The Number of Faces in a Minimal Foam

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1 Introduction

A *compound bubble* is a partition of a Riemannian 3-manifold M into domains whose boundaries are smooth constant mean curvature surfaces, meeting 3 to a smooth edge or 6 to an isolated vertex, at equal angles. Observe that the equal angle condition means that at any point on the *support* Σ (that is, the union of the boundary surfaces) of the compound bubble the *tangent cone* is either a plane, the product of a line with an equiangular "Y" or the central cone over the 1-skeleton of a regular tetrahedron.

These local regularity properties of compound bubbles were first articulated by Plateau in the 1800's, and only recently derived from an area minimizing property by Jean Taylor [10] (cf. also [2, 3, 4, 6]). An important problem is whether they imply any global properties. For example, R. Gulliver (Problem 1.2 [1]) has asked if a partition of $\mathbf{R}^3$ into domains of equal volume by minimal surfaces must be *periodic*, in the sense that all domains translates of a single fundamental domain?

In this note we consider a special type of compound bubble Σ which arises from a *cell decomposition* of M: in this case we call Σ a *foam*, and say that a foam is *minimal* if each 2-cell (or *face*) is a minimal surface. Minimal foams arise naturally from the least boundary area fundamental domains of irreducible 3-manifolds (for example, spaceforms) considered by J. Choe [6].

The main result here is a formula for the average number of faces per cell in a minimal foam. In case of a periodic minimal foam in $\mathbf{R}^3$ this formula implies that the fundamental cell has at least 14 faces, a lower bound realized by Lord Kelvin's [12] remarkable example of 1887 (see below). The formula also provides constraints on minimal foams in other 3-manifolds, such as the sphere $\mathbf{S}^3$.

While the existence and regularity results cited above rely on geometric measure theory, the methods employed here are classical: differential geometry of curves and surfaces, most notably, the Gauss equation and Gauss-Bonnet formula. Thus, in principle, our face formula could have been derived in Plateau's day.

2 The face formula

For simplicity in stating the formula assume that M is a compact Riemannian 3-manifold and that $C_1, \ldots, C_n$ are the cells of a minimal foam Σ in M. Given a cell C_i of Σ, its (abstract) boundary ∂C_i is a 2-sphere containing f_i faces, e_i edges, and v_i vertices. From the Euler formula

$$f_i - e_i + v_i = 2\,, \tag{1}$$

and from the incidence relation

$$2e_i = 3v_i \tag{2}$$

one sees that any one of f_i, e_i, or v_i determines the other two.

Let A denote the second fundamental form of a face of Σ, and R, the sectional curvature of M in planes tangent to the face. Using $\delta = arcsec(3)$ for the dihedral angle of a regular tetrahedron, and a "tilde" to denote the average of a quantity over the collection of cells (for example, $\displaystyle \fint_{\partial C} =: \frac{1}{n}\sum_{1\le i\le n}\int_{\partial C_i}$) one has the following formula.

Theorem 1 *Let Σ be a minimal foam in a compact Riemannian 3-manifold M. The average number of faces in Σ is*

$$\widetilde{f} = (12\delta + \fint_{\partial C}(|A|^2/2 - R))/(6\delta - 2\pi)\,. \tag{3}$$

In particular, if the ambient manifold M is nonpositively curved $(R \le 0)$ then

$$\widetilde{f} \ge 6\delta/(3\delta - \pi) = 13.39733\ldots \tag{4}$$

Moreover, if the foam is also periodic, then (1), (2) *and* (4) *imply*

$$(f, e, v) = (\widetilde{f}, \widetilde{e}, \widetilde{v}) \ge (14, 36, 24)\,. \tag{5}$$

Proof. Let C be any cell of Σ. The derivation of (3) begins with the Gauss equation relating the intrinsic curvature K of a face $F \subset C$ to the extrinsic and ambient curvatures:

$$K = \det(A) + R = (H^2 - |A|^2)/2 + R = -|A|^2/2 + R\,, \tag{6}$$

where the mean curvature $H = trace(A) = 0$ since F is minimal. Write κ for the geodesic curvature of an edge $E \subset F$ and note that any two edges meet a vertex with exterior angle $\delta = arcsec(3)$.

Integrating (6) over F, and applying the Gauss-Bonnet formula, this yields

$$\int_F (-|A|^2/2 + R) = 2\pi - \delta v(F) - \sum_{E\subset F} \int_E \kappa\,, \tag{7}$$

where $v(F)$ is the number of vertices in ∂F. Now sum (7) over the f faces $F \subset \partial C$ (using (1) and (2) to write $6(f-2) = 3v = \sum_{F\subset C} v(F)$) and rearrange terms to obtain

$$(6\delta - 2\pi)f = 12\delta + \int_{\partial C} (|A|^2/2 - r) - \sum_{F\subset C} \sum_{E\subset F} \int_E \kappa\,. \tag{8}$$

By averaging over the collection of cells, this gives the face formula (3) provided we can show that the "error" vanishes:

$$\sum_{1\le i\le n} \sum_{F\subset C_i} \sum_{E\subset F} \int_E \kappa = 0\,. \tag{9}$$

To prove (9) one must recall that the geodesic curvature κ of an edge $E \subset F$ is related to the space curvature vector $\mathbf{k}$ of $E \subset M$ by the formula

$$\kappa = \eta \cdot \mathbf{k} \tag{10}$$

where η is the (inward) conormal vector to $E \subset F$; that is, η is tangent to F and normal to E. Because each edge E meets exactly 3 faces (say) F, F', and F'', with their corresponding conormals η, η', η'', observe that (with aid of (10)) formula (9) can be rewritten as

$$2\sum_E \int_E (\eta + \eta' + \eta'') \cdot \mathbf{k} = 0\,, \tag{11}$$

which is true pointwise because the equiangular "Y" configuration along E implies $(\eta + \eta' + \eta'') = \mathbf{0}$.

3 Extensions, examples and remarks

i) When M is complete and noncompact the same derivation applies, provided one makes some additional assumptions on the minimal foam Σ. For instance, if one assumes Σ is "quasi-periodic" in the sense that Σ is invariant under a discrete group of isometries acting freely on M with compact quotient, then the face formula follows from the obvious lifting argument.

A special case is the periodic minimal foam arising from a fundamental domain with least boundary area mentioned in the Introduction.

ii) In the absence of a symmetry group in (i), one can instead impose certain uniformity conditions on the cells of Σ (for example, uniformly bounded volume ratios, diameters, and curvatures) so that the "error" (9) averaged over the cells meeting a ball B_ρ of radius ρ in M is bounded by

$$C|\partial B_\rho|/|B_\rho|\,,$$

which is $O(\frac{1}{\rho})$ if M is flat ($R = 0$) or asymptotically flat. Letting ρ tend to ∞, this "error" term decays to 0.

iii) The standard 3-sphere $\mathbf{S}^3$ (with constant curvature $R = 1$) supports a variety of minimal foams with totally geodesic ($|A| = 0$) faces: these include cell decompositions defined by the equator $\mathbf{S}^2$, the "spherical simplex", and the "Poincare dodecahedral decomposition". One may ask if these foams correspond to the least boundary area fundamental domains for the appropriate spherical-space-forms. Note that the area of each face is easily computed from the face formula. For example, the Poincare foam in $\mathbf{S}^3$ whose 60 cells are totally geodesic regular dodecahedra has total surface area $720\pi - 1800\delta$. An important problem is to determine the minimal foams in $\mathbf{S}^3$ for which the associated 3-cone in $\mathbf{R}^4$ is volume-minimizing (Problem 5.14 [1]).

iv) W. Thomson (Lord Kelvin) [12] provided an example of a periodic minimal foam Σ in $\mathbf{R}^3$ with period lattice Λ generated by the vectors (2,0,0), (0,2,0) and (1,1,1). The fundamental cell of Σ is (combinatorially) a truncated octahedron with 14 faces (6 flat quadrilaterals, 8 curved hexagons), 36 edges, and 24 vertices, showing that the lower bound (5) is sharp. Choe [6] has speculated that this example may give the least boundary area fundamental domain for the flat torus $\mathbf{R}^3/\Lambda$. Some computer simulations of this foam have been carried out recently by K. Brakke and J. Sullivan.

v) When M is flat the inequality (4) cannot be sharp: this would require each face to be totally geodesic, which is easily excluded (for example, using the proof of the face formula). In fact, estimating the total curvature $\int |A|^2$ of the faces is equivalent (via (3)) to finding an *upper* bound on the average number of faces. If the foam is actually area minimizing, perhaps the stability (second variation) inequality can be employed to get the required integral curvature estimate.

vi) H. Coxeter has noted that Kelvin's example is closely related to a lattice packing of equal spheres; he also describes a sequence of delightful "condensation" experiments (cf. [7] and the references therein, especially [5]) which seem to indicate that the average number of faces for a minimal partition is closer to 13.5, possibly (via (5)) precluding periodicity. In fact, the coincidence between the lower bound (4) and Coxeter's proposed upper bound ([7], p.66) on the density of a unit sphere packing suggests that there may be some interesting geometry lurking in this condensation phenomenon. (It is still unknown whether the most dense unit sphere packing in $\mathbf{R}^3$ can be realized by a lattice packing.)

vii) It is hoped that formulas such as (3), (4), (5) will be of some interest to researchers in fields other than mathematics. In addition to soap froth — or the "head" on a fresh beer! — compound bubbles appear to model a number of interesting natural systems: the classical monatomic fluid (such as liquid argon) [5], the cellular structure of an organism [11], and the overall distribution of galaxies in the universe [8].

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