The Geometry and Conformal Structure of Properly Embedded Minimal Surfaces of Finite Topology in R^3

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April 6, 1993

1 Introduction

In this paper we study the conformal structure and the asymptotic behavior of properly embedded minimal surfaces of finite topology in ^{3}. One consequence of our study is that when such a surface has at least two ends, then it has finite conformal type, i.e., it is conformally diffeomorphic to a compact Riemann surface punctured in a finite number of points.

Except for the helicoid and a recently found one-ended genus example of Hoffman, Karcher and Wei [9] of a helicoid with one handle attached, every known example M of such a properly embedded minimal surface of finite topology satisfies the strong geometric constraint of having finite total Gaussian curvature $C(M) = \int_M K dA$. When M has finite total curvature, then it always has finite conformal type. Furthermore, the coordinate functions of a finite total curvature M can be defined analytically in terms of a meromorphic one-form and a meromorphic function on the conformal compactification. In many ways the finite total curvature examples are well-understood;

*The research described in this paper was supported by research grant DE-FG02-86ER250125 of the Applied Mathematical Science subprogram of Office of Energy Research, U. S. Department of Energy, and National Science Foundation grant DMS-9204535.

for example, each end of such a surface is asymptotic to the end of a plane or catenoid in $\mathbb{R}^3$. In recent years major progress has been made in constructing new examples of embedded complete minimal surfaces of finite total curvature in $\mathbb{R}^3$ (see [3], [8], [11]).

The constraint that a properly embedded minimal surface M have finite total curvature leads to important analytical and geometrical results. For example, it has been shown by Schoen [18] that if such an M has two ends, then M is a catenoid; it was shown by Lopez-Ros [13] that if M is a planar domain, then M is a plane or a catenoid; and finally Costa [4] has classified the genus 1 examples with three ends. These non-existence and classification theorems strongly use the property of finite total curvature. A natural and fundamental conjecture is:

Conjecture 1.1 *Every properly embedded minimal surface of finite topology with at least two ends has finite total curvature.*

An important partial result on the above conjecture was given by Hoffman-Meeks [10] who proved that a properly embedded minimal surface in $\mathbb{R}^3$ can have at most two distinct annular ends with infinite total curvature (also see [5] for related results). Our main theorem gives some detailed information on the behavior of the possible two ends of M of infinite total curvature.

Theorem 1.1 *Suppose $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with more than one end. If A is an annular end of M, then, after a rotation of M, either A is smoothly asymptotic to a horizontal plane or $x_3|A$ is a proper harmonic function on A. In particular, every such A is conformally diffeomorphic to the punctured disk $D^* = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$.*

Corollary 1.1 *If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface of finite topology and with at least two ends, then M has finite conformal type.*

Corollary 1.2 *If $M \subset \mathbb{R}^3$ is a properly embedded minimal annulus, then, after a possible rotation of M, M intersects every horizontal plane in a single simple closed curve.*

Corollary 1.2 is closely related to the famous Nitsche conjecture that states that if a complete embedded minimal annulus intersects every horizontal plane in a simple closed curve, then it is a catenoid. Thus, one sees that when M is a properly embedded minimal annulus in $\mathbb{R}^3$, then M is a catenoid if and only if the Nitsche conjecture holds. More generally, Theorem 1.1 reduces the general finite total curvature conjecture to the following:

Conjecture 1.2 *(Generalized Nitsche Conjecture): For $t \geq 0$, let P_t denote the horizontal plane of height t over the x_1x_2-plane. Suppose that $A \subset \mathbb{R}^3$ is a minimal annulus with $\partial A \subset P_0$ and that A intersects every P_t in a simple closed curve. Then A has finite total curvature.*

We remark that Rosenberg and Toubiana [16] proved that there exist proper minimal immersions $X\colon \mathbb{C} - \{0\} \to \mathbb{R}^3$ of infinite total curvature with third coordinate function $X_3(z) = \ln |z|$. Such a surface intersects every horizontal plane in a single immersed closed curve. Examples such as X show the need for the annulus in the above conjecture to be embedded.

2 Proof of the Main Theorem

In [12] Hoffman and Meeks proved that two disjoint properly immersed minimal surfaces in $\mathbb{R}^3$ must be parallel planes. A key step in proving this theorem is to prove the special case where one of the minimal surfaces is a plane; in other words, a key step is to show that if a properly immersed minimal surface is contained in a halfspace, then the surface is a plane. This special result is called the Halfspace Theorem. The technique of proof of the Halfspace Theorem of using catenoid barriers will be used to prove the following lemma.

For the remainder of this section we use the following notation:

$$\begin{array}{rcl} H^+ & = & \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}, \\ P_t & = & \{x_3 = t\}, \\ P & = & \{x_3 = 0\}\,. \end{array}$$

Lemma 2.1 *Suppose $\Sigma \subset H^+$ is a properly immersed minimal surface with nonempty, possibly noncompact, bounday $\partial\Sigma$. If $x_3(\partial\Sigma) \geq \delta$, then $x_3(\Sigma) \geq \delta$.*

Proof. Suppose that $x_3(\Sigma) \geq \delta$ fails, i.e., that $\varepsilon = \inf\{x_3(p) \mid p \in \Sigma\} < \delta$. After translating Σ down by ε we may assume that $\varepsilon = 0$. Consider the unit disk $D \subset P$. Since D is compact, D is a positive distance d from Σ. Hence, after a small upward translation of D by a distance of $\min(\frac{1}{2}, \frac{d}{2})$ we obtain a new disk $\widetilde{D}$. For $t \geq 1$, let $S(t)$ denote the circle of radius t in P centered at the origin in P. For each t, $S(t) \cup \partial\widetilde{D}$ bounds a stable catenoid $C(t)$. (See Figure 1.) These catenoids vary continuously with t and $C(1)$ is disjoint from Σ. Since the sets $\widetilde{D} \cup C(t)$ converge continuously (on compact subsets of ^{3}) to the plane at height $\min(\frac{1}{2}, \frac{d}{2})$ and Σ is a closed subset of ^{3}, there exists a smallest t_0 such that $C(t_0) \cap \Sigma \neq \emptyset$. Since $\partial C(t_0) \cap \Sigma = \emptyset$, $C(t_0)$ and Σ intersect at an interior point and so the maximum principle gives a contradiction. This contradiction proves the lemma. $\square$

Figure 1

Lemma 2.2 *If $M \subset {}^3$ is a properly embedded minimal surface with more than one end, then there exists an end of a catenoid or an end of a plane in one of the closed complements of M in ^{3}.*

Proof. This result is well-known and essentially appears in several papers (see for example, [1], [7]). For completeness, we will sketch the proof of the lemma.

If M has finite total curvature, then the proof of the lemma is clear. Assume now M has infinite total curvature. Since M has more than one end and infinite total curvature, it is easy to prove that there exists a simple closed curve α in M that separates M into two noncompact components; one of which contains a compact unstable domain and the other component has infinite total curvature, which implies it also is unstable [6]. Separation properties imply that there is a closed complement N of M in $\mathbb{R}^3$ such that α does not bound with $\mathbb{Z}_2$-coefficients in N.

Let Δ be one of the closed complements of α in M and choose a smooth compact exhaustion $\alpha \subset \Delta_1 \subset \Delta_2 \subset \ldots$ of Δ. Since ∂N has nonnegative mean curvature, ∂N is an appropriate barrier for solving Plateau type problems in N. Let $\widetilde{\Delta}_i$ denote the least-area $\mathbb{Z}_2$-current in N with $\partial \widetilde{\Delta}_i = \partial \Delta_i$. Standard regularity results [19] imply that $\widetilde{\Delta}_i$ is smooth and that a subsequence of the $\widetilde{\Delta}_i$ converges to a properly embedded area-minimizing surface Σ in N with $\partial\Sigma = \alpha = \Sigma \cap (M = \partial N)$. Separation properties imply Σ separates N and hence Σ is orientable. Since Σ is stable and orientable, a theorem of Fischer-Colbrie [6] implies Σ has finite total curvature. Since α does not bound a compact cycle in N, the surface Σ is noncompact.

Consider an annular end representative A of Σ with $\partial A \cap M = \emptyset$. The maximum principle at infinity [14] implies that $dist(A, M) > 0$. Since A has finite total curvature, it is asymptotic to an end of a catenoid or a plane. Since A is a positive distance from M, we can pick an end E representative of this plane or catenoid to also be a positive distance from M and hence E is contained in a closed complement of M in $\mathbb{R}^3$. This completes our sketch of the proof of Lemma 2.2. $\square$

Lemma 2.3 *Suppose A is the image of a proper minimal embedding of $S^1 \times [0, \infty)$ into $\mathbb{R}^3$ such that A is disjoint from P and A has infinite total curvature. Then the third coordinate function of A is a proper harmonic function on A.*

Proof. The annular end theorem in [10] shows that if the third coordinate function of a minimal annulus is bounded, then the annulus has finite total curvature. Thus the annulus A must have unbounded third coordinate function.

Suppose that the lemma fails. We will derive a contradiction by using the annulus A as a barrier to construct a closed simply connected minimal graph G over a domain in P with nonempty boundary and such that G is a bounded graph of bounded gradient. Furthermore, outside of some compact set, the boundary values of G are contained in a plane $\widetilde{P}$ parallel to P but G is not asymptotic to its projection onto $\widetilde{P}$. Since G is simply connected, the boundary components of G are each noncompact. By Theorem 3.1, such a G must be asymptotic to its projection onto $\widetilde{P}$, which will give the required contradiction. We now proceed to our construction of G.

We first want to extend the annulus A so that it has its boundary on the plane P. (The extended annulus will not be minimal.) Let $B^+(R) = \{x \in H^+ \mid \|x\| \leq R\}$ for some fixed R that is large enough so that $\partial A \subset B^+(R)$ and so that $\partial B^+(R)$ is transverse to A. The maximum principle implies that $Q = A \cap (\mathbb{R}^3 - B^+(R))$ is a connected planar domain with one end. Let $\mathcal{D}$ be a pairwise disjoint collection of disks in $B^+(R)$ with $\partial\mathcal{D} = \partial Q$ and such that $\mathcal{D} \cup Q$ is a smooth embedded surface. Inside $B^+(R)$ we can join some point in the interior of a disk of $\mathcal{D}$ to a point on P by an arc α that intersects $\mathcal{D} \cup Q$ only at its end point. Removing disk neighborhoods of the points $\alpha \cap (\mathcal{D} \cup Q)$ and replacing these disks by the annular boundary of a tubular neighborhood of α, one obtains a smooth annulus $\widehat{A}$ with $\widehat{A} \cap P = \partial\widehat{A}$. See Figure 2. Note that $\widehat{A}$ will not be minimal on the compact subdomain $\widehat{A} - Int(A)$. Let x_3 denote the third coordinate function of $\widehat{A}$.

Figure 2

Since the lemma fails, we can define the finite number $L = \sup\{t \mid x_3^{-1}[0,t]$ is compact$\}$. We will show that the set $x_3^{-1}[0,L]$ is noncompact. If not, then the closed subset $x_3^{-1}(L)$ is compact. Remove a small compact neighborhood of $x_3^{-1}(L)$ from $x_3^{-1}([L, \infty))$ to obtain a surface Σ with compact boundary and $x_3(\partial\Sigma) \geq L + \varepsilon$ for some $\varepsilon > 0$. However since $x_3^{-1}[L, L + \varepsilon/2]$ is noncompact, there exist points in $x_3^{-1}[L, L+\varepsilon/2]$ that are in Σ. Hence, $x_3(\Sigma) \ngeq L+\varepsilon/2$, in contradiction to Lemma 2.1. This proves that $x_3^{-1}([0, L])$ is noncompact. It follows that $x_3^{-1}(L)$ is noncompact, a fact we will later use.

Note that the L defined above is positive. Now choose a regular value δ of $x_3|A$ such that δ is much larger than L and much larger than the height of $\partial(\widehat{A} - A)$. Assume that δ is chosen large enough so that the component of $A(\delta) = x_3^{-1}([0, \delta]) \subset \widehat{A}$ that contains $\partial\widehat{A}$ also contains the compact set $\widehat{A} \cap B^+(R)$. This choice of δ forces any other component of $A(\delta)$ to be contained in A, an impossibility by Lemma 2.1. Therefore, $A(\delta)$ is connected. Note that the choice of δ is now fixed for the remainder of the proof.

We now show that every component of $\partial A(\delta)$ except $\partial\widehat{A}$ is noncompact. If not, then there exists a compact component ϕ of $\partial A(\delta)$ at height δ. This ϕ separates $\widehat{A}$ into a noncompact domain and a compact domain F. By the maximum principle $F \subset A(\delta)$ and since $A(\delta)$ is connected $F = A(\delta)$. But $A(\delta)$ is noncompact which is a contradiction and proves that $\partial\widehat{A}$ is the only compact component of $\partial A(\delta)$.

Choose an embedded arc in $A(\delta)$ joining $\partial\widehat{A}$ to one of the other boundary components of $\partial A(\delta)$ and remove a small open regular neighborhood of the arc from $A(\delta)$ to obtain a simply connected subdomain Δ of $A(\delta)$. See Figure 3.

Figure 3: Note Δ is slightly shaded

Later on we will use Δ to construct simply connected minimal graphs and use $\widehat{A}$ as a barrier. Let Ω denote the exterior annulus in P bounded by $\partial\widehat{A}$. Now $\widehat{A} - A$ is not a minimal submanifold of ^{3}. However, we can take a compact ball B containing $\widehat{A} - A$ and modify the flat metric in the interior of B so that $\Omega \cup \widehat{A}$ is a good barrier for solving the Plateau problem. In other words, the smooth points of $\Omega \cup \widehat{A}$ will be mean convex as the boundary of the subdomain N of H^+ bounded by $\Omega \cup \widehat{A}$ and N will have convex angles along $\partial\widehat{A}$. If $\widehat{A}$ were minimal, this change of metric would not be necessary. The point is that some minimal surfaces constructed in this new metric will provide minimal graphs for the flat metric outside a suitably chosen region containing B, and these graphs will eventually suffice to give our needed contradiction. With these motivating remarks in mind, we return to the construction of the minimal graphs.

Let $C(t)$ denote the solid cylinder of radius t around the x_3-axis. Now choose an increasing sequence of numbers $t_1 < t_2 < \dots$ diverging to infinity such that $\partial C(t_i)$ is transverse to $\widehat{A}$. Also assume that t_1 is chosen large enough so that the ball $B \subset C(t_1)$ and the part of $\partial\Delta$ strictly below height δ is contained in $C(t_1)$ (see Figure 4).

Figure 4: Note that $D(t_i)$ lies below $\Delta(t_i)$

Let $\Delta(t_i)$ denote the component of $\Delta \cap C(t_i)$ that contains $\partial\widehat{A} \cap \Delta$. We want to show that $\Delta(t_i)$ is a disk. Since $\Delta(t_i)$ is a planar domain, it suffices to show $\partial\Delta(t_i)$ is connected. If $\partial\Delta(t_i)$ is not connected, then there exists a boundary curve σ such that σ is completely contained in $\partial C(t_i)$. Since Δ is simply connected, σ bounds a disk F in Δ that goes out of $C(t_i)$ near σ. But this is impossible since, by the convex hull property, the disk F is contained in $C(t_i)$. Thus, we see that $\Delta(t_i)$ is a disk.

Since ∂N has nonnegative mean curvature and $\partial\Delta(t_i)$ bounds a disk in N, the geometric Dehn's lemma [15] implies that $\partial\Delta(t_i)$ is the boundary of an embedded least-area disk $D(t_i)$ in N with $\partial D(t_i) = \partial\Delta(t_i)$. (Recall that N is the exterior component of $\widehat{A}$ in H^+.) Recall that Δ actually depended on the choice of a large δ. By the curvature estimates of Schoen [17] for stable minimal surfaces, there exists a universal constant c such that for all points $p \in D(t_i)$ of distance $d(p)$ from $\partial D(t_i)$, the Gaussian curvature satisfies the inequality $|K(p)| \leq \frac{c}{d^2(p)}$. Since $D(t_i)$ is minimal, this estimate gives good upper bounds on the second fundamental form on that portion of $D(t_i)$ uniformly far from $\partial D(t_i)$. More precisely, consider a point $p \in \partial C(t) \cap D(t_i)$. When $\delta \gg L, t_1 \ll t \ll t_i$ and $x_3(p) \leq L+1$, then p is far from $\partial D(t_i)$ and hence the second fundamental form is arbitrarily small at p. Notice that these estimates imply that the tangent plane at such a p is close to the horizontal. Otherwise, since $D(t_i)$ is geometrically almost a plane in a large geodesic disk E around p in $D(t_i)$, points on this tangent plane relatively close to p, and hence also on E, would lie below P, which is false.

Now assume δ and t_k are sufficiently large, so that for $t_k \leq t \ll t_i$, and $p \in \partial C(t) \cap D(t_i)$ with $x_3(p) \leq L+1$, the tangent plane at p makes an angle of less than $\pi/4$ radians with the horizontal. We will say such a pair (δ, t_k) is *admissible.*

Assertion 2.1 *Let (δ, t_k) be admissible. Then for $t > t_k$ and t_i sufficiently large, each component of $D(t_i) \cap (C(t) - C(t_k)) \cap \{x_3 \leq L+1\} = W_i(t)$ is a minimal graph for $t \leq t_i/2$.*

Proof. We know that t_i can be chosen sufficiently large so that for each $p \in W_i(t)$ the tangent plane of $D(t_i)$ at p makes an angle of at most $\pi/4$ with P. Hence, for each $s \in [t_k, t], W_i(t)$ is transverse to $\partial C(s)$ and the slope of the intersection curves $\Gamma(s)$ on $\partial C(s)$ are less than $\pi/4$. The endpoints of these intersection curves on $\partial C(s)$ are all at height $L+1$; hence each such connected curve is a graph over the horizontal circle (this uses the fact that these curves are embedded). Suppose $\beta_1(s)$ and $\beta_2(s)$ are two components of $\Gamma(s)$, graphs over circular arcs $\gamma_1(s)$ and $\gamma_2(s)$ with $\gamma_1(s) \subset \gamma_2(s)$.

Then as s varies between t_k and t, $\beta_1(s)$ and $\beta_2(s)$ must remain in distinct components of $W_i(t)$ since the endpoints of $\beta_1(s)$ and $\beta_2(s)$ can not join together as s varies. This proves the assertion that each component of $W_i(t)$ is a graph. $\square$

Since $x_3^{-1}(L) \subset \widehat{A}$ is noncompact and $\overline{\widehat{A} - A}$ is compact, there exists a proper noncompact connected arc on $x_3^{-1}(L)$ such that $C(t_k)$ intersects this arc in exactly one point, the endpoint of the arc. Call this arc η.

Now assume that t_j is chosen large enough so that $\Delta \cap C(t_k) \subset \Delta(t_j)$ and $t_j \gg t_k$. (Such a t_j exists since Δ is proper.) Consider the annulus $\widetilde{A}(t_j)$ obtained by adding the compact strip $A(\delta) - \Delta$ to $\Delta(t_j)$. Let $S(t_j)$ denote the intersection of $C(t_j)$ with the slab bounded by the planes P and P_δ. The annulus $\widetilde{A}(t_j)$ separates $S(t_j)$ into two regions and let $N(t_j)$ denote the closed complement that is disjoint from the open disk $F \subset P$ bounded by $\partial\widehat{A}$. Note that $D(t_j) \subset N(t_j)$ and $\Delta(t_j) \cup D(t_j)$ bounds a compact region $R(t_j)$ of $N(t_j)$ that is disjoint from F. (See Figure 5.)

Figure 5

Consider the endpoint e of η and the projection $e_p = (x_1(e_1), x_2(e), 0) \in P$ and the vertical arc β joining e_p to e. Separation properties imply that $\beta \cap D(t_j)$ is nonempty (odd if in general position) and has a lowest point which we will call $q(j)$. The point $q(j)$ is on a component $G(t_j)$ of $W_j(t_j/2)$.

Since the $G(t_i)$ satisfy uniform curvature as well as local area bounds (since they are graphs with uniformly bounded gradients) a subsequence of the $G(t_i)$ converge to a properly embedded connected graph G. Note that $\partial G \subset P_{L+1} \cup \partial C(t_k)$ and we know that this boundary is nonempty and smooth except at the intersection points with the circle $C(t_k) \cap P_{L+1}$. Note that η is disjoint from G (except in the case $G \subset A$ and $\eta \subset G$). Also, for each point of η the vertical downward segment intersects some point of G, even if $G \subset A$ in which case the points coincide. Since the arc η is noncompact, G lies below η along η (i.e., the vertical segments going down from η always intersect G). Hence, G can not be asymptotic to the plane P_{L+1}.

We now show that G is simply connected. Consider a closed curve γ in G. For i sufficiently large we can lift γ to a curve $\widetilde{\gamma} \subset G(t_i) \subset D(t_i)$. Since $D(t_i)$ is simply connected, $\widetilde{\gamma}$ bounds a disk $F(t_i)$ in $D(t_i)$ and a subsequence of these disks will converge to a disk in G bounded by γ. This proves G is simply connected.

Since G is simply connected and noncompact, ∂G is composed of a noncompact boundary arcs (possibly disconnected) at height $L+1$ together with compact arcs on $\partial C(t_k)$. However, outside of any compact subset of $\mathbb{R}^3$, G has points below η and hence below P_L. Furthermore, G has bounded gradient since it is the limit of graphs with a fixed bound on their gradients. By Theorem 3.1, such a graph G can not have this asymptotic behavior; instead, it must be asymptotic to the plane at height $L+1$. This contradiction proves the lemma. $\square$

Lemma 2.4 *Suppose A is the image of a proper minimal embedding of $S^1 \times [0, \infty)$ into $\mathbb{R}^3$ and A is disjoint from the end E of some catenoid with vertical limit normal vector. If A has infinite total curvature, then the third coordinate function of A is a proper harmonic function on A.*

Proof. Assume the lemma fails, i.e., $x_3|A$ is not proper. After a rigid motion of A and E and replacement of E by a subend of E, we may assume that E is centered along the x_3-axis, E is a graph over the complement of a disk $D \subset P$, $\partial E = \partial D$ and E has negative logarithmic growth. Note that $E \cup D$ separates $\mathbb{R}^3$ into two regions, an upper region and a lower region. After the removal of a compact subannulus of A, we may assume that A is disjoint from D and hence $A \cap (E \cup D) = \emptyset$. This implies A either lies above or below $E \cup D$. If A lies below $E \cup D$, then the restriction of the third coordinate function would be a proper function on the region below $E \cup D$ and hence a proper function on A (since A is a closed subset of this lower region). Thus, we may assume that A lies above $E \cup D$. Since A lies above $D \cup E$, after an upward translation of A, we may assume $x_3(\partial A) > 0$.

Now for $t \in [0, 1)$ consider a smooth family $E(t)$ of catenoidal graphs with $\partial E = \partial E(t)$ such that $E(0) = E$, $E(t)$ is above $E(s)$ when $t > s$ and they converge to P as $t \to 1$. First note that $E(t) \cap A \neq \emptyset$ for some $t > 0$. Otherwise A is disjoint from the plane P to which the $E(t)$ limit as $t \to 1$, in which case Lemma 2.3 implies Lemma 2.4.

Let $T_0 = \sup\{t \mid E(t) \cap A = \emptyset\}$. If $E(T_0)$ intersected A, $E(T_0)$ intersects A transversely at an interior point $p \in E(T_0)$. However, since the $E(t)$ vary smoothly with $t, E(t)$ would intersect A transversely near p for t near T_0, a contradiction of the definition of T_0. Thus, $D \cup E(T_0)$ is disjoint from A. Since $x_3(\partial A) > 0$, it is clear that $A \cap E(t) \neq \emptyset$ for all $t > T_0$. (See Figure 6.)

Figure 6

Assertion 2.2 *There exists an $s_1 > T_0$ such that $E(s_1)$ is transverse to A and the region $R(s_1)$ in the lower halfspace bounded by $E(s_1) \cup E(T_0)$ intersects A in some simply connected component Δ.*

Proof of Assertion 2.2. First choose any value $s' > T_0$ such that $E(s')$ intersects A transversely. Let C be a component of $R(s') \cap A$.

We will now prove that ∂C has at most one compact component ∂ and if ∂ exists then ∂ is homotopically nontrivial on A. Suppose ∂ is a compact component of ∂C

such that ∂ is the boundary of a disk F on A which near its boundary lies above A. Since ∂F is below P, F is also below P by the maximum principle. By translating F downward one achieves a last point of contact with $E(s')$, which contradicts the maximum principle. If ∂C contained two compact curves ∂_1, ∂_2 then together they bound a compact annulus on A and a vertical translation of this annulus will again contradict the maximum principle. This proves our above affirmation concerning the possible compact component of boundary C. (See Figure 7.)

Figure 7

Since $C \subset A$ and A is an annulus, C will be simply connected if the possible compact component ∂ does not exist. Suppose now that ∂ of ∂C does exist. If $\partial C = \partial$, then A contains an end representative, namely C, that lies below $E(s')$ which implies $x_3|A$ is proper, a contradiction to our assumption. Thus ∂C also has some noncompact boundary component. Choose an arc α in C joining ∂ to some point in a noncompact boundary component of ∂C. Since α is compact, the $E(t)$ converge smoothly to $E(T_0)$ as $t \to T_0$, and $E(T_0) \cap A = \emptyset$, we can choose a value $s_1 > T_0$ such that α lies above $E(s_1)$. But it then follows that there is a proper noncompact arc β in A, beginning at ∂A, that lies above $E(s_1)$. Hence, $A \cap R(s_1) = (A - \beta) \cap R(s_1)$.

Let Δ be a component of $R(s_1) \cap A$ and note that Δ is contained in the simply connected region $A - \beta$. Thus, to prove Δ is simply connected, it suffices to prove that every boundary curve in Δ is noncompact. If ∂_1 were a compact boundary component of Δ, then ∂_1 bounds a disk $D_1 \subset A - \beta$ whose boundary lies in $E(s_1)$

and which lies above $E(s_1)$ near its boundary. Again the usual application of the maximum principle proves that D_1 cannot exist which completes the proof that Δ is simply connected. This completes the proof of Assertion 2.2. $\square$

We will use Δ as a barrier to construct a minimal graph G with boundary in $E(s_2)$ for some s_2, $s_1 > s_2 > T_0$, in a way similar to how we used the surface Δ defined in the proof of Lemma 2.3 as a barrier to construct a minimal graph there. We will use $\partial R(s_2)$ also as a barrier so that $G \subset R(s_2)$. As in the proof of Lemma 2.3, let $C(t)$ denote the solid cylinder of radius t and choose a divergent sequence $t_1 < t_2 < t_3 < \dots$ such that $C(t_i)$ is transverse to Δ. Assume that $C(t_1)$ contains ∂E and some fixed point $q \in \Delta$.

Let N be the closed complement of $R(s_1) - \Delta$ that contains $E(T_0)$ in its boundary. Let $\Delta(t_i)$ be the component of $\Delta \cap C(t_i)$ containing the point q. The same proof as in Lemma 2.3 shows $\Delta(t_i)$ is a disk. Since ∂N is a good barrier for solving Plateau problems, we can replace $\Delta(t_i)$ by an embedded disk $D(t_i)$ of least area in N.

Now consider a sequence $s_1 > a_1 > a_2 > a_3 > \dots$ converging to T_0 and consider the catenoids $E(a_i)$. The curvature estimates of Schoen imply that for n sufficiently large, there are curvature estimates for $D(t_i) \cap R(a_n) \cap \partial C(t)$ when $t \ll t_i$.

Assertion 2.3 *For $n \gg 0$, $0 \ll t \ll t_i$, and $p \in D(t_i) \cap \partial C(t) \cap R(a_n)$, the tangent plane at p is within $\pi/4$ of the horizontal.*

Proof. It is sufficient to prove that the tangent plane $D(t_i)$ at p is close to the tangent plane of $E(T_0)$ at the projection point $\pi(p) \in E(T_0)$ on the same vertical as p. (Close here means that the angle between the two planes is small.) This is true since if t is sufficiently large, the tangent plane to $E(T_0)$ along the circle $E(T_0) \cap \partial C(t)$ is almost horizontal.

For fixed large n consider the vertical interval $I(n,p)$ joining $\pi(p)$ to $E(s_1)$. Let T_p be the translation of ^{3} sending $\pi(p)$ to the origin. Let H be the homothety of ^{3} that sends $T_p(I(n,p))$ to the vertical interval I of height 1 at the origin. Let $f_{(p,n)} = H \circ T_p$.

Fix a compact cylinder

$$K = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1,\ -1 \leq x_3 \leq 2\}.$$

Consider the intersection $Q(p) = f_{(p,n)}(R(s_1)) \cap K$. As $t \to \infty, Q(p)$ converges to the subcylinder $\overline{K}$ of K of height 1 with axis I. The image of $E(s_1)$ converges to the top disk of $\overline{K}$, i.e., $\overline{K} \cap \{x_3 = 1\}$, the image of $E(T_0)$ in K converges to the bottom disk of $\overline{K}$ and the image of $E(a_n)$ in K converges to the horizontal disk of $\overline{K}$ of height $h_n = \frac{\widetilde{s}_1 - \widetilde{a}_n}{\widetilde{a}_n - \widetilde{T}_0}$ where $\sim$ means the logarithmic growth of the catenoids in question.

It follows that when t and t_i are large that the distance from $f_{(p,n)}(R(a_n)) \cap K$ to $f_{(p,n)}(\partial D(t_i)) \cap K$ is greater than $\frac{1-h_n}{2}$. This gives uniform curvature estimates, using Schoen's inequality, for points of $\Sigma_n = f_{(p,n)}(D(t_i) \cap R(a_n)) \cap K$ and hence the second fundamental forms of Σ_n are uniformly bounded. Thus, there exists an $\varepsilon > 0$ and an $h > 0$ independent of n such that for each $q \in \Sigma_n, \Sigma_n$ is a graph of height at most h over the disk of radius ε in the tangent plane to Σ_n at q. Since $h_n \to 0$ as $n \to \infty$, it follows that the tangent planes to Σ_n converge to the horizontal as $n \to \infty$, otherwise the Σ_n would cross $f_{(p,n)}(E(T_0)) \cap K$. This proves Assertion 2.3. $\square$

Fix k and n sufficiently large and $t_k \ll t_i$. Then by Assertion 2.3, the tangent plane to points $p \in D(t_i) \cap (C(t) - C(t_k)) \cap R(a_n)$ are $\pi/4$ close to the horizontal, for $t_k \leq t \leq t_i/2$. A small modification of the arguments at the end of the proof of Lemma 2.3 shows that the components of $D(t_i) \cap (C(t_i/2) - C(t_k)) \cap R(a_n)$ are graphs. Furthermore, the arguments there show that some subsequence of these graphs converges to a simply connected minimal graph G contained in the region between $E(a_n)$ and $E(T_0)$ and that ∂G consists of a compact part in $C(t_k)$ and the rest in $E(a_n)$. Furthermore, we have G is not contained in $E(a_n)$.

We now apply Theorem 3.1 to G and the part of the catenoid $E(a_n)$ which is a graph over the same domain of P as G. Since the difference of these two graphs grows at most logarithmically, we have a contradiction which proves Lemma 2.4. $\square$

Proof of Theorem 1.1. Suppose $M \subset {}^3$ is a properly embedded minimal surface with more than one end. Assume, after a possible rotation of M, that the limiting normal line of E is vertical. By Lemma 2.2 there exists an end E of a catenoid or the end of a plane in one of the closed complements of M in ^{3}. Let A be an annular end of M. If A has finite total curvature, then the theorem holds for A. Suppose now A has infinite total curvature. If E is the end of a plane Q, then $Q \cap A$ is compact. Hence, A has a subend $\widetilde{A}$ that is disjoint from a horizontal plane or else A is disjoint from the end of a vertical catenoid. Thus, Lemmas 2.3 and 2.4 now imply that $x_3|A$ is a proper harmonic function, which completes the proof of Theorem 1.1. $\square$

3 The Uniqueness of Certain Minimal Graphs

Theorem 3.1 *Let u_1, u_2 be solutions of the minimal surface equation on a domain Ω with some component of $\partial\Omega$ noncompact. Assume $u_1 - u_2 = u$ is of compact support on $\partial\Omega$ and $\nabla u_1, \nabla u_2$ are bounded. Let $M(r) = \sup\{|u(x)|/|x| = r\}$.*

1. *If $M(r)$ is bounded, then $M(r) \to 0$ as $r \to \infty$;*
2. *if $M(r)$ is not bounded, then for every integer $n > 0$,*
$$\liminf \frac{M(r)}{(\ln(r))^n} > 0\,.$$
In particular, for any constant K, $\liminf \frac{M(r)}{K \ln r} > 0$.

Proof. We shall use the techniques in [2]. Let $\Omega_r = \{x \in \Omega/|x| < r\}$, $C_r = \Omega_r \cap \{|x| = r\}$ and $u_1 = u_2$ on $\partial\Omega \cap \{|x| \geq \widetilde{r}_0\}$. Let $p = \frac{\partial}{\partial x}(u_1 - u_2) = u_x\,, q = \frac{\partial}{\partial y}(u_1 - u_2) = u_y$, $w = \sqrt{1 + |\nabla u|^2}$ and $p_i = \frac{\partial}{\partial x} u_i, q_i = \frac{\partial}{\partial y} u_i$ for $i = 1, 2$.

Since u_1, u_2 satisfy the minimal surface equation, the form
$$(\frac{p_1}{w_1} - \frac{p_2}{w_2})\, dy - (\frac{q_1}{w_1} - \frac{q_2}{w_2})\, dx$$
is closed; we write this form as $d\psi, \psi$ a (multi-valued) function on Ω.

A direct calculation (cf. [2]) shows:
$$\iint_{\Omega_r} |d\psi|^2 \leq \int_{\partial\Omega_r} u\, d\psi\,.$$
Let $R > R_0 > \widetilde{r}_0$, and $\mu(r) = \iint_{\Omega_r} |d\psi|^2$, we have:
$$\mu(R_0) + \iint_{\Omega_R - \Omega_{R_0}} |d\psi|^2 \leq \int_{\partial\Omega_R} u\, d\psi = \int_{\partial\Omega_R \cap \{|x| \leq \widetilde{r}_0\}} u\, d\psi + \int_{C_R} u\, d\psi$$
since $u = 0$ on $\partial\Omega_r \cap \{|x| > \widetilde{r}_0\}$. Let $c_0 = \int_{\partial\Omega_R \cap \{|x| \leq \widetilde{r}_0\}} u\, d\psi$ and $\widetilde{\mu}(R) = \mu(R) - c_0$. Then the last inequality may be written:
$$\begin{aligned} &\widetilde{\mu}(R_0) + \iint_{\Omega_R - \Omega_{R_0}} |d\psi|^2 \leq \int_{C_R} u\, d\psi \leq \sup_{|x|=R} |u(x)| \eta(R)\,, \\ &\text{where } \eta(R) = \int_{C_R} |d\psi|\,. \end{aligned} \tag{3.1}$$

By Cauchy-Schwarz we have:

$$\frac{\eta(r)^2}{2\pi r} \leq \int_{C_r} |d\psi|^2 .$$

Then inequality 1. becomes:

$$\widetilde{\mu}(R_0) + \int_{R_0}^{R} \frac{\eta(r)^2}{2\pi r} dr \leq M(R)\eta(R) . \tag{3.2}$$

Since $|\nabla u_1|, |\nabla u_2|$ are bounded, we assert that there is a constant $\overline{d}_0 > 0$, so that

$$\overline{d}_0 |\nabla u| \leq |d\psi| .$$

We postpone the proof of this assertion until the end of the proof of Theorem 3.1.

On $C_r, r > \widetilde{r}_0$, u is zero on ∂C_r (which is nonempty by our hypothesis that $\partial\Omega$ has a noncompact component) and its' maximum value on C_r is $M(r)$, so $\int_{C_r} |\nabla u| \geq \overline{d}_1 M(r)$, for some positive constant $\overline{d}_1$, and

$$d_0 M(r) \leq \eta(r),\ d_0 = \overline{d}_0\, \overline{d}_1 . \tag{3.3}$$

Since u is the difference of two solutions of the minimal surface equation, $M(r)$ can not have a local maximum for $r > \widetilde{r}_0$, by the maximum principle. Hence if $M(r), r > \widetilde{r}_0$ is monotone increasing at one point, it remains monotone increasing for larger values of r.

In particular, if either $M(r)$ does not tend to zero as $r \to \infty$ or if $M(r)$ is unbounded, there is a positive constant c_1 such that $M(r) \geq c_1$ for r larger than some $R_0 \geq \widetilde{r}_0$. Thus $\eta(r) \geq d_0 c_1 = c_2 > 0$ for $r \geq R_0$ by 3.3. When $\eta(r) \geq c_2 > 0$ for $r \geq R_0$, we have for $R \geq R_0$; $\mu(R) \geq \int_{R_0}^{R} \int_{C_r} |d\psi|^2 dr \geq \int_{R_0}^{R} \frac{\eta(r)^2}{2\pi r} dr \to \infty$, as $R \to \infty$. In particular, $\widetilde{\mu}(R) = \mu(R) - c_0 > 0$ for R large.

Now we shall prove 1. Assume, on the contrary, that $M(r)$ does not tend to zero and $M(r)$ is bounded, $M(r) \leq A$. Let R_0 be large so that $\widetilde{\mu}(R_0) > 0$. Inequality 3.2 yields:

$$\widetilde{\mu}(R_0) + \int_{R_0}^{R} \frac{\eta(r)^2}{2\pi r}\, dr \leq A\eta(R) . \tag{3.4}$$

Let J be the interval $[R_0, R_0 \exp(4\pi A^2/\widetilde{\mu})]$, $\widetilde{\mu} = \widetilde{\mu}(R_0)$, and $\xi(r)$ the function on J:

$$\frac{2A}{\widetilde{\mu}} - \frac{1}{\xi(r)} = \frac{1}{2\pi A} \ln\left(\frac{r}{R_0}\right) .$$

Then

$$\xi'(r) = \xi(r)^2/2\pi A r \text{ and}$$

$$\xi(R_0) = \frac{\widetilde{\mu}}{2A} < \frac{\widetilde{\mu}}{A} \le \eta(R_0) \text{ (by 3.4).}$$

Hence R_0 is in the set $\{R \ge R_0 \mid \text{for all } R', R_0 \le R' \le R,\ \xi(R') < \eta(R')\}$, and this set is open. It is closed (by 3.4 and) since $A\xi(R) = \frac{\widetilde{\mu}}{2} + \int_{R_0}^{R} \frac{\xi(r)^2}{2\pi r}\,dr$.

It follows that this set is $J \cap [R_0, R]$. Since $\xi(r) \to \infty$ as $r \to R_0 \exp(\frac{4\pi A^2}{\widetilde{\mu}})$ from below, and η is defined there, $\xi(r) < \eta(r)$ is impossible at this point, so $R < R_0 \exp(\frac{4\pi A^2}{\widetilde{\mu}})$. This is a contradiction since R is arbitrary in the above argument, and Case 1 of Theorem 3.1 is proved.

Notice that the above argument proves the following. Suppose that $M(r)$ is monotone increasing from some $\widetilde{R}_0$ on (e.g., if $M(r)$ is unbounded as in case 2 of our theorem). Choose $\widetilde{R}_0$ so that $\widetilde{\mu}(\widetilde{R}_0) > 0$ and let $R_1 > \widetilde{R}_0$, $M(R_1) = A$. Then

$$R_1 < \widetilde{R}_0 \exp(\frac{4\pi A^2}{\widetilde{\mu}})\,, \text{ i.e.}$$

$$A = M(R_1) > (\frac{\widetilde{\mu}}{4\pi} \ln(\frac{R_1}{\widetilde{R}_0}))^{1/2}\,, \widetilde{\mu} = \widetilde{\mu}(\widetilde{R}_0)\,. \tag{3.5}$$

We estimate $\mu(R_0), R_0 \ge \widetilde{R}_0$:

$$\mu(R_0) \ge \int_{\widetilde{R}_0}^{R_0} \int_{C_r} |d\psi|^2 \ge \int_{\widetilde{R}_0}^{R_0} \frac{\eta(r)^2}{2\pi r}\,dr \ge \int_{\widetilde{R}_0}^{R_0} \frac{(d_0 M(r))^2}{2\pi r}\,dr$$

(the last inequality by 3.3). By 3.5, one has

$$\mu(R_0) \ge c_3 \widetilde{\mu}(\widetilde{R}_0) \ln(\frac{R_0}{\widetilde{R}_0})^2\,, c_3 = \frac{d_0^2}{16\pi^2 \widetilde{R}_0}\,. \tag{3.6}$$

Now $\widetilde{\mu}(R_0) = \mu(R_0) - c_0$, so estimate $M(R_1)$ with 3.5, 3.6 (with R_0 in place of $\widetilde{R}_0$) and $R_0 = \sqrt{R_1 \widetilde{R}_0}$:

$$\begin{aligned} M(R_1) &> \left(\tfrac{\widetilde{\mu}(R_0)}{4\pi} \ln\left(\tfrac{R_1}{R_0}\right)\right)^{1/2} = \left(\tfrac{(\mu(R_0)-c_0)}{4\pi} \ln\left(\tfrac{R_1}{R_0}\right)\right)^{1/2} \\ &\ge \tfrac{1}{2\sqrt{\pi}} \left(c_3 \widetilde{\mu}(\widetilde{R}_0) \ln\left(\sqrt{\tfrac{R_1}{\widetilde{R}_0}}\right)^2 \ln\left(\sqrt{\tfrac{R_1}{\widetilde{R}_0}}\right) - c_0 \ln\left(\sqrt{\tfrac{R_1}{\widetilde{R}_0}}\right)\right)^{1/2}\,. \end{aligned}$$

The growth of this last expression is like

$$\ln\left(\sqrt{\frac{R_1}{\widetilde{R}_0}}\right)^{3/2} \sim \ln\left(\frac{R_1}{\widetilde{R}_0}\right)^{3/2}\,.$$

Now it is clear that one can iterate this procedure (bound $\mu(R_0)$ from below by a higher power of $\ln\left(\frac{R_0}{\widetilde{R}_0}\right)$ and then $M(R_1)$) to obtain any power of $\ln(R)$ as a growth rate. $\square$

We now prove our earlier assertion that when $|\nabla u_1|, |\nabla u_2|$ are bounded that there is a constant $\overline{d}_0 > 0$, so that $\overline{d}_0|\nabla u| \le |d\psi|$.

The normals to the graphs of u_1, u_2 are $n_1 = (-\frac{p_1}{w_1}, -\frac{q_1}{w_1}, \frac{1}{w_1})$ and $n_2 = (-\frac{p_2}{w_2}, -\frac{q_2}{w_2}, \frac{1}{w_2})$, and they are bounded away from the equator of S^2 since $|\nabla u_1|, |\nabla u_2|$ are bounded. Thus the vector $n_1 - n_2$ is bounded away from the vertical whenever $n_1 \neq n_2$. It follows that $|n_1 - n_2|$ and the length of the projection of $n_1 - n_2$ (which is $|d\psi|$) are of the same order; i.e. there exists a $c > 0$ such that

$$\frac{1}{c}|d\psi| \le |n_1 - n_2| \le c|d\psi|\,.$$

So it suffices to show $|n_1 - n_2|$ and $|\nabla u|$ are of the same order. Consider the map I that radially projects the open upper hemisphere of S^2 to the plane $P = \{x_3 = 1\}$. With respect to the metric $d(p,q) = |p-q|$ on S^2, I is a quasi-isometry on any compact subset of the open hemisphere. Since $I(n_1) = (-p_1, -q_1, 1)$ and $I(n_2) = (-p_2 - q_2, 1)$, then $|\nabla u| = |I(n_1) - I(n_2)|$ has the same order as $|n_1 - n_2|$.

References

[1] M. Callahan, D. Hoffman, and W. H. Meeks III. The structure of singly-periodic minimal surfaces. *Inventiones Math.*, 99:455–481, 1990.

[2] P. Collin and R. Krust. Le probleme de Dirichlet pour l'equation des surfaces minimales sur des domaines non bornes. *Bull. Soc. Math. France*, 120:101–120, 1991.

[3] C. Costa. Example of a complete minimal immersion in ^{3} of genus one and three embedded ends. *Bull. Soc. Bras. Mat.*, 15:47–54, 1984.

[4] C. Costa. Uniqueness of minimal surfaces embedded in ^{3} with total curvature 12π. *Journal of Differential Geometry*, 30(3):597–618, 1989.

[5] Y. Fang and W. H. Meeks III. Some global properties of complete minimal surfaces of finite topology in ^{3}. *Topology*, 30(1):9–20, 1991.

[6] D. Fischer-Colbrie. On complete minimal surfaces with finite Morse index in 3-manifolds. *Inventiones Math.*, 82:121–132, 1985.

[7] C. Frohman and W. H. Meeks III. The ordering theorem for the ends of properly embedded minimal surfaces. preprint.

[8] D. Hoffman. The discovery of new embedded minimal surfaces: Elliptic functions, symmetry; computer graphics. In *Proceedings of the Berlin Conference on Global Differential Geometry*, Springer Lecture Series 1748, 1985.

[9] D. Hoffman, H. Karcher, and F. Wei. The genus-one helicoid, and the surfaces that led to its discovery. in preparation.

[10] D. Hoffman and W. H. Meeks III. The asymptotic behavior of properly embedded minimal surfaces of finite topology. *Journal of the AMS*, 2(4):667–681, 1989.

[11] D. Hoffman and W. H. Meeks III. Embedded minimal surfaces of finite topology. *Annals of Mathematics*, 131:1–34, 1990.

[12] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. *Inventiones Math.*, 101:373–377, 1990.

[13] F. J. Lopez and A. Ros. On embedded complete minimal surfaces of genus zero. *Journal of Differential Geometry*, 33(1):293–300, 1991.

[14] W. H. Meeks III and H. Rosenberg. The maximum principle at infinity for minimal surfaces in flat three-manifolds. *Commentari Mathematici Helvetici*, 65:255–270, 1990.

[15] W. H. Meeks III and S. T. Yau. The equivariant Dehn's lemma and loop theorem. *Commentari Mathematici Helvetici*, 56:225–239, 1981.

[16] H. Rosenberg and E. Toubiana. A cylindrical type complete minimal surface in a slab of ^{3}. *Bull. Sc. Math. III*, pages 241–245, 1987.

[17] R. Schoen. *Estimates for Stable Minimal Surfaces in Three Dimensional Manifolds*, volume 103 of *Annals of Math. Studies*. Princeton University Press, 1983.

[18] R. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. *Journal of Differential Geometry*, 18:791–809, 1983.

[19] L. Simon. Lectures on geometric measure theory. In *Proceedings of the Center for Mathematical Analysis*, volume 3, Canberra, Australia, 1983. Australian National University.