Hypersurfaces of Constant Curvature in Space Forms

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Introduction

In this paper we shall discuss hypersurfaces M of space forms of constant curvature; where curvature means one of the symmetric functions of curvature associated to the second fundamental form. The values of the constant will be chosen so that the linearized equation will be an elliptic equation on M. For example, for surfaces in ^{3} the two possible curvatures are the mean curvature H and the Gaussian curvature K. The linearized equation for H is always elliptic and for K it is elliptic when the constant K is positive. In hyperbolic 3-space, the constant $K > -1$ yields an elliptic equation. Hypersurfaces of constant scalar curvature S_2 will be elliptic when $S_2 > 0$.

We obtain height estimates for compact embedded hypersurfaces of ${}^{m+1}$ whose $r+1$'st symmetric function of curvature, S_{r+1}, is a positive constant (§6.1.): Given such a hypersurface M, with ∂M contained in a hyperplane P, then the maximum distance of a point of M to P is

$$2 \times \left(\frac{C_{r+1}}{S_{r+1}} \right)^{\frac{1}{r+1}}, C_k = \left(\begin{array}{c} m \\ k \end{array} \right).$$

So for constant mean curvature in ^{3}, it is $2/H$ (proved by Heinz: cf. §6), and for constant $K > 0$, it is $2/\sqrt{K}$.

As an application of this height estimate we prove (§8.1.), a theorem of P. Hartman [10]: Let M be a complete embedded hypersurface of ${}^{m+1}$ with S_{r+1} a positive constant. If M has nonnegative sectional curvature then M is isometric to $S^p \times {}^{\ell}$, S^p a round sphere (in a linear $p+1$ plane of ${}^{m+1}$). For scalar curvature ($r = 1$) this result was obtained by Cheng and Yau [6].

In Section 1, we discuss an obstruction for a smooth closed curve in $\mathbb{R}^3$ to bound a surface of positive Gaussian curvature; not necessarily constant. Perhaps the main points of §1 are that there exists such an obstruction and the questions that arise.

In Section 2, we state the existence theorems that are relevant to us. Our main interest is the boundary value problem: when does a given curve bound a positive constant Gaussian curvature surface?

In Section 3, we discuss the equations of Reilly and the variational problem whose critical points are the hypersurfaces S_{r+1} equals constant.

We derive the (known) first variation equation of the second fundamental form and use this to calculate the first variation of the symmetric functions S_{r+1} of curvature. We discuss the linearized operator L_r in this equation and its Jacobi fields. An important point is that L_r is a divergence operator: $L_r(f) = div(T_r \nabla f)$, T_r the r'th Newton transformation of the second fundamental form.

In Section 5, we derive the formula for $L_r(X)$ and $L_r(n)$ (in vector form), X position and n the unit normal. Each is a linear combination of X and n whose coefficients involve S_1, S_r, S_{r+1}, and S_{r+2}, (§5.2.).

In Section 6, we apply the above formulae to obtain the height estimates. A basic tool here are inequalities of Newton:

$$H_{i-1}\ H_{i+1} \leq H_i^2 \qquad (1 \leq i < m)\,,$$

where H_i is the i'th mean curvature, $H_i \begin{pmatrix} m \\ i \end{pmatrix} = S_i$, and all real values are allowed for the principal curvatures (positive and negative!).

In Section 7, we derive some applications of the height estimates: Hartman's theorem, a balancing formula and a theorem concerning embedded constant curvature hypersurfaces with boundary a sphere.

1 An Obstruction to the Existence of Surfaces of Positive Curvature

Let $C \subset \mathbb{R}^3$ be a smooth immersed closed curve with no inflection points (points of zero curvature). Let $n: C \to S^2$ denote the unit principal normal to C. Let $I(C)$ be the regular homotopy class of $n(C)$, i.e., homotopies of $n(C)$ in S^2 among curves with nonvanishing tangent vector (this is the same as the class represented by $n(C)$ in the unit tangent bundle of the sphere).

It is well known that $I(C)$ takes its values in Z_2; embedded curves on S^2 represent the zero of Z_2, and the nonzero element is represented by a figure eight. If one adds a loop to a figure eight, this is zero as indicated by the regular homotopy in figure 1.

Figure 1

Theorem 1.1 *Let* $f\colon M \to {}^3$ *be an immersion,* $C = f(\partial M)$ *and* $f(M)$ *of positive curvature. Then* $I(C) = 0$*.*

Proof. Assume first that M is a disc $\{0 \leq x^2+y^2 \leq 1\}$. Let $C(t) = f(x^2+y^2 = t)$, for $0 < t \leq 1$. For $t = \varepsilon$ sufficiently small, $C(\varepsilon)$ is embedded and close to a convex planar curve in ^{3}, since our positive curvature hypothesis implies that fM is strictly locally convex in a neighborhood of $f(0,0)$. Thus $n(C(\varepsilon))$ is an embedded curve close to a great circle of S^2 and consequently $I(C(\varepsilon)) = 0$. Since $n(C) = n(C(1))$ is regularly homotopic to $n(C(\varepsilon))$, we have $I(C) = 0$. We have used here the fact that if $\alpha(s)$ is a curve on a surface of positive curvature with nonzero tangent at each point, then the curvature $\kappa(s)$ of α is nonzero at each point hence the derivative (with respect to arc length) of the unit normal $n_\alpha(s)$ of α is never 0 (it equals $-\kappa(s)t_\alpha(s) + \tau(s)b_\alpha(s)$). Thus a homotopy of α on the surface of positive curvature yields a regular homotopy on S^2 of $n_\alpha(s)$. □

In general, M is an orientable surface with $\partial M = C$ since $K(M) > 0$ so the mean curvature vector orients M. We remark that the class $I(C)$ can equally well be defined as the regular homotopy class of the image of the tangent vector of C: C has nonvanishing curvature at each point so $(\cos\theta)t(s) + (\sin\theta)n(s)$ is a regular homotopy between the tangent map $t(s) \in S^2$ of C and the normal map $n(s)$. The two regular homotopy classes are then represented by a circle on S^2 with its' tangent and a figure eight on S^2 with its' tangent.

Now let C_1, C_2 be immersed closed curves on M with $K(M) > 0$. Let $x \in C_1, y \in C_2, x \neq y$, and let α be an arc on M joining x to y, transverse to C_1 at x and C_2 at y. Let $C_1 \# C_2$ be a smooth curve on M obtained by thickening α, to obtain two arcs

α_1, α_2 joining C_1 and C_2, and then $C_1 \# C_2$ is $\widetilde{C}_1\alpha_1\widetilde{C}_2\alpha_2$ smoothed along the corners; cf. figure 2.

Figure 2

We claim that $I(C_1 \# C_2) = I(C_1) + I(C_2)$. This can be seen as follows. Let z be a point on α distinct from x and y. Perform a regular homotopy of C_1 by sliding a neighborhood of x on C_1, along α until a point just before z. Similarly do a regular homotopy of C_2 by sliding a neighborhood of y on C_2 along α until z. Now $C_1 \# C_2$ can be constructed with support in a neighborhood N of z; cf. figure 3.

Figure 3

Next perform regular homotopies of C_1 and C_2, leaving N fixed, to standard circles or figure eights. Then the formula $I(C_1 \# C_2) = I(C_1) + I(C_2)$, in Z_2, is clear.

Now we can prove Theorem 1. Decompose M into pairs of pants $P_1, \cdots, P_n$, and an annulus A (a pair of pants is topologically a disc minus two (interior) open discs); cf. figure 4.

Figure 4

Consider the pair of pants P_1, let C_1, C_2 be the other boundary curves of P_1, besides C. Let α be an arc on P_1 joining a point of C_1 to a point of C_2. Then $C_1 \# C_2$ and C bound an annulus on P_1 hence $I(C) = I(C_1 \# C_2)$, and we conclude $I(C) = I(C_1) + I(C_2)$. The same reasoning in P_2 implies $I(C) = I(\widetilde{C})$ where $\widetilde{C}$ is $\partial P_2 - (C_1 \cup C_2)$. Continuing in this way we obtain $I(C) = I(\partial A)$. Since A is an annulus, the two components ∂_1, ∂_2 of ∂A are isotopic. M has positive curvature so the normal images of ∂_1, ∂_2 are regularly homotopic. Hence $I(C) = I(\partial A) = 2\, I(\partial_1) = 0$.

Now we can give an example of a Jordan curve with no inflection points which bounds no surface with $K(M) > 0$.

Consider the curve C of figure 5.

Figure 5

The arc $\overrightarrow{pq}$ is chosen to be part of a helice and the arc $\overrightarrow{ba}$ part of a circle. Then the tangent (or normal) spherical image has one point of self intersection hence $I(C) \neq 0$.

We remark that in the proof of Theorem 1, it is rather easy to see why C bounds no immersed disk of positive curvature when $I(C) \neq 0$. For the C of figure 5, it is also simple to show that C bounds no higher genus immersed surface of positive curvature. One observes that C can be constructed so the height function has exactly two critical points on C. Then if M is an immersed surface with $\partial M = C$ and the height function is a Morse function on C (which we can assume) then one has the formula:

$$i(M) + \frac{1}{2}(T(M) - F(M)) = 1 - 2g\,.$$

Here $i(M)$ are the number of interior critical points of M, $T(M)$ the number of true extremum of the height function on ∂M, $F(M)$ the number of false extremum, and $g =$ the genus of M. We used $K(M) > 0$ in this formula as each interior critical point has index one. Now using the fact that there are exactly two critical points on C, it's easy to see that M can not exist.

1.1 The self-linking number

Christian Bonatti has suggested another invariant $J(C)$: the linking number of C and the curve $C(\varepsilon)$ obtained by going ε along the normal n of C. For ε small and C with no inflection points, $C(\varepsilon)$ is disjoint from C and $J(C)$ is a well defined integer. Bonatti observed that $J(C)$ is zero if C bounds an embedded surface of positive curvature. For if M is such a surface then the scalar product of $n(C)$ and the normal N of M along C is never zero. Therefore, the curve $C(\varepsilon)$ is isotopic, in the complement of C, to the curve $\widetilde{C}$ on M obtained by going ε along the normal to M. Since M is embedded, $\widetilde{C}$ is disjoint from M, hence the intersection number of $\widetilde{C}$ and M is zero. Thus the same holds for $C(\varepsilon)$ and $J(C) = 0$. Notice that M need have positive curvature only near C for this argument, and be orientable.

For the curve C of figure 5, $J(C) = 1$, so C bounds no embedded M of positive curvature.

We observe that $J(C)$ mod 2, is an obstruction to the existence of an immersed surface of positive curvature with boundary C. For if we suppose M is such a surface, we can assume the double points of M are in general position; we consider $f\colon M \to {}^{3}$ the immersion. The double points, $D(M)$, is a one dimensional submanifold of M, composed of embedded cycles in $\mathrm{Int}(M)$ and embedded arcs joining two points of ∂M. Now, as in the previous paragraph, we must show the intersection number of $\widetilde{C}$ and M is even. If $\widetilde{C}$ is disjoint from $D(M)$ then as before, the intersection number is zero. $\widetilde{C}$ can be chosen close enough to C so that the only points of intersection of $\widetilde{C}$ with $D(M)$ are the arcs of $D(M)$ joining two points of ∂M. Assuming $\widetilde{C}$ is transverse to $D(M)$, it is clear that this number is even. When $\widetilde{C}$ is lifted off M by N, each intersection point of $\widetilde{C}$ and $D(M)$ gives rise to one intersection point of $\widetilde{C}$ and M. Thus $J(M) = 0$ mod 2.

1.2 $I(C) = J(C) \bmod 2$

That the two invariants are the same is a consequence of a theorem of Feldman: there are two classes of immersed closed curves in $\mathbb{R}^3$ with no inflection points; the circle represents one class and the curve of figure 5 represents the other class [8].

Naturally, two immersions in the same class means they are homotopic through immersions with no inflection points. Clearly $I(C)$ and $J(C)$ mod 2 do not change during such a homotopy so to prove they are equal it suffices to check this for the circle and the curve of figure 5.

1.3 Vertices of C

Generically, an immersed curve C in $\mathbb{R}^3$ has nonvanishing curvature and torsion that vanishes at isolated points. It is reasonable to consider vertices of a space curve as points where the torsion vanishes. For planar curves, vertices are points where the curve is locally on one side of its osculating circle. Clearly, vertices are preserved under Möbius transformations so when one projects a plane curve stereographically into the unit sphere, the image curve is locally on one side of its osculating plane at the image of the vertex, i.e., it is a point of vanishing torsion.

Now if C is a space curve that bounds a surface M of positive curvature, there are two questions one can pose:

- does C have at least four vertices?
- does $n(C)$ have at least four vertices?

The classical four vertex theorem answers the first question affirmatively for C an embedded curve on the sphere S^2. Indeed, Barner has proved this to be true for embedded curves on strictly convex bodies [2].

Segre has proved that an embedded curve C in $\mathbb{R}^3$ whose tangent spherical image is an embedding and has at least four vertices [22].

A curve C that projects injectively onto a planar convex curve has an injective tangential image hence at least four vertices. Such a curve also bounds surfaces of positive curvature.

1.4 Higher dimensions

Let C be a codimension two immersed sphere in $\mathbb{R}^{m+1}$. When does C bound an immersed submanifold $M \subset \mathbb{R}^{m+1}$ of positive curvature? We now have a choice of curvatures for which to pose this question; any symmetric function of the second fundamental form is a candidate. The obstruction $I(C)$ generalizes directly to the Gauss-Kroneker curvature, the determinant of the second fundamental form. This is equivalent to finding a locally convex hypersurface M with $\partial M = C$. The mean curvature vector H of C would then be nonvanishing and point locally to the convex side of M. As before, the map $C \mapsto S^m$ given by the (normalized) mean curvature vector, is regularly homotopic to the equatorial inclusion map, when M is an immersed m-ball that is locally convex. Thus, the regular homotopy class is trivial.

It would be interesting to find obstructions for the scalar curvature. When the scalar curvature is positive, the Gauss map of M has rank at least two at each point.

2 Some Existence Theorems

Now consider curves $C \subset \mathbb{R}^3$ that do bound surfaces of positive curvature. When can one find such a surface M of constant positive curvature? This should be true for curves C on the boundary of a convex body.

Let us denote a surface M of constant positive Gaussian curvature a K-surface. We now state some of the results we know about this problem.

The fundamental theorem was obtained by Caffarelli, Nirenberg and Spruck.

Theorem 2.1 **([4])** *Let C be a smooth curve that admits an orthogonal injective projection onto a strictly convex planar curve $\partial\Omega$. Then C bounds a K-surface M. M is a graph over Ω.*

In fact, they proved the above theorem in dimension n. The value of K depends on C and if one K-graph with boundary C exists, then they also exist for all K' in the interval $(0, K)$.

Their technique to find the K-surface is the continuity method. Start with some graph u^0 over Ω with C as boundary values and of positive curvature; this is easily obtained from the convexity of $\partial\Omega$. Consider the equation:

$$\det(u^t_{ij}) = ct(1 + |\nabla u^t|^2)^{3/2} + (1 - t)K_0(1 + |\nabla u^0|^2)^{3/2}\,.$$

Here K_0 is the Gaussian curvature of u^0, c a constant, $0 < c < \inf K_0$, and u^t, $0 < t \leq 1$ solutions one looks for, each u^t has the same boundary values C. By the formula for curvature, if u^1 exists its curvature is c and the problem is solved.

One proves the set of t for which one can solve the equation for u^t is open and closed. The implicit function theorem in Banach space yields open rather easily. The difficult point is closed; here they work hard to obtain *a priori* C^2 estimates for the solutions.

Using the same techniques, Hoffman, Rosenberg and Spruck have proved:

Theorem 2.2 ([11]) *Let Ω be a planar annulus with boundary two strictly convex curves. Let C be a curve (with two components) that admits an orthogonal injective projection onto $\partial\Omega$. Assume C bounds some graph over Ω of positive curvature. Then C bounds a graph over Ω that is a K-surface.*

We remark that the hypothesis that C bound some surface of positive curvature is necessary for $\partial\Omega$ bounds no such surface.

Guan and Spruck have announced a generalization of Theorem 2 to radial graphs over certain annular domains on the sphere S^2. Their result will have many interesting consequences. We mention one:

- If C_1, C_2 are convex curves in parallel planes then there exists a K-surface M with boundary $C_1 \cup C_2$; M is topologically an annulus (this is unknown for H-surfaces, H the mean curvature).

In this spirit there are some natural problems that arise:

- How many K-surfaces span a given link Γ? Is it finite for most Γ? (we do not know the answer even for $\Gamma = C_1 \cup C_2$, convex curves in parallel planes). A major difficulty is the lack of boundary regularity: there are K-surfaces of revolution that are bounded by two circles that are not C^2 up to the boundary.

- Let C be a convex planar curve and M_1 a K-graph with boundary C (which always exists for $K > 0$, small enough, by 2.1). Does C bound a distinct K-surface M_2? (The analogous result for H-surfaces was proved by Bresis and Coron, and Struwe [3, 23]: the existence of a small bubble with boundary C implies the existence of a large such bubble).

We believe there is a Bridge principle: Let M_1 and M_2 be K-surfaces in $\mathbb{R}^3$ with nonempty boundary. There is an arc α joining ∂M_1 to ∂M_2 and one can thicken α to form $\partial M_1 \# \partial M_2 = \Gamma$ so that Γ bounds a K-surface M. M is near $M_1 \cup M_2 \cup$ (the bridge along α).

The results we have discussed should also hold in hyperbolic space. We mention some of the work we have done.

Theorem 2.3 ([18]) *Let P be a horosphere in ^{3} and $\Omega \subset P$ a compact domain with smooth boundary. We parametrize ^{3} by $(x,t), x \in P$ and t the hyperbolic distance along the geodesic through x, orthogonal to P. If C is a graph over $\partial\Omega$ (in the (x,t) coordinate system) then C bounds a K-surface M for some $K > -1$, and M is a graph over Ω. Moreover, for any $K \in (-1,0)$, there exists a K-surface M (a graph over Ω) such that $\partial M = \partial\Omega$.*

It is natural to ask what curves at infinity of ^{3} are the asymptotic boundary of K-surfaces. Consider the unit ball model of ^{3} with the sphere at infinity the boundary of the ball. Assume P is the equatorial plane and the (x,t) coordinate system is the latitude-longitude system on S_∞. We prove:

Theorem 2.4 ([18]) *Let $C \subset S_\infty$ be a smooth curve which is a graph over the equator of S_∞. Then there is a K-surface M, for some $K \in (-1,0)$, with asymptotic boundary C, and M is a graph over P. Moreover, any K-surface with asymptotic boundary $C, K > -1$, and embedded, is a graph over P. For a given $K > -1$, there are at most two such M.*

Francois Labourie has done much interesting work on K-surfaces in hyperbolic manifolds [14, 15]. One of his results provides many complete K-surfaces in ^{3}, for each $K \in (-1,0)$:

Theorem 2.5 *Each end of a hyperbolic 3-manifold is foliated by a family of K-surfaces, K varying from -1 to 0.*

3 The Equations of Variation of Reilly

Let M be an oriented hypersurface of a space N^{m+1} of constant sectional curvature c; $c = -1, 0$, or 1. Let n be a unit vector field normal to M in N and let $Y = fn$ be a vector field along M for some smooth function f on M. We think of Y as a variation vector field of M in N and let $\psi_t\colon M \to N$ satisfy $\psi_0(x) = x$ for $x \in M$ and $\frac{d\psi_t(x)}{dt}\Big|_{t=0} = Y(x)$, for all $x \in M$. When $\partial M \neq \emptyset$, we assume f and ∇f vanish on ∂M.

Let $A(x)$ be the endomorphism of the tangent space $T_x(M)$ defined by the second fundamental form of M in N and for each r, $0 \leq r \leq m$, let $S_r(x)$ be the r'th symmetric function of curvature of M; $S_r = \begin{pmatrix} n \\ r \end{pmatrix} H_r$, H_r the r'th mean curvature of M.

Robert Reilly has derived the variation formulas for integrals of the form:

$$\int_M f(S_1, \ldots, S_m, h, Q)dV\,,$$

where h is the support function and $2Q$ is the square of the length of the position vector ($c = 0$) [17]. In particular:

$$\frac{d}{dt}\int_M S_r dV = \int_M f\left\{-(r+1)S_{r+1} + c(m-r+1)S_{r-1}\right\}dV = \delta_Y\left(\int_M S_r\right). \tag{3.1}$$

The last equality is simply notation. Now let $c = 0$ and let $h(x) =< x, n(x) >$ be the support function of $M \subset {}^{m+1}$. It is well known:

$$\delta_Y\left(\int_M h\right) = \frac{1}{m+1}\int_M f\,. \tag{3.2}$$

Thus for any real number a, we have:

$$\delta_Y\left(\int_M (S_r + ah)\right) = \int_M f\left\{-(r+1)S_{r+1} + \frac{a}{m+1}\right\}. \tag{3.3}$$

Hence the critical points of the functional $M \mapsto \int_M (S_r + ah)dV$ are those manifolds M for which S_{r+1} is the constant $\frac{a}{(r+1)\,(m+1)}$. We remind the reader that we assumed the variation was normal to M and C^1 fixed on ∂M.

In Reilly's calculations appears an operator "$T_r^{ij}\lambda_{,ij}$" which is fundamental to our problem. We will introduce and derive the equations of this operator in a form suitable to our problem. We refer the reader to [24] and [25], where calculations of this nature appear.

We now assume $Y = fn$ is a normal variation of $M^m \subset N^{m+1}(c) = N$ and $M(t)$ the hypersurfaces arising from this variation, $M = M(0)$. We make no hypothesis on $f/\partial M$. Let $A(t)$ be the second fundamental form of $M(t)$. Then one has the known equation:

$$\frac{d}{dt}\Big|_{t=0} A(t) = \overset{\bullet}{A}(0) = D^2 f + cfI + fA^2\,. \tag{3.4}$$

For the readers convenience, we now indicate how 3.4 is derived. Let $\psi_*\colon M \times \quad \to N$ be the variation of M, $\frac{\partial}{\partial t} \mapsto fn$ at $t = 0$, and for $u, v \in T_x(M)$ we abbreviate the metric on $M(t) < \psi_{t_*}(x), \psi_{t_*}(v) >=< u, v >$, and $II(t)(u,v) = - < \nabla_u n, v >=< A_t(u), v >$. Then

$$\begin{aligned}
-\overset{\bullet}{II}(t) &=< \nabla_{\frac{\partial}{\partial t}}\nabla_u n, v > + < \nabla_u n, \nabla_{\frac{\partial}{\partial t}} v > \\
&=< \nabla_u \nabla_{\frac{\partial}{\partial t}} n, v > - < R\left(\frac{\partial}{\partial t}, u\right)n, v > + < \nabla_u n, \nabla_v\left(\frac{\partial}{\partial t}\right) >,
\end{aligned}$$

the latter part of the equation holds since $\frac{\partial}{\partial t}$ and v commute. Now $<\nabla_{\frac{\partial}{\partial t}} n, n>= 0$ so $<\nabla_{\frac{\partial}{\partial t}} n, u> = -<n, \nabla_{\frac{\partial}{\partial t}} u> = -<n, \nabla_u(fn)> = -df(u)$, hence $\nabla_{\frac{\partial}{\partial t}} n = -$ grad $f = -\nabla f$. Thus

$$
\begin{aligned}
-\overset{\bullet}{II}(t) &= -<\nabla_u \nabla f, v> - f<R(n,u)n, v> + <\nabla_u n, \nabla_v(fn)> \\
&= -D^2 f(u,v) - fc<u,v> + f<A_t^2(u), v>
\end{aligned}
$$

since $<\nabla_u n, df(v)n>= 0$.

Now $II(t) = <A_t(u), v>$ so

$$
\begin{aligned}
\overset{\bullet}{II}(t) &= <\nabla_{\frac{\partial}{\partial t}}(A_t(u)), v> + <A_t(u), \nabla_{\frac{\partial}{\partial t}} v> \\
&= <\overset{\bullet}{A}_t(u), v> + <A_t(\nabla_{\frac{\partial}{\partial t}} u), v> + <A_t(u), \nabla_{\frac{\partial}{\partial t}} v> \\
&= <\overset{\bullet}{A}_t(u), v> - 2f<A_t^2(u), v> \ .
\end{aligned}
$$

This together with the last equation for $\overset{\bullet}{II}(t)$ yields 3.4.

4 The Linearized Equations and their Jacobi Fields

We now assume $M^m \subset N^{m+1}(c)$ and $M(t)$ is a normal variation of M, as in the previous paragraph, with $Y = fn$ and no assumption on the boundary values of f.

We have $S_m(t) = \det(A(t))$ and $S_1(t) = trace(A(t))$. Differentiating these equations at $t = 0$ we obtain:

$$
\begin{aligned}
\overset{\bullet}{S}_m(0) &= \det(A) \cdot trace(\overset{\bullet}{A}(0) A^{-1}) \\
\overset{\bullet}{S}_1(0) &= trace(\overset{\bullet}{A}(0)) \,.
\end{aligned}
$$

Now using equation 3.4, we have

$$\overset{\bullet}{S}_m(0) = trace(|A| D^2 f \circ A^{-1}) + fcS_{m-1} + f|A|S_1 \tag{4.1a}$$

$$\overset{\bullet}{S}_1(0) = \Delta(f) + mcf + \| A \|^2 f \,. \tag{4.1b}$$

Here $|A| = \det A$ and $\|A\|^2 = tr(A^2)$ is the square of the length of the second fundamental form. When $m = 2$, $K = \det A + c$, is the intrinsic curvature of M.

In general, to calculate $\overset{\bullet}{S}_{r+1}(0)$, one introduces the r'th Newton transformation: $T_r = S_r I - S_{r-1} A + \cdots + (-1)^r A^r$, or inductively, $T_r = S_r I - AT_{r-1}$, $T_0 = I$ [17].

The properties of T_r we need are [17]:

- Newton's formula: $(r+1)S_{r+1} = trace(AT_r)$
- $trace(T_r) = (m-r)S_r$
- $trace(T_r A^2) = S_1 S_{r+1} - (r+2)S_{r+2}$
- the eigenvalues of T_r are $\frac{\partial S_{r+1}}{\partial \lambda_j}$, $\lambda_j, 1 \leq j \leq m$, the eigenvalues of A
- $\overset{\bullet}{S}_{r+1}(0) = trace(\overset{\bullet}{A}(0)T_r)$

These properties are algebraic, except for the last; we will subsequently prove this last property for the reader's convenience.

Theorem 4.1 *$\overset{\bullet}{S}_{r+1}(0)(f) = L_r(f) + (c(m-r)S_r)f + (S_1S_{r+1} - (r+2)S_{r+2})f$ where $L_r(f) = div(T_r\nabla f)$.*

Proof. Combining equation 3.4 and the properties of T_r one has:

$$\overset{\bullet}{S}_{r+1}(0) = trace(D^2 f \cdot T_r) + (c(m-r)S_r + S_1S_{r+1} - (r+2)S_{r+2})f\,.$$

Hence one needs to show $trace(D^2 f \circ T_r) = div(T_r \nabla f)$. This clearly follows from:

$$trace(u \to \nabla_{T(u)}v) = trace(u \to \nabla_u Tv) \tag{4.2}$$

for all $v \in T(M)$, $T = T_r$. Fix a point $x \in M$ and let $e_j, 1 \leq j \leq m$, be a orthonormal frame in a neighborhood of x such that $\nabla_{e_i} e_j = 0$ at x, for all i and j.

A simple calculation shows that if equation 4.2 is true for a vector field v, then it is also true for ϕv, for any smooth function ϕ. So it suffices to establish 4.2 for $v = e_j, 1 \leq j \leq m$. We do this for $v = e_1$ for notational convenience. When $v = e_1$, the left side of 4.2 vanishes at x, so we show the right side vanishes too.

We have $T_r = S_r I - T_{r-1}A$ so the right side of 4.2 vanishes provided:

$$trace(u \mapsto \nabla_u T_{r-1}Ae_1) = trace(u \mapsto \nabla_u(S_r e_1))\,. \tag{4.3}$$

The left side of this equation, we calculate assuming 4.2 has been established for $r-1$:

$$\begin{aligned}\sum_{i=1}^{m}\langle e_i, \nabla_{e_i} T_{r-1}Ae_1\rangle &= \sum_{i=1}^{m}\langle e_i, T_{r-1}\nabla_{e_i}Ae_1\rangle \\ = \sum_{i=1}^{m}\langle e_i, T_{r-1}\nabla_{e_1}Ae_i\rangle &= trace(T_{r-1}\nabla_{e_1}A)\,.\end{aligned}$$

(Here we used Codazzi and $\nabla_{e_i}e_j(x) = 0$.) The right side of 4.3 is:

$$\sum_{i=1}^{m}\langle e_i, \nabla_{e_i}(S_r e_1)\rangle = \sum_{i=1}^{m}\langle e_i, \nabla_{e_i}(S_r)e_1\rangle = \nabla_{e_1}(S_r)\,.$$

Hence it remains to show:

$$trace(T_{r-1}\nabla_{e_1}A) = \nabla_{e_1}(S_r)\,, \tag{4.4}$$

(notice that this proof also yields: $\dot{S}_{r+1}(0) = trace(\dot{A}_r(0)T_r)$.)

We prove 4.4 by calculating in a basis that diagonalizes A, $A = diag(\lambda_1, \cdots, \lambda_m)$. Then

$$\begin{aligned} T_{r-1} &= diag(t_1, \cdots, t_m), \text{ where} \\ t_i &= \sum \lambda_{i_1} \cdots \lambda_{i_{r-1}} \\ & \quad i_1 < i_2 < \cdots < i_{r-1} \\ & \quad i_j \neq i\,. \end{aligned}$$

We have

$$trace(T_{r-1}\nabla_e A) = \sum_i t_i \nabla_e \lambda_i = \nabla_e S_r\,.$$

This completes the proof of 4.2 (we leave to the reader to verify that one has 4.2 valid for $r = 1$, i.e. 4.3 holds for $r = 1$).

The point of view we wish to pursue is that the equations 4.1 (r varying) are of the same nature in many situations.

For example, for surfaces in ^{3} the equations become:

a) $\dot{K}(0) = L(f) + 2KHf, L = L_1$,

b) $\dot{H}(0) = \Delta(f) + (4H^2 - 2K)f$

When the eigenvalues of A are positive, i.e., M is strictly locally convex, a) is an elliptic equation and the operator L is a self adjoint Fredholm operator of index 0; just as the equation b).

Similarly, for surfaces in S^3 or ^{3}, L is elliptic provided the eigenvalues of A are positive.

Therefore, theorems and problems concerning the mean curvature of surfaces and using the equation 4.1b), have their analogous statement for K-surfaces, using 4.1a).

A Jacobi field of (an H or K) surface is a normal variation fn of $M, f = 0$ on ∂M, which is a solution of the equation

$$\begin{array}{ll} L(f) + 2KHf = 0, & (K\text{-surface}) \\ \Delta(f) + (2c + \|A\|^2)f = 0, & (H\text{-surface}). \end{array}$$

In the elliptique case, M compact, the dimension of the vector space of Jacobi fields is finite. A natural source of Jacobi fields comes from the vector fields of the ambient space.

For any (not necessarily normal) vector field Y on N, the normal component of $Y, < Y, n > Y$, changes the curvature (H or K) of M the same way Y changes the curvature of M, when the curvature function of M is constant. Since in this case, the tangent component of Y leaves the curvature of M fixed. Hence the function $\phi =< Y, n >$ satisfies the equation 4.1a) if M is a K-surface and equation 4.1b), if M is an H-surface.

For example if a is a fixed point of $\mathbb{R}^3$ and Y is the constant vector field a, then Y does not change curvature, hence the function $\phi =< a, n >$ satisfies:

$$\begin{array}{ll} L(\phi) + 2HK\phi = 0, & \text{if } M \text{ is a } K\text{-surface}\,, \\ \Delta\phi + \|A\|^2\phi = 0, & \text{if } M \text{ is an } H\text{-surface}\,. \end{array}$$

Locating the components of M where ϕ is zero provides Jacobi fields on these components.

Taking a over a basis of $\mathbb{R}^3$ we can write these equations as vector equations for n:

$$\begin{array}{ll} L(n) + 2HKn = 0 & (K\text{-surface}) \\ \Delta(n) + \|A\|^2 n = 0 & (H\text{-surface})\,. \end{array}$$

Let Y be the vector field arising from homothety: $Y(x) = x$, $x \in \mathbb{R}^3$. Then $\psi_t(x) = (1+t)x$, and we know how Y changes curvature: $H(t) = \frac{H}{1+t}$, $K(t) = \frac{K}{(1+t)^2}$, hence the function (support) $\phi =< Y, n >=< x, n >$ satisfies:

$$L(\phi) + 2HK\phi = -2K(=\overset{\bullet}{K}(0)) \text{ for a } K\text{-surface}\,,$$

$$\Delta(\phi) + \|A\|^2\phi = -H \text{ for a } H\text{-surface}\,.$$

For hypersurfaces of $\mathbb{R}^{m+1}$, with S_{r+1} constant, the same reasoning as above, together with equation 4.1, shows the support function ϕ satisfies:

$$L_r(\phi) + (S_1 S_{r+1} - (r+2)S_{r+2})\,\phi = -(r+1)S_{r+1}\,.$$

One has analogous formulae in $N^{m+1}(c)$ which we shall derive shortly.

We remark that these formulae are well known for H-surfaces.

Of particular interest are those manifolds M for which every compact subdomain admits only zero as Jacobi field (M is then stable). In $\mathbb{R}^3$, a complete stable minimal surface is necessarily a flat plane [21]. A K-surface M in $\mathbb{R}^3, K > 0$, is compact, so if $\partial M = \emptyset$, M is a round sphere. This means the interesting K-surfaces in $\mathbb{R}^3$ are those for which $\partial M \neq \emptyset$. When is such a surface stable? For minimal surfaces M in $\mathbb{R}^3$ one has the useful stability condition of Barbosa-do Carmo: if the total curvature of M is less than 2π then M is stable [1].

In ^{3}, K-surfaces M, with $K \in (-1,0)$, are stable and the operator L is elliptic. The equation is

$$L(f) + 2HKf = 0\,.$$

Since $K = \det A - 1$, we have $\det A \in (0,1)$ so (for an appropriate choice of the normal to M) both eigenvalues of A are positive, L is elliptic and $2H = \text{trace } A > 0$. Hence HK is negative and the usual maximum principle for elliptic operators shows there are no Jacobi fields on compact domains with boundary [14]. In particular, a complete K-surface M in ^{3} with $K \in (-1,0)$ is stable.

As an application of the linearized operator L_r, we have generalized the theorem of B. White concerning the structure of the space of constant mean curvature surfaces [26].

Let $\mathcal{M}$ be the space of parametrized compact immersed hypersurfaces M in ^{n} that are locally convex and S_{r+1} is a fixed positive constant. We identify two such M if they differ by a parametrization leaving the boundary fixed. We have:

Theorem 4.2 ([19]) *$\mathcal{M}$ is a C^∞ Banach manifold modelled on $C^{j+3,\alpha}(\partial M, \,^n)$, and the projection $\pi\colon \mathcal{M} \to C^{j+3,\alpha}(\partial M, \,^n)$, $[f] \mapsto [f/\partial M]$, is a C^∞ Fredholm map of index zero. Using this we prove: generically a Jordan curve on the boundary S of a convex body B, bounds at most a finite number of K-surfaces in B, for K fixed, $0 < K < \inf(K(S))$.*

5 Some Formulas for L in $\mathbf{R}^{m+1}$ and $\mathbf{H}^{m+1}$

Let $M^m \subset \,^{m+1}(c)$ where $N = N^{m+1}(c) = \,^{m+1}$ for $c = 0$ and $^{m+1}$ for $c = -1$. We take the Minkowski model of $^{m+1}$:

$$\{X = (x_0, x) \in \,^{m+2} \mid x \in \,^{m+1}, x_0 > 0, |x|^2 - x_0^2 = -1\}\,.$$

H^{m+1} is the "unit sphere" in $(m+2)$ dimensional Minkowski space with the Lorentz metric $-dx_0^2 + dx_1^2 + \cdots + dx_{m+1}^2$.

Then $M \subset \,^{m+1}$ can be thought of as a codimension-two submanifold of $^{1,m+1} = \,^{m+2}$. By position vector X of a point of M we mean the position vector of the point in $^{m+2}$. Let n denote a unit normal vector field to M in $^{m+1}$. Then it is easy to see that the normal bundle of M in $^{m+2}$ is generated by n and X.

We have the formulas for $L_r(X)$ and $L_r(n)$.

Theorem 5.1

$$\begin{aligned} L(X) &= -c_0(r)(H_{r+1}n - H_r X) \\ L(n) &= -(S_1 S_{r+1} - (r+2)S_{r+2})n + (r+1)S_{r+1}X\,, \end{aligned} \tag{5.1}$$

this formula for $L(n)$ assumes S_{r+1} constant. Here

$$L = L_r, c_0(r) = (m-r)\left(\begin{array}{c} m \\ r \end{array}\right).$$

Proof of Theorem 5.1. When $M \subset {}^{m+1}$ and $r = 1$, these formulae are derived by Cheng and Yau [6], once one identifies their operator

$$\Box f = \sum_{i,j}(mH\delta_{ij} - a_{ij})f_{,ij}$$

with $L_2(f)$. Also they can be found in Reilly, for $M \subset {}^{m+1}$ and S^{m+1}.

We derive the formulae in ${}^{m+1}$.

Fix a point $p \in M$ and let $e_1, \cdots, e_m$ be an orthonormal frame tangent to M in a neighborhood of p, chosen so that, for all i, j, $\overline{\nabla}_{e_i}(e_j)(p)$ is normal to M in ${}^{m+2}$ ($\overline{\nabla}$ the connection of ${}^{m+2}$) and at p, the second fundamental form of M is diagonal: $< \overline{\nabla}_{e_i}(e_j), n(p) >= -\lambda_i \delta_i^j$. Then

$$\begin{aligned} L(X)(p) &= trace(u \mapsto \nabla_{T(u)}(\nabla X)) \\ &= \sum_{i=1}^{m} < e_i, T\nabla_{e_i}\nabla X >= \sum_{i=1}^{m} \alpha_i < e_i, \nabla_{e_i}\nabla X > \end{aligned}$$

(where $T = T_r = diag(\alpha_i, \cdots, \alpha_m)$)

$$\begin{aligned} &= \sum \alpha_i < e_i, \overline{\nabla}_{e_i}\left(\sum_{j=1}^{m} e_j(X)e_j\right) > \\ &= \sum_i \alpha_i < e_i, \sum_{j=1}^{m} \overline{\nabla}_{e_i}(e_j(X))e_j > \end{aligned}$$

(since the tangent part of $\overline{\nabla}_{e_i}(e_j)(p) = 0$)

$$\begin{aligned} &= \sum_{i=1}^{m} \alpha_i \overline{\nabla}_{e_i}(e_i(X)) = \sum_i \alpha_i \overline{\nabla}_{e_i}(e_i) \\ &= \sum \alpha_i < \overline{\nabla}_{e_i}(e_i), n > n - \sum \alpha_i < \overline{\nabla}_{e_i}(e_i), X > X \end{aligned}$$

(since n and X span the normal bundle and $< X, X >= -1$)

$$\begin{aligned} &= (-\sum_i \alpha_i\lambda_i)n + (\sum_i \alpha_i)X = -(trace(TA))n + trace(T)X \\ &= -(r+1)S_{r+1}n + (m-r)S_rX = -c_0(r)(H_{r+1}n - H_rX). \end{aligned}$$

The calculation of $L(n)$ is similar but one uses S_{r+1} = constant:

$$
\begin{aligned}
L(n)(p) &= \sum_i \alpha_i < e_i, \overline{\nabla}_{e_i}(\nabla n) > \\
&= \sum \alpha_i < e_i, e_i(\sum_j e_j(n) e_j) > \\
&= \sum \alpha_i \overline{\nabla}_{e_i}(e_i(n))\,.
\end{aligned}
$$

Now $\overline{\nabla}_{e_i}(n)$ is orthogonal to n and X, so

$$
\begin{aligned}
\overline{\nabla}_{e_i}(e_i(n)) &= \overline{\nabla}_{e_i}(\sum_j < e_j, \overline{\nabla}_{e_i}(n) > e_j) \\
&= \sum_j \overline{\nabla}_{e_i}(a_{ij}) e_j + a_{ij} \overline{\nabla}_{e_i} e_j \\
&= \sum_j \overline{\nabla}_{e_j}(a_{ii}) e_j + \sum_j a_{ij} \overline{\nabla}_{e_i} e_j \quad \text{(Codazzi)}\,.
\end{aligned}
$$

We observe the first term will not contribute to $L(n)$; for e any vector, we have:

$$
\sum_i \alpha_i \nabla_e(a_{ii}) = trace(T \nabla_e A) = \nabla_e(S_{r+1}) \text{ by 4.4}\,.
$$

Since S_{r+1} is constant, this is zero.

$$
\begin{aligned}
L(n)(p) &= \sum_i \alpha_i \lambda_i (\overline{\nabla}_{e_i}(e_i)) \\
&= \sum_i \alpha_i \lambda_i (-\lambda_i n + X) = -trace(TA^2) n + trace(TA) X \\
&= -(S_1 S_{r+1} - (r+2) S_{r+2}) n + (r+1) S_{r+1} X\,.
\end{aligned}
$$

Remark 5.1 *Let $M \subset S^{m+1} \subset {}^{m+2}$ be a codimension one submanifold of the unit sphere S^{m+1} and let X be the position vector of M in ${}^{m+2}$, n the normal vector of M in S^{m+1}. The above calculation of $L_r(X)$ and $L_r(n)$ is exactly the same here, except when one expresses $\overline{\nabla}_{e_i}(e_i)$ in terms of X and n, the sign of the coefficient of X changes since $\langle X, X \rangle = 1$; i.e.*

$$
\overline{\nabla}_{e_i}(e_i) = \langle \overline{\nabla}_{e_i}(e_i), X \rangle X + \langle \overline{\nabla}_{e_i}(e_i), n \rangle n\,.
$$

Thus the formulae for $L(X)$ and $L(n)$ are the same as in ${}^{m+1}$ except the coefficients of X change sign. We can now write these equations for $M^m \subset {}^{m+1}, S^{m+1}$ or ${}^{m+1}(c = 0, 1, \text{ or} -1)$ in the form:

$$
\begin{aligned}
L_r(X) &= -((r+1) S_{r+1}) n - c(m-r) S_r X \\
&= -c_0(r)(H_{r+1} n + c H_r X) \qquad (5.2) \\
L_r(n) &= -(S_1 S_{r+1} - (r+2) S_{r+2}) n - c(r+1) S_{r+1} X\,.
\end{aligned}
$$

The latter equation assumes S_{r+1} is constant.

Notice that when M is a closed hypersurface, the fact that L_r is a divergence operator (and Stoke's theorem) gives the Minkowski formulae in ${}^{m+1}$, S^{m+1} and ${}^{m+1}$:

$$\int_M (H_{r+1}n + cH_rX) = 0\,. \tag{5.3}$$

□

6 Height Estimates

Heinz obtained an estimate for the maximum height a graph of constant mean curvature in ${}^{m+1}$ can obtain, provided the graph is defined on a compact domain of ^{m} and has zero boundary values. The answer is $\frac{m}{S_1} = \frac{1}{H}$ and a hemisphere shows this is the best possible. The proof of this is a simple application of the maximum principle and equations 5.2 for $r = 0$.

Now if M is any compact embedded hypersurface in ${}^{m+1}$, of constant mean curvature, whose boundary is in ${}^{m} = {}^{m} \times 0$, then the maximum height M can rise above ^{m} is $\frac{2}{H}$. This follows from Heinz's estimate and the Alexandrov reflection principle applied to M, using horizontal hyperplanes coming down from the hightest point of M: the part of M above such a plane, must remain a graph, at least until the hyperplane is halfway down to the hyperplane ^{m}.

This result is generalized to constant mean curvature hypersurfaces in ${}^{m+1}$ in [13].

We shall now obtain height estimates for other curvature functions.

First, let M be a hypersurface in ${}^{m+1}$ with S_{r+1} a positive constant. M a graph over a compact domain in ^{m}, $\partial M \subset {}^{m}$. M is the graph of the function x_{m+1} and the normal n to M is chosen so that $n_{m+1} \geq 0$.

It is clear there is at least one interior point of M where M is strictly locally convex (englobe M with spheres of large curvature until such a sphere touches M on one side at an interior point). Just as it is proved in [12], it follows that L_r is elliptic at every point of M and S_j is positive for $1 \leq j \leq r$, at each point of M.

More precisely, let $f(\lambda_1, \cdots, \lambda_m) = S_r^{1/r}(\lambda_1, \cdots, \lambda_m)$. Let Γ be the connected component in ^{m}, containing $1 = (1, \ldots, 1)$, where $f > 0$. It is proved in [[5], pp. 269-270] that $\frac{\partial S_r}{\partial \lambda_i} > 0$ on Γ for $1 \leq i \leq m$. Hence if S_r is positive on M and $p \in M$, is strictly convex, so that $\lambda(p) \in \Gamma$, ($\lambda(p)$ the m principal curvatures of M at p), and if $q \in M$ is any point that can be joined to p by a path on M, then $q \in \Gamma$, and $\frac{\partial S_r}{\partial \lambda_i}(q) > 0$ for all i. This means L_{r-1} is elliptic on M.

We remark that $S_2 > 0$ is always elliptic on any hypersurface in a space form. $S_1^2 = \sum_{j=1}^{m} \lambda_j^2 + 2S_2 > \lambda_j$, for each j. Hence $S_1 - \lambda_j = \frac{\partial S_2}{\partial \lambda_j} > 0$, and the eigenvalues of T_1 are positive.

Let H_i be the i'th mean curvature function of M: $S_i = \begin{pmatrix} m \\ i \end{pmatrix} H_i$. It is always true that

$$H_{i-1}H_{i+1} \leq H_i^2 \qquad (1 \leq i < m)$$

and

$$H_1 \geq H_2^{1/2} \geq H_3^{1/3} \geq \cdots \geq H_i^{1/i},$$

provided $H_1, H_2, \cdots, H_i$ are nonnegative, [page 52 of [9]].

On $M, H_i > 0$ for $i \leq r+1$, so the above inequalities yield:

$$(*) \qquad (m-i-1)S_1S_{i+1} - m(i+2)S_{i+2} \geq 0 \qquad (H_1H_{i+1} \geq H_{i+2}),$$

for $i \leq r-1$. Morever, at points of M where $S_{r+2} \geq 0$, the inequality is valid for $i = r$ as well and where $S_{r+2} < 0$, it holds, so $(*)$ is valid for $i = r$.

Let $a > 0$ be a constant to be chosen later. Define $f = ax_{m+1} - n_{m+1}$. We calculate $L_r(f)$ with the equations 5.2:

$$L_r(f) = (S_{r+1}(S_1 - a(r+1)) - (r+2)S_{r+2})n_{m+1}.$$

We wish to choose a so that $L_r(f) \geq 0$ on M. This will yield a height estimate since on ∂M, $f \leq 0$, and L_r is elliptic so by the maximum principle, $f \leq 0$ on M, and $ax_{m+1} \leq n_{m+1} \leq 1$ so $x_{m+1} \leq \frac{1}{a}$.

We know that:

$$(*) \qquad (m-r-1)S_1S_{r+1} - m(r+2)S_{r+2} \geq 0.$$

Add and substract $(\frac{m-r-1}{m})S_1S_{r+1}$ to the coefficient of n_{m+1} in $L_r(f)$ and use $(*)$ to obtain:

$$\begin{aligned} L_r(f) &\geq S_{r+1}\left[(S_1 - a(r+1)) - \left(\frac{m-(r+1)}{m}\right)S_1\right]n_{m+1} \\ &= S_{r+1}\left[S_1\left(\frac{r+1}{m}\right) - a(r+1)\right]n_{m+1}. \end{aligned}$$

Hence $L_r(f) \geq 0$ when $a \leq \frac{S_1}{m}$. Since $H_1 \geq H_{r+1}^{1/r+1}$, one can choose $a = (\frac{S_{r+1}}{c_{r+1}})^{1/r+1}$, $c_i = \begin{pmatrix} m \\ i \end{pmatrix}$.

For $r = 0$, $a = S_1/m$, $L_0 = \Delta$ is elliptic and this is Heinz's estimate. Thus we have proved:

Theorem 6.1 *Let $M \subset {}^{m+1}$ be a compact embedded hypersurface with $\partial M \subset {}^{m} = {}^{m} \times 0$. If S_{r+1} is a positive constant on M, then the maximal distance of M to the hyperplane ^{m} is $2\left(\frac{c_{r+1}}{S_{r+1}}\right)^{1/r+1}$.*

Remark 6.1 *The above height estimate gives an estimate (the same) for the maximum distance of points of M to the convex hull of its boundary. Here M is a compact embedded hypersurface in ${}^{m+1}$ with S_{r+1} a positive constant. Again, to prove this, one does Alexandrov reflection with planes coming from the furthest point q of M to the boundary of its convex hull, parallel to the tangent plane at q.*

In ${}^{m+1}$, one can also obtain height estimates for compact embedded hypersurfaces with boundary in a totally geodesic hyperplane, having one curvature function S_{r+1} a positive constant. Let M be a graph in the Minkowski model, of the function x_{m+1}, with $\partial M \subset \{x_{m+1} = 0\}$ and the normal oriented so that $n_{m+1} \geq 0$ on M.

Let $f = ax_{m+1} - n_{m+1}$ as before, and calculate $L_r(f)$. The coefficient of n_{m+1} is the same as in the Euclidean calculation so it is nonnegative for $a \leq \frac{S_1}{m}$; in particular, for $a = \left(\frac{S_{r+1}}{c_{r+1}}\right)^{1/r+1}$.

The coefficient of X is nonnegative provided:

$$(r+1)S_{r+1} \leq a(m-r)S_r\,. \tag{6.1}$$

Using the inequality: $H_{r+1}^{r/r+1} \leq H_r$, one verifies that 6.1 is satisfied for $a = \left(\frac{S_{r+1}}{c_{r+1}}\right)^{1/r+1}$.

Now we can state:

Theorem 6.2 *Let M be a connected compact embedded hypersurface in ${}^{m+1}$, with ∂M contained in a totally geodesic hyperplane P. Assume S_{r+1} of M is a positive constant and let $a = \left(\frac{S_{r+1}}{c_{r+1}}\right)^{1/r+1}$. Then the maximum distance between a point of M and P is 2 arctanh $\left(\frac{1}{a}\right)$. When ∂M is not necessarily in a hyperplane, the estimate applies to the maximum distance between points of M and the convex hull of ∂M.*

Proof. Let $p \in M$ be a point of maximal distance from P, and let γ be the unit speed geodesic with $\gamma(0) \in P$ and $\gamma(d) = p; \gamma$ is orthogonal to P at $\gamma(0)$. Let Y be the Killing field generated by the hyperbolic isometries h_t (translations along γ) with h_0 the identity. Let $R_t\colon {}^{m+1} \to {}^{m+1}$ be hyperbolic reflection in the geodesic hyperplane $P_t = h_t(P)$ (we remark that this Alexandrov reflection part of the proof is done in [13] where the theorem is proved for constant mean curvature. We repeat it here for the reader's convenience).

Let $M_t = \cup_{t' \geq t}(P_{t'} \cap M)$. The Alexandrov reflection principle implies that for $t \geq d/2, R_t(M_t) \cap M = \partial M_t$ and the intersection is transverse. Hence $\widetilde{M} = h_{\frac{-d}{2}}(M_{\frac{d}{2}})$ is a graph with respect to the Killing coordinate t over a domain $\Omega \subset P$.

Notice that $\widetilde{M}$ is also a graph over Ω with respect to the distance function s from P, since, by the previous discussion, if β is any geodesic orthogonal to P then $\widetilde{M}$ is a graph over Ω with respect to the Killing coordinate t defined by β: Along β, t coincides with s.

It remains to show $s \leq \operatorname{arctanh}(\frac{1}{a})$.

Let P be the geodesic hyperplane $x_{m+1} = 0$, in the Minkowski model of $\;^{m+1}$. Then the distance s from P is $s = \operatorname{arcsinh}(x_{m+1})$. $\widetilde{M}$ is a graph with respect to x_{m+1} and $x_{m+1} = 0$ on $\partial\widetilde{M}$. On $\widetilde{M}$

$$n_{m+1} \geq 0 \text{ and } n_{m+1} \leq \sqrt{1 + x_{m+1}^2} = \cosh s\,.$$

This last inequality follows from:

$$\nabla x_{m+1} = e_{m+1} + x_{m+1}X - n_{m+1}n, \qquad X \cdot X = -1\,,$$

so

$$|\nabla x_{m+1}|^2 = 1 + x_{m+1}^2 - n_{m+1}^2\,.$$

Now $\sinh(s) = x_{m+1} \leq \frac{n_{m+1}}{a} \leq \frac{\cosh(s)}{a}$ which proves Theorem 6.2. $\square$

7 Some Applications

The height estimates enable us to simplify the proof of a theorem of Hartman [10]. We have:

Theorem 7.1 *Let M be a complete embedded hypersurface in $\;^{m+1}$ with nonnegative sectional curvature. If some curvature S_{r+1} of M is a positive constant then $\;^{m+1}$ can be expressed as a metric product $\;^{k+1} \times \;^{\ell}$ and M is isometric to $S^k \times \;^{\ell}, S^k$ a round sphere in $\;^{k+1}$. If one assumes M is complete and strictly convex and S_{r+1} is a positive constant, then M is compact hence a round sphere.*

Proof.

If at each point of M, some principal curvature were zero, then by Sacksteder, we could write M as a metric product: $M_1 \times \;$, and we then would consider M_1 [20]. So we can assume M is strictly convex at some point p. After a Euclidean motion of M, we can suppose p is the origin of $\;^{m+1}$, $n(p) = e_{m+1} = (0, \cdots, 0, 1)$, and locally, near

p, M is the graph of a nonnegative strictly convex function defined in a neighborhood of p in the hyperplane $P = \mathbb{R}^m \times 0$.

Let $P(t) = \mathbb{R}^m \times (t)$ for $t > 0$. We claim that $M \cap P(t)$ is compact for all $t > 0$ and $M \cap P(T)$ is empty for T large. Hence M is compact and a round sphere by the theorem of Ros-Montiel [16] (one does Alexandrov reflection starting with any plane).

By strict local convexity at P, the component $S(t)$ of $M \cap P(t)$, that is near p, is compact for $t > 0$, t small. Also $S(t)$ is diffeomorphic to a sphere S^{m-1}, for $t = t_0$ a small positive number. Let $T > t_0$.

For each $x \in S(t_0)$, let $A(x)$ be the intersection of the tangent space of M at x with $P(T)$. $A(x)$ is a codimension one affine subspace of $P(T)$. The envelope $E(T)$ of these subspaces, as x varies in $S(t_0)$, is a compact codimension one submanifold of $P(T)$. $E(T)$ bounds a domain Ω in $P(T)$ and by convexity of M, we have $S(T) = M \cap P(T) \subset \Omega$. Hence $S(T)$ is compact.

By the height estimates $S(T)$ must be empty for T large and since M is in the "cone" E, we have M compact. So the theorem of Ros-Montiel applies. $\square$

7.1 A balancing formula

Let $M \subset \mathbb{R}^{m+1}$ be a compact hypersurface with some curvature function S_{r+1} a positive constant. Let D be a compact hypersurface with $\partial M = \partial D$ and assume $M \cup D$ is an oriented m-cycle of $\mathbb{R}^{m+1}$, M oriented by its mean curvature vector. Let n_D be the normal that orients D (so that $M \cup D$ is an oriented cycle when M is oriented by its mean curvature vector). Then for any constant vector field Y on $\mathbb{R}^{m+1}$, we have:

Theorem 7.2 (Balancing Formula)

$$\int_D < Y, n_D >= \frac{1}{(r+1)S_{r+1}} \int_{\partial M} < Y, T_r(\nu) > \ .$$

Here ν is the inner pointing conormal to M along ∂M.

Proof. Let $a = -(r+1)S_{r+1}, L = L_r, T = T_r$, so that $L(X) = -an, n$ the "outer" pointing normal to M, i.e., the opposed orientation of the mean curvature vector. Since L is a divergence operator:

$$\begin{aligned}\int_M < Y, L(X) > &= \int_M L(X \cdot Y) = -\int_{\partial M} < T(\nabla(X \cdot Y)), \nu > \\ &= -\int_{\partial M} < \nabla(X \cdot Y),\, T(\nu) >= -\int_{\partial M} < Y, T(\nu) >,\end{aligned}$$

(we used the fact that T is symmetric).

The flux of any constant vector field going into $M \cup D$ along D equals the flux of the field going out of M. It follows that $\int_D n_D = \int_M n$. Combining this equation with $L(X) = -an$ we obtain:

$$\int_D < Y, n_D > = \int_M < Y, n >= -\frac{1}{a} \int_M < Y, L(X) >$$
$$= \frac{1}{a} \int_{\partial M} < Y, T(\nu) > \ .$$

As an application of this formula we generalize Theorem 1 of [7].

Theorem 7.3 *Let $S \subset {}^{m} = {}^{m} \times 0 \subset {}^{m+1}$ be a strictly convex hypersurface and $M \subset {}^{m+1}$ a compact embedded hypersurface with $\partial M = S$. Assume some curvature function S_{r+1} of M is a positive constant and M is transverse to $P = {}^{m}$ along S. Then M is contained in one of the half-spaces of ${}^{m+1}$ determined by P and M has all the symmetries of S. In particular, if S is a round sphere then M is part of a round sphere. If $r = m - 1$ then one does not need to assume M is transverse to P along S.*

Proof. The proof uses the ideas in [7]. The hypothesis $\partial M \subset P$ a convex hypersurface implies either $M \subset P$ or there is some point of M that is strictly convex. One sees this coming down (and up) to P with very flat spheres (i.e., small curvature); a first point of contact with the interior of M is strictly convex. If there is no such first point then $M \subset P$ which contradicts $S_{r+1} > 0$. Thus the operator $L = L_r$ is elliptic on M and S_1 never vanishes on M. Also ellipticity of L means $\frac{\partial S_{r+1}}{\partial \kappa_k}$ is positive at each point of M and for each $j, 1 \leq j \leq m$; these numbers are the eigenvalues of $T = T_r$. In particular, for each nonzero tangent vector v to M, we have $< v, T(v) >> 0$. Also, Alexandrov reflection applies to M so once M is contained in one of the half-spaces determined by P, one does Alexandrov reflection with vertical planes to show that M inherits the symmetries of ∂M.

Now to prove M is indeed contained in a half space, one proceeds exactly as in the proof of Theorem 1 of [7]. We do not give the details here; we will briefly sketch the proof.

Apply the balancing formula to M and Y the unit vertical field to show that $M \cap (P - \partial P)$ cannot be entirely contained in the domain Ω of P bounded by ∂M. This uses our above remark: $< v, T(v) >> 0$ for every nonzero tangent vector v to M.

Next one observes that if A is a component of $F = M \cap (P - \overline{\Omega})$, then A cannot bound in F. If so, one uses vertical planes coming from infinity which sweep out A before meeting $\partial M(\partial M$ is convex so this is possible). Alexandrov reflection yields a position of such a plane which is a symmetry plane of M, yet the plane is disjoint from ∂M; this is impossible.

Thus, a component A of F is homologous to ∂M in F and again by Alexandrov reflection there is at most one such component.

Finally, the balancing formula shows no such component A exists.

The eigenvalues of T_{m-1} are $\kappa_1 \cdots \widehat{\kappa}_i \cdots \kappa_m$, $1 \leq i \leq m$, κ_i the eigenvalues of A. Since there is one point where the κ_i are all positive, it follows that they are all positive at every point when S_m is a positive constant. Hence M is strictly convex at each point.

Now come towards P from above with horizontal hyperplanes $P(t)$. $P(t) \cap M$ is strictly convex for all t hence M is contained in one halfspace and topologically a disc. $\square$

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