoped plane intersects M transversely in a fewest number of components and this number is either zero or one. Thus, after a bounded isotopy of P, there is a new isotoped plane that intersects Σ in a simple closed curve that separates Σ into two noncompact surfaces (since a bounded isotopy of P can not fail to separate the previously separated ends of Σ). □

Lemma 3.3 *Suppose Σ satisfies the hypotheses of M in Theorem 1.1. Suppose $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3$ correspond to three of the ends of Σ naturally ordered as $\overline{\alpha}_1 < \overline{\alpha}_2 < \overline{\alpha}_3$. Then there exist properly embedded planes P_1, P_2 such that:*

1. *The ends of P_1 and P_2 are ends of catenoids or planes, with horizontal limit tangent planes;*
2. *P_1 lies below P_2;*
3. *$P_i \cap \Sigma$ is a simple closed separating curve for $i = 1, 2$;*
4. *$\overline{\alpha}_1$ lies below $P_1, \overline{\alpha}_2$ lies between P_1 and P_2 and $\overline{\alpha}_3$ lies above P_2.*

Proof. Since $\overline{\alpha}_1 < \overline{\alpha}_2 < \overline{\alpha}_3$, the proof of the ordering theorem implies that there exist pairwise disjoint two properly embedded planes $\widetilde{P}_1, \widetilde{P}_2$, each of which is a graph over the xy-plane, with $\widetilde{P}_1$ below $\widetilde{P}_2$, the ends of $\widetilde{P}_1, \widetilde{P}_2$ are ends of catenoids or planes and $\widetilde{P}_i \cap \Sigma$ is compact. Furthermore, $\overline{\alpha}_1$ lies below $\widetilde{P}_1$, $\overline{\alpha}_2$ lies between $\widetilde{P}_1$ and $\widetilde{P}_2$ and $\overline{\alpha}_3$ lies above $\widetilde{P}_2$. Lemma 3.2 implies that after a bounded isotopy of $\widetilde{P}_1$, we obtain a new plane P_1, that intersects Σ in a simple closed curve. Since the end of P_1 is disjoint from the end of $\widetilde{P}_2$, we can replace a compact domain of $\widetilde{P}_2$ so that the new $\widetilde{P}_2$ is disjoint from P_1.

Let H be the closed halfspace of R^3 with boundary P_1 that contains $\widetilde{P}_2$. The surface $H \cap \Sigma$ separates H into two closed components and the fundamental group of this surface maps onto each of these components. Again, application of the proof of Haken's lemma shows that after a bounded isotopy of $\widetilde{P}_2$ in H, we can move $\widetilde{P}_2$ to a new plane P_2 that intersects $\Sigma \cap H$ in a simple closed curve. □

Lemma 3.4 *Suppose M is as in the statement of Theorem 1.1. Suppose P_1 and P_2 are two properly embedded, pairwise disjoint, planes in $\mathbb{R}^3$ such that for $i = 1, 2$, $P_i \cap M = \Gamma_i$ is a simple closed nonseparating curve on M. Let E_1 be the closed complementary domain of $P_1 \cup P_2$ that has boundary P_1, E_2 the domain with $\partial E_2 = P_2$ and let R be the closed complementary domain with boundary $P_1 \cup P_2$. Suppose $\overline{\alpha}_1, \overline{\alpha}_2$ are ends of M with end representatives contained in E_1, E_2, respectively. Suppose $\overline{\alpha}_3$ is an end of M with an end representative contained in R. Then in the ordering of the ends of M, either $\overline{\alpha}_1 < \overline{\alpha}_3 < \overline{\alpha}_2$ or $\overline{\alpha}_1 > \overline{\alpha}_3 > \overline{\alpha}_2$.*

Proof. We will first replace P_1 and P_2 by new pairwise disjoint planes $\widetilde{P}_1, \widetilde{P}_2$ such that $\Gamma_i = P_i \cap \Sigma = \widetilde{P}_i \cap \Sigma$ and such that the ends of $\widetilde{P}_1$ and $\widetilde{P}_2$ are equal to the ends of planes and catenoids. First replace the disks in P_1, P_2 bounded by Γ_1, Γ_2 by least area disks D_1, D_2 in respective closed complements of Σ in $\mathbb{R}^3$. The proof of Lemma 3.1 shows that we can replace the annuli A_i in P_i bounded by Γ_i by least-area embedded annuli $\widetilde{A}_i$ in the closed complements of Σ in $\mathbb{R}^3$. By carrying out this minimization argument simultaneously for $\widetilde{A}_1$ and $\widetilde{A}_2$ we can be sure that $\widetilde{A}_1$ and $\widetilde{A}_2$ are disjoint. (To prove disjointness one applies well known disjointness properties for least-area compact planar domains as described in [15].) The usual disk replacement argument shows that $(D_1 \cup D_2) \cap (\widetilde{A}_1 \cup \widetilde{A}_2) = \Gamma_1 \cup \Gamma_2$. Define $\widehat{P}_i = D_i \cup \widetilde{A}_i$.

Since $A_i \cup \widetilde{A}_i$ is a properly immersed piecewise smooth surface in a complement of Σ, it bounds a piecewise smooth domain in the complement that intersects Σ only along Γ. It follows that an end representative of $\overline{\alpha}_j$ that lies on one side P_i lies on the same side of $\widehat{P}_i$ for any i, j. In particular it follows that each of the three complements of $\widehat{P}_1 \cup \widehat{P}_2$ contains the end representative of one and only one of the ends $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3$ and furthermore that $\overline{\alpha}_2$ has an end representative contained in the complement with boundary $\widehat{P}_1 \cup \widehat{P}_2$.

In the proof of the uniqueness portion of geometric ordering in Theorem 1.1 we demonstrated that if F_1 and F_2 are catenoid or planar type ends contained in $\mathbb{R}^3 - \Sigma$ and an end $\overline{\beta}_1$ lies below $F_1 \cup F_2$, an end $\overline{\beta}_2$ lies between F_1 and F_2, and an end $\overline{\beta}_3$ lies above $F_1 \cup F_2$, then $\overline{\beta}_1 < \overline{\beta}_2 < \overline{\beta}_3$. By choosing F_1 to be the higher catenoid-type end of $\widehat{P}_1 \cup \widehat{P}_2$ and F_2 to be the lower catenoid-type end of $\widehat{P}_1 \cup \widehat{P}_2$, we conclude from the discussion in the previous paragraph that $\overline{\alpha}_2$ must lie between $\overline{\alpha}_1$ and $\overline{\alpha}_3$, which completes the proof of the lemma. $\square$

Proof of Theorem 1.2. Suppose M_1 and M_2 satisfy the hypotheses of Theorem 1.1 and $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism such that $F(M_1) = M_2$ but such that F fails to preserve or reverse the natural ordering of the ends of M_1 and M_2. This means that there exist three ends $\overline{\alpha}_1 < \overline{\alpha}_2 < \overline{\alpha}_3$ of M_1 such that either $\overline{F \circ \alpha_1} < \overline{F \circ \alpha_3} < \overline{F \circ \alpha_2}$ or $\overline{F \circ \alpha_3} < \overline{F \circ \alpha_1} < \overline{F \circ \alpha_2}$.

By Lemma 3.3 there exist pairwise disjoint planes P_1 and P_2 such that $\overline{\alpha}_1$ lies below P_1, $\overline{\alpha}_2$ lies between P_1 and P_2, and $\overline{\alpha}_3$ lies above P_3. Since $\overline{F \circ \alpha_2}$ lies in the region between $F(P_1)$ and $F(P_2)$, Lemma 3.4 implies that either $\overline{F \circ \alpha_1} < \overline{F \circ \alpha_2} < \overline{F \circ \alpha_3}$ or $\overline{F \circ \alpha_3} < \overline{F \circ \alpha_2} < \overline{F \circ \alpha_1}$, which contradicts our earlier conclusion. This contradiction completes the proof of Theorem 1.2. $\square$

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