Abstract

We describe first the analytic structure of Riemann's examples of singly-periodic minimal surfaces; we also characterize them as extensions of minimal annuli bounded by parallel straight lines between parallel planes. We then prove their uniqueness as solutions of the perturbed problem of a punctured annulus, and we present standard methods for determining finite total curvature periodic minimal surfaces and solving the period problems.

Résumé

Nous exposons d'abord la structure complexe de la famille de surfaces minimales simplement périodiques découverte par Riemann; elles sont caractérisées comme extensions analytiques des anneaux minimaux bordés par deux droites parallèles dans deux plans parallèles. Nous montrons alors leur unicité en tant que solutions du problème généralisé aux anneaux épointés. Nous présenterons ce faisant les méthodes usuelles de détermination des surfaces minimales simplement périodiques de courbure totale finie, et d'élimination des périodes.

A Rigidity Theorem for Riemann's minimal surfaces

Pascal Romon

Department of Mathematics
University of Massachusetts
Amherst, MA 01003

A RIGIDITY THEOREM
FOR RIEMANN'S MINIMAL SURFACES

1 Introduction

We are interested here in properly embedded minimal surfaces in $\mathbb{R}^3$, which are invariant under a translation τ. Much has been proven about complete embedded minimal surfaces in general (see [6], [7], [8] for example) and singly-periodic ones in particular ([1], [2], [5]). However the classification of examples satisfying certain geometric constraints, like fixed genus or number of ends, is still open in almost all cases.

The most famous of these surfaces are Riemann's Examples (see fig 1), which are, with the catenoid, the only minimal surfaces fibered by circles. However, while the catenoid is fibered by circles only, Riemann's Examples' circles tend periodically to parallel lines. Two successive lines bound an annulus. Using Schwarz Reflection, we can extend this annulus by rotation around these lines and their images, in order to reconstruct the complete surface $\mathcal{R}$. The translation τ induced by rotation around two successive lines leaves the surface invariant. In fact there is a one-parameter family of singly-periodic minimal surfaces. A more detailed construction of $\mathcal{R}$ will be found in section 2 and in [3].

A natural question to ask is whether there are any non trivial ϵ-deformations of Riemann's Examples (keeping some straight lines fixed), or if all minimal planar domains bounded by straight lines and lying in a slab belong in this family. The latter was proven in [3] to be true if the lines are parallel and the total curvature

Figure 1: domaine fondamental d'un exemple de Riemann avec ses lignes de niveau, qui sont des cercles.

of the domain is 4π [1]. We will now generalize that result to the following

Theorem 1 : *If A is a properly embedded minimal annulus, punctured in a single point, that lies between two parallel planes, and is bounded by two parallel lines in these planes, then A is a piece of a Riemann Example.*

Equivalently, we can state a result about minimal surfaces properly embedded in $\mathbb{R}^3$ modulo a translation, since they are in 1-1 correspondance with singly-periodic embedded minimal surfaces.

Theorem 2 : *Any complete minimal torus embedded in some $\mathbb{R}^3/\tau\mathbb{Z}$ with planar ends and two lines, and total curvature 16π or less, is a covering of a Riemann Example.*

Proof of Theorem 2:

Let M be such a surface. After a rotation in $\mathbb{R}^3$, if necessary, we may assume that its ends are all horizontal. The number of ends is even because M is embedded and thus orientable (see [3]). Using the Callahan-Hoffman-Meeks formula (see [2])

$$\int_M |K|\,dA = 4\pi(r+\gamma-1)$$

where γ is the genus of M and r the number of ends, we deduce the following: if the total curvature of M is 8π, there are only two ends, which obviously correspond to each of the two lines. Furthermore, the degree of the Gauss map g is 2, by the Gauss-Bonnet Theorem. If, on the other hand, the total curvature is 16π, then there are four ends and the degree of the Gauss map g is 4. Since the ends are planar, the Gauss map is branched at each one of them, and takes the values 0 and ∞ alternatively. Thus their branching order is exactly 1.

Since genus(M)=1, any fundamental domain is planar, and so is its intersection with a horizontal slab. Namely, if there were topology in any subdomain, say genus γ, then the genus if M would be at least $\gamma+1$. So if $\mathcal{P}$ is a limit plane for an end, its intersection with M must be a simple infinite curve. The genus 1 assumption forbids any closed loop, and if there were another curve going

[1] It was also proven by Toubiana [11] and then by Pérez and Ros [9] that the lines have to be parallel, provided they lie in parallel planes.

to infinity, then the branching order of g would be strictly greater than 1, which is excluded. Recall that the order of the Gauss map at a finite point equals the number of level curves crossing at that point, minus 1, and at a planar end, the order is equal to the number of curves going to that end, minus 2. Consider now the two horizontal lines: rotation around them generates the whole surface, so that they bound a fundamental piece of the surface, lying between two horizontal planes. Furthermore, it is a planar domain, punctured at most once. So we are back to Theorem 1.

I also want to thank Harold Rosenberg, David Hoffman and Fu Sheng Wei for their support and fruitful discussions.

2 Riemann's example

2.1 Some notations

We will first recall briefly the construction of Riemann's example. Let $\mathbb{T}$ denote the rectangular torus whose vertices are $(0, \mu, \mu + i, i)$ in the complex plane ($\mu \in \mathbb{R}_+$). We can associate to this torus the lattice generated by $(2\omega_1, 2\omega_3)$, where $(\omega_1, \omega_2, \omega_3)$ are defined as $(\frac{\mu}{2}, \frac{\mu+i}{2}, \frac{i}{2})$ as usual. Let P be the unique elliptic function having a double pole at 0, a double zero at ω_2 (and no other pole or zero), and two other branch points at ω_1 and ω_3, whose respective values are $\lambda, -1/\lambda$, where λ is a positive constant characterizing the lattice. Then P has the following properties of symmetry: $P(-z) = P(z)$ and $P(\bar{z}) = \overline{P(z)}$. Furthermore it satisfies the functional equation: $P'(z)^2 = C_\lambda P(P - \lambda)(P + 1/\lambda)$, for some positive real C_λ depending on the lattice. The function P will allow us to define the Weierstrass-Enneper representation of Riemann'example on this torus. For more details about this function, see [3].

2.2 The Weierstrass-Enneper representation

We define a minimal immersion $R : \mathbb{T} \to \mathbb{R}^3$ up to translation by:

$$R(z) = Re \int^z (1 - g^2, i(1 + g^2), 2g)\eta$$

where $g = P$ is a degree two meromorphic map from $\mathbb{T}$ to S^2 corresponding to the stereographic projection of the Gauss map of the surface, and $\eta = idz/g$ is a holomorphic 1-form defined on $\mathbb{T}$ minus the ends. The ends will naturally be 0 and ω_2, and then we define τ to be the translation of vector

$$Re \oint (1 - g^2, i(1 + g^2), 2g)\eta$$

the integration being taken along any "vertical" simple loop, that is homologous to the segment $[0, i]$ oriented upward. Then R induces a proper finite total curvature embedding of the torus in $\mathbb{R}^3/\tau$. This embedding also has the following properties: each horizontal curve on $\mathbb{T}$ is mapped to a horizontal circle, except for the heights 0 and 1/2, where the curves is mapped to a horizontal line. There is a natural rotational symmetry around these lines (see fig. 1). Conversely, we can state the uniqueness theorem proven in [3]

Theorem 2 : *Let A be a properly embedded minimal annulus bounded by two parallel lines; assume further that A lies between two planes with one line in each plane. Then there exist a real constant μ such that A lies in the image of R, up to obvious isometries.*

3 General properties

3.1 Extension of the surface

Let us now prove Theorem 1. We will start with two horizontal parallel lines L_0 and L_1 bounding a properly embedded minimal annulus A with one end. We also suppose that A lies between two horizontal planes, each containing L_0 or L_1 respectively. Applying Schwarz Reflection Principle, we can rotate by 180^o around those lines in order to extend the surface. Repeating this procedure, we get a complete orientable minimal surface S, with parallel horizontal ends. If we define the translation τ as the composition of the two basic rotations around L_0 and around L_1, then the quotient $M = S/\tau\mathbb{Z}$ is an *orientable, genus 1, embedded minimal surface, with four horizontal ends.* It is a general result in minimal surface theory ([8]) that any complete finite topology properly embedded minimal surface is conformally equivalent to a punctured Riemann surface. Our purpose will now be to prove that M is conformally a rectangular torus.

3.2 The first conformal representation

Let σ denote the 180^o rotation in space around L_1; then $B = A \cup \sigma(A)$ is a fundamental domain of M and is topologically an annulus bounded by two lines, L_0 and $L_2 = \sigma(L_0)$, minus three points, namely l_1, l_2 and $\sigma(l_2)$. Here l_1 is the end at the height of L_1, and we will write abusively $l_1 \in L_1$; l_2 is the additional end in between the two original lines. Since B is minimal, it is conformally isomorphic to an annulus in the complex plane, that is $\{z \in \mathbb{C}; 1 \leq |z| \leq r_2\}$; from now on, we will see B as that particular annulus. Furthermore the third coordinate X_3 of the embedding is harmonic and constant on the boundary (here circles); thus it has to be something like

$$X_3(z) = a \log |z| + b$$

An obvious consequence of this is that L_1 is also a circle in this conformal representation, whose radius is the geometric mean of the radii of the boundary circles.

Considered as an isometry of $\mathbb{R}^3$, σ is orientation-preserving; but by construction of M, we see that σ sends the normal vector at any point $q \in M$ to the opposite of the normal at $\sigma(q)$:

$$\sigma_*(n(q)) = -n(\sigma(q)) \tag{1}$$

We can now conclude that σ is an orientation-reversing isometry of the surface. But the only anti-conformal map from B into itself that sends L_0 to L_2 and leaves the circle L_1 pointwise invariant is the affine inversion $z \mapsto k/\bar{z}$, where k is some real constant uniquely determined by the position of L_1. So that M is conformally this annulus in the complex plane with boundary identified through σ. However we will use another representation.

3.3 Parameterization on the flat rectangular torus

We now consider the rectangle in $\mathbb{C}$ whose vertices are $0, \mu, \mu + i, i$ (as in section 2, whose notations we will use). It is conformally equivalent to the annulus defined above, provided we identify the vertical edges. The equivalence mapping is $z \mapsto e^{icz}$ for some real constant c. We see also that the lines L_0, L_1, L_2 correspond to horizontal segments of height $0, 1/2, 1$ respectively (see figure 2). For the sake of simplicity, we will keep calling these lines L_0, L_1 and L_2.

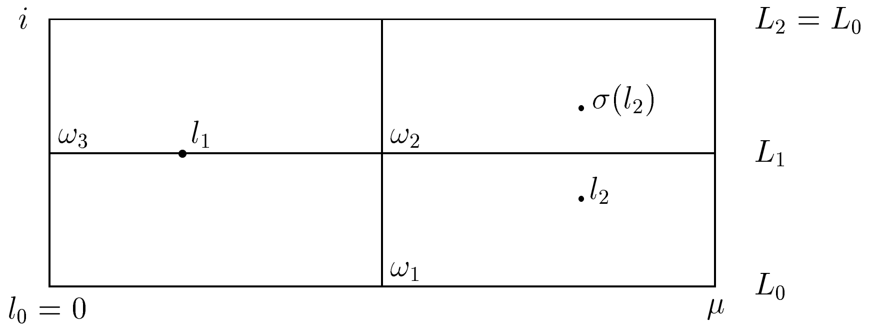

fig. 2

Note that X_3 is here an affine function of $Im(z)$, therefore X maps the horizontal lines to horizontal curves in $\mathbb{R}^3$. L_1 is the middle horizontal line, and the rotation σ in space induces a line of symmetry on the torus. It is easy to see that σ is here the usual reflection around L_1, which identifies any point of L_0 with the point of L_2 with the same real part. Therefore the conformal model of M is a *rectangular* torus. In that quotient, σ can be seen as the usual complex conjugation. Then we have to remove the four ends. We will keep on calling l_k the end that is "on" the line L_k, for $k = 0, 1$ and we can always suppose $l_0 = 0$. Let l_2 still be the end between L_0 and L_1 and, because of the symmetry, $\bar{l_2}$ is the other one. So that $M \simeq \mathbb{T} - \{l_0, l_1, l_2, \overline{l_2}\}$ conformally. We deduce from section 2 that in the case of the double covering of Riemann's example, the four ends are located at $(0, \omega_3, \omega_2 - i/4, \omega_2 + i/4)$ (provided of course we locate l_0 at 0). We will prove that these points are the only possible choices.

3.4 The Weierstrass-Enneper representation

Now that we have a proper conformal model for M, we will try to determine the Weierstrass-Enneper parametrization of the immersion $X : M \to \mathbb{R}^3$. Let us sum up what we know about M: it is a finite topology, orientable, embedded minimal surface; its genus is equal to 1, with four ends. Then the Euler characteristic $\chi(M) = -4$; the total curvature is finite and satisfies the Callahan-Hoffman-Meeks formula:

$$\int_M K\, dM = 2\pi(\chi(M) - 4) = -16\pi = -4\pi d(g)$$

Here g denotes the *extended* Gauss map, which is a meromorphic map from $\mathbb{T}$, the completion of M as a Riemann surface, to S^2, the extended complex plane. Thus the degree of g is 4. Using the orientability of M, we may suppose that the normal map at l_0 and l_1 is vertical in the upward direction while at p and its conjugate it is vertical but goes downward. In terms of the Gauss map, l_0 and l_1 are poles, while l_2 and $\bar{l_2}$ are zeros.

Up to a scaling, we have $X_3(z) \equiv Re \int^z 2idz \pmod 1$. Thus if η denotes the holomorphic 1-form associated to g in the Weierstrass representation, then $g\eta = idz$.

$$\eta = \frac{idz}{g}$$

Equivalently, we can write $dh = g\eta = idz$ where h is the "height function". Furthermore, the ends are double points; namely, at a flat end, g must have branching order greater or equal to 1; but the degree of g is 4, so *all the ends of M have branching order 1 for g.* We can now easily write the coordinates of the immersion:

$$\begin{cases} X_1(z) = Re \int^z (1-g^2)\eta = Re \int^z (g^{-1} - g) idz = Im \int^z (g - g^{-1}) dz \\ X_2(z) = Re \int^z i(1+g^2)\eta = -\, Re \int^z (g^{-1} + g) dz \\ X_3(z) = Re \int^z 2g\eta = Re \int^z 2idz = -2\, Im(z - z_0) \end{cases}$$

3.5 Properties of the Gauss map g

We have seen that M has by construction a symmetry, namely the complex conjugation corresponding to rotation around the lines in space. Whether we choose rotation around L_0 or around L_1 does not matter, since they yield the same isometry on the torus. The most important point is the behaviour of g under conjugation. If we choose the lines L_0 and L_1 to follow the X_2 direction, the property stated in (1) in terms of the normal vector translates as

$$g(\bar{z}) = \overline{g(z)}.$$

We check easily that $z = \bar{z}$ on the torus is equivalent to saying that z belongs to L_0 or L_1, and in that case g has to be real, which means exactly that the normal map is orthogonal to the X_2 direction.

Since g is an elliptic function of degree 4, we know it can be written as $F(\wp(z)) + \wp'(z) G(\wp(z))$, where F and G are rational functions, and $\wp$ is the Weierstrass p-function associated with the lattice defining this torus. Rather than $\wp$, we will use the function P defined in 2.1, which similarily generates the field of elliptic functions, and has the advantage of having a double zero at $\omega_2 = \frac{\mu+i}{2}$. Then

Proposition : *The end l_1 is one of the two branch points of P on L_1, namely ω_2 or ω_3. F is a degree 2 fraction with simple poles at infinity and $P(l_1)$, while G is only a degree 1 fraction, with a pole at $P(l_1)$ and a zero at infinity, unless it vanishes everywhere.*

Corollary 1 : *There exist a degree 2 polynomial E, which does not vanish at $P(l_1)$, and a complex constant ν such that :*

$$g(z) = \frac{E(P(z)) + \nu P'(z)}{P(z) - P(l_1)}$$

Corollary 2 : *Since g is invariant under complex conjugation, then ν and all the coefficients in the expansion of E are real.*

Proof of proposition :
– By considering a neighbourhood of zero in the z-plane, we can see that F has a simple pole at infinity, whereas G vanishes there. Namely, if $F(w) = cw^k + o(w^k)$ and $G(w) = cw^h + o(w^h)$ when $w \to \infty$ (here and afterwards, c will denote some non zero complex constant), then $g(z) \sim cz^{-2k} + cz^{-3-2h}$. This is compatible with the double pole requirement if and only if $h \leq -1$ and $k = 1$.

– We will now prove by contradiction that l_1 equals ω_2 or ω_3 (in other words $2l_1 \equiv 0$ on the torus). Indeed let us suppose that l_1 is not a branch point of P. We can find a function g having double poles at 0 and l_1, and double zeros at l_2 and $\bar{l_2}$, provided these points satisfy $2(l_1 - l_2 - \bar{l_2}) \equiv 0$ on the torus, by the Abel-Jacobi theorem. However, in order to eliminate the periods, the Gauss map must also satisfy the following criterion (proven in 4.1): both g and g^{-1} have no residues. We will prove that under the hypothesis $2l_1 \not\equiv 0$, the function $f = g^{-1}$ has non-vanishing residues at l_2. We proceed in technical lemmas.

Lemma 1 : $f'''(0) = 0$.

Proof: This condition is actually equivalent to $\mathrm{Res}(g, 0) = 0$. Indeed, let us write the Taylor expansion of $f = 1/g$ in a neighbourhood of the origin: if $f(z) = az^2 + bz^3 + O(z^4)$ then:

$$g(z) = \frac{1}{f(z)} = \frac{1}{az^2}\Big(1 - \frac{bz}{a} + O(z^2)\Big)$$

So $b = -a^2\,\mathrm{Res}(g, 0)$, which vanishes by hypothesis. Hence $f'''(0) = 0$. Q.E.D.

Lemma 2 : *Let $\hat{f}$ be the odd part of f; then:*

$$\hat{f} = \frac{\alpha P'}{[(P - P(l_2))(P - P(\bar{l_2}))]^2}$$

where $\alpha \in \mathbb{C}^*$ is a constant.

Proof: By hypothesis $2l_1 \not\equiv 0$, hence $l_2 + \bar{l_2} \not\equiv 0$. Consequently, $\pm l_2, \pm\bar{l_2}$ are four distinct points on $\mathbb{T}$. $\hat{f}$ is defined by $\hat{f}(z) = \frac{1}{2}(f(z) - f(-z))$, and has double poles at each of these four points, and nowhere else: hence $d(\hat{f}) = 8$. Being odd, $\hat{f}$ must vanish at $0, \omega_1, \omega_2, \omega_3$; using lemma 1, we see that $\hat{f}$ has a zero of order at least 5 at the origin. We conclude that it has simple zeros at ω_j, $j = 1, 2, 3$. Finally we consider $\hat{f}/P'$, which has the same poles as $\hat{f}$ with the same multiplicity, and only one zero of order 8 at the origin. It can only be $\alpha[(P - P(l_2))(P - P(\bar{l_2}))]^{-2}$ for some $\alpha \in \mathbb{C}^*$. Q.E.D.

Lemma 3 : $\mathrm{Res}(\hat{f}, l_2) = \frac{1}{2}\mathrm{Res}(f, l_2)$

Proof: Since $f(-z)$ is non singular at $-l_2$, then in some neighbourhood of l_2, one can write:

$$\hat{f}(z) = \frac{1}{2}(f(z) - f(-z)) = \frac{1}{2}(f(z) + O(1))$$

Q.E.D.

Lemma 4 : *The residues of $\hat{f}$ and f at l_2 never vanish.*

Proof: Let us write the Taylor expansion of P in a neighbourhood of l_2; $P(l_2+\zeta) = a + b\zeta + c\zeta^2 + O(\zeta^3)$. Notice that neither a nor b vanish. Then

$$(P - P(l_2))^{-2} = \frac{1}{b^2\zeta^2}\Big(1 - \frac{2c}{b}\zeta + O(\zeta^2)\Big)$$

$$(P - P(\bar{l_2}))^{-2} = \frac{1}{a'^2}\Big(1 - \frac{2b}{a'}\zeta + O(\zeta^2)\Big)$$

where $a' = a - \bar{a}$ ($a' \neq 0$ because $\bar{l_2} \not\equiv -l_2$).

$$\frac{P'}{[(P-P(l_2))(P-P(\bar{l_2}))]^2} = \frac{1}{ba'^2\zeta^2}\Big(1 + \frac{2c}{b}\zeta + O(\zeta^2)\Big)\Big(1 - 2(\frac{b}{a'} + \frac{c}{b})\zeta + O(\zeta^2)\Big)$$

$$= \frac{1}{ba'^2\zeta^2}(1 - \frac{2b}{a'}\zeta + +O(\zeta^2))$$

The residue of $\hat{f}$ is $-2\alpha a'^{-3}$ which never vanishes under our hypotheses. Using lemma 3, we see that f itself has non vanishing residues, which contradicts section 4.1. We must exclude this case. Q.E.D.

Therefore we know that $l_1 = \omega_2$ or $l_1 = \omega_3$; equivalently, $P(l_1) = 0$ or $P(l_1) = -1/\lambda$. One deduces that F has another simple pole at $P(l_1)$, while G can have at most a simple pole there.

– Let us finally look for the other possible poles of G. There must be at least one pole, since we have shown G vanishes at infinity (unless $G \equiv 0$). If G has a pole at some $P(z_0)$, then z_0 must be a pole for g. In fact, if it were not, then $P'(z)G(P(z))$ would have to stay bounded near z_0; this is never possible, whether z_0 is a branch point of P or not. Thus $P(z_0)$ is either infinity or $P(l_1)$. The first is excluded, so we can conclude that G has a simple pole at $P(l_1)$. The corollaries follow. Q.E.D.

4 The period constraints

4.1 The different types of periods

In order for X to be well defined, we need to check that there are no periods for X on the torus. These real periods fall into two distinct categories: "analytical" periods coming from poles of a differential form, and "geometric" periods coming from the homology. For the latter, we will take into account only the real periods along any horizontal loop (by horizontal loop, we mean homologous to L_0).

It is obvious already that $\mathrm{d}X_3 = -2\, Im\, dz$ has no period, either around the ends, or along a horizontal loop. However $\mathrm{d}X_1 = Im((g - g^{-1})dz)$ and $\mathrm{d}X_2 = -\,Re((g + g^{-1})dz)$ may have periods around the poles and zeros of g. Clearly their analytical periods vanish if and only if g and g^{-1} have no residues. There remains then to check the homological periods.

4.2 The planar symmetry

We will show that there has to be a plane of symmetry for the immersion X, and how this translates into analytical terms.

Proposition : *The function g is even; this implies that there is a plane of symmetry perpendicular to the X_2 direction.*

We will need the following

Lemma : *The residue of g at 0 is -2ν .*

Proof of the lemma :

We have found an expression of g as

$$g(z) = \frac{E(P(z)) + \nu P'(z)}{P(z) - P(l_1)}$$

where l_1 is the pole of g on L_1, and is one of the two branch points of P on L_1. Obviously, $\frac{E(P(z))}{P(z)-P(l_1)}$ is even, and thus has no residue at 0; then

$$\mathrm{Res}(g, 0) = \nu \mathrm{Res}(P'/(P - P(l_1)), 0)$$

Near zero, we know that

$$P'(z)/(P(z) - P(l_1)) \sim -2/z.$$

This proves the lemma.

Proof of the proposition :
According to the previous section, it is necessary for ν to vanish; thus $g(-z) = g(z)$. One checks easily that it amounts to saying that there is a vertical plane of symmetry, orthogonal to the X_2 direction. QED.

Finally, we deduce

Corollary : *$P(l_2) \in \mathbb{R}$ and there is a real number α such that :*

$$g(z) = \alpha \, \frac{(P(z) - P(l_2))^2}{P(z) - P(l_1)}.$$

Proof :
Now that we have eliminated the odd part in the expansion of g, we can easily determine its accurate expression. In fact, let us compute the derivative of g :

$$g'(z) = P'(z) \, \frac{E'(P(z))(P(z) - P(l_1)) - E(P(z))}{(P(z) - P(l_1))^2}$$

and apply the condition $g(l_2) = g'(l_2) = 0$. Since obviously $P(l_2) \neq P(l_1)$ and $P'(l_2) \neq 0$, $P(l_2)$ has to be a double zero of E. Furthermore all parameters in the expansion of g must be real in order to satisfy the requirement that $g(\bar{z}) = \overline{g(z)}$. QED.

4.3 The value of l_1

We have two possibilities for l_1: either ω_2 or ω_3, whose values under P are respectively 0 and $-1/\lambda < 0$. We also know from section 3.5 that $g(L_1)$ lies either in $\overline{\mathbb{R}}_+^*$ or $\overline{\mathbb{R}}_-^*$. Namely the image of each of these lines lies in $\overline{\mathbb{R}}$ but doesn't include zero, and reaches ∞ only once (though with order 1). We will prove that the sign of g along both L_0 and L_1 must be the same, namely the sign of α. As a corollary this will force l_1 to be ω_3.

Proof :
A necessary condition on g is that neither g nor g^{-1} have residues on $\mathbb{T}$. Thus

$$\int_{L_1} g^{-1}dz = \int_{L_0} g^{-1}dz$$

both lines having the same orientation (say increasing real part). It is clear then that g has the same sign on both of them. Suppose now that $l_1 = \omega_2$, or equivalently, $P(l_1) = 0$. Let us evaluate g in the neighbourhood of this pole. It is easy to compute that for small ζ, $P(l_1+\zeta) \sim -\frac{C_\lambda}{4}\zeta^2$. Hence $g(l_1+\zeta) \sim -\frac{C_\lambda}{4}\alpha P(l_2)^2/\zeta^2$. Meanwhile, near zero, on L_0, $g(z) \sim \alpha/z^2$. Both expressions have opposite signs, we get a contradiction (recall $C_\lambda > 0$). We can now write :

$$g(z) = \alpha\,\frac{(P(z)-P(l_2))^2}{P(z)+1/\lambda}$$

4.4 The period condition at the l_2 end

Let us now consider periods around l_2 (or $\overline{l_2}$ which is equivalent). We need only check that the residue of g^{-1} vanishes. Therefore we write for small u,

$$P(l_2+u) = P(l_2) + au + bu^2/2 + O(u^3)$$

then

$$(P(l_2+u)-P(l_2))^2 = a^2u^2(1+bu/a+O(u^2))$$

and

$$\begin{aligned} g^{-1}(l_2+u) &= \frac{\big((P(l_2)+1/\lambda)+au+O(u^2)\big)\big(1-bu/a+O(u^2)\big)}{\alpha a^2u^2} \\ &= \frac{\big(P(l_2)+1/\lambda\big)+\big(a-b(P(l_2)+1/\lambda)/a\big)u+O(u^2)}{\alpha a^2u^2} \end{aligned}$$

Therefore the residue is $\frac{a^2-b(P(l_2)+1/\lambda)}{\alpha a^3}$. Note that $a = P'(l_2) \neq 0$.

We will now show that only two possible choices of l_2, up to conjugation, make this residue vanish. The condition is:

$$a^2 = b(P(l_2)+1/\lambda)$$

where a, b are respectively the first and second derivative of P at l_2. Let us first compute b in terms of the other quantities. The structural equation for P being $P'^2 = C_\lambda\, P(P+1/\lambda)(P-\lambda)$, derivation yields :

$$2P'P'' = C_\lambda\, P'(P(P+1/\lambda)+P(P-\lambda)+(P+1/\lambda)(P-\lambda)),$$

therefore

$$\begin{cases} 2b = C_\lambda\,(P(l_2)(P(l_2)+1/\lambda)+P(l_2)(P(l_2)-\lambda)+(P(l_2)+1/\lambda)(P(l_2)-\lambda)) \\ a^2 = C_\lambda\, P(l_2)(P(l_2)+1/\lambda)(P(l_2)-\lambda) \\ a^2 = b(P(l_2)+1/\lambda) \end{cases}$$

This implies:

$$2P(l_2)(P(l_2)-\lambda) = (P(l_2)(P(l_2)+1/\lambda)+P(l_2)(P(l_2)-\lambda)+(P(l_2)+1/\lambda)(P(l_2)-\lambda))$$

namely

$$P(l_2)^2 + 2P(l_2)/\lambda - 1 = 0$$

whose solutions are $P(l_2) = \frac{\pm\sqrt{\lambda^2+1}-1}{\lambda}$.

4.5 The homology period

We have hencewith verified that no real period can arise from loops around $l_2, \overline{l_2}, 0$ or l_1. This leaves us with two distinct families of solutions. However there remains to check the homological condition, which will eventually reduce the solution to the known Riemann's example.

Suppose γ is a simple horizontal loop. We must have :

$$\begin{cases} Im \int_\gamma (g^{-1} - g)\, dz = 0 \\ Re \int_\gamma (g^{-1} + g)\, dz = 0 \end{cases}$$

We suppose of course that γ does not meet any zero or pole of g; however, in order to evaluate the integral, we can split it and consider homologous curves. The first integral is always zero, because of the invariance of g under conjugation. Namely, the integrals of g and g^{-1} along γ are both real. The second equation can be written as

$$\alpha \int_\gamma \frac{(P(z) - P(l_2))^2}{P(z) + 1/\lambda}\, dz + \frac{1}{\alpha} \int_\gamma \frac{P(z) + 1/\lambda}{(P(z) - P(l_2))^2}\, dz = 0$$

where we must find a real α, for one of the two possible $P(l_2)$. A necessary and sufficient condition is that both integrals have opposite signs (or both 0). But we see at once that $\int_\gamma g^{-1} dz$ is positive if α is, by integrating along L_0. Thus $\int g dz$ and α must have opposite signs. However this integral can be seen as a polynomial in $P(l_2)$; in fact let us define:

$$\begin{aligned} \mathcal{P}(w) &= \tfrac{1}{\alpha} \int_\gamma g\, dz = \int \tfrac{(P(z)-w)^2}{P(z)+1/\lambda} dz \\ &= \int [(P(z) + 1/\lambda) - 2(w + 1/\lambda) + \tfrac{(w+1/\lambda)^2}{P(z)+1/\lambda}] dz \\ &= \Big(- \mu(2w + 1/\lambda) + \int P(z) dz + (w + 1/\lambda)^2 \int \tfrac{dz}{P(z)+1/\lambda} \Big) \end{aligned}$$

The study of $\mathcal{P}$ shows that : $\mathcal{P}$ is a upward-oriented parabola which is positive at $-1/\lambda$, negative at 0, and attains its minimum *after* λ. Therefore $\mathcal{P}(\frac{-\sqrt{\lambda^2+1}-1}{\lambda}) > 0$, which excludes the smallest possible value for $P(l_2)$. On the other hand,

$$\mathcal{P}(\frac{\sqrt{\lambda^2 + 1} - 1}{\lambda}) < 0,$$

which corresponds to the known Riemann's example.

Proof :

Note first that the leading term in $\mathcal{P}$ is positive. In fact we can integrate $\int \frac{dz}{P(z)+1/\lambda}$ along L_0 instead of γ, where we have the following inequality : $P(z) \geq \lambda$ (remember $\lambda > 0$). So $P(z) + 1/\lambda \geq \lambda + 1/\lambda > 0$; this yields the desired result but also a bound on the integral, namely :

$$\int_{L_0} \frac{dz}{P(z) + 1/\lambda} < \int_{L_0} (\lambda + 1/\lambda)^{-1} dz = \frac{\mu\lambda}{1 + \lambda^2}$$

Let us now evaluate the position for the minimum of this parabola, which we will call w_m :

$$w_m = -1/\lambda + \frac{\mu}{\int \frac{dz}{P(z)+1/\lambda}}$$

It is very easy to verify that the previous inequality is equivalent to

$$w_m > \lambda$$

Third and last point, we would like to have an estimate of $\mathcal{P}$ at 0 and $-1/\lambda$.

$$\mathcal{P}(0) = -\frac{\mu}{\lambda} + \int P(z)dz + \frac{1}{\lambda^2} \int \frac{dz}{P(z) + 1/\lambda}$$

By integrating along L_1, it is obvious that $\int P(z)dz < 0$; we know more, namely $-\mu/\lambda < \int P(z)dz < 0$. Therefore to prove that $\mathcal{P}(0) < 0$ we need only show that

$$\frac{1}{\lambda^2} \int \frac{dz}{P(z) + 1/\lambda} < \frac{\mu}{\lambda}$$

that is

$$\int \frac{dz}{P(z) + 1/\lambda} < \mu\lambda$$

Using the previous result, we know that

$$\int \frac{dz}{P(z) + 1/\lambda} < \frac{\mu\lambda}{1 + \lambda^2}$$

which is sufficient. And when $w = -1/\lambda$,

$$\mathcal{P}(-1/\lambda) = \frac{\mu}{\lambda} + \int P(z)dz$$

which is positive, as seen previously. QED.

4.6 Conclusion

The Riemann Example is only minimal surface which is conformally a once-punctured annulus lying between two parallel planes, bounded by two lines in those planes. Up to scaling, there is only a one-parameter family of such examples, parameterized by μ (or λ).

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