Adding Handles to the Helicoid*

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We have constructed two minimal surfaces of theoretical interest. The first is a complete, embedded, singly-periodic minimal surface (SPEMS) that is asymptotic to the helicoid, has infinite genus, and whose quotient by translations has genus one. The quotient of the helicoid by translations has genus zero and the helicoid itself is simply-connected.

Theorem 1. *There exists an embedded singly-periodic minimal surface* $\mathcal{W}_1$*, asymptotic to the helicoid and invariant under a translation* T*. The quotient surface* $\mathcal{W}_1/T$ *has genus equal to one and two ends.*

$\mathcal{W}_1$ contains a vertical axis, as does the helicoid, and $\mathcal{W}_1/T$ contains two horizontal lines.

The second surface is a complete, properly-embedded minimal surface of finite topology with infinite total curvature. It is the first such surface to be found since the helicoid, which was discovered in the 18th century.

Theorem 2. *There exists a complete, properly-embedded minimal surface,* $\mathcal{H}e_1$*, of genus-one, whose one end is of helicoidal type.*

$\mathcal{H}e_1$ contains a vertical line, like the helicoid, and one horizontal line that crosses it. Schwarz reflection in these two lines generates the symmetry group of the surface.

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Figure 1: The surfaces $\mathcal{W}_1$ (left) and $\mathcal{H}e_1$ (right)

1 History and context

Except for the plane, the helicoid is the only ruled minimal surface. It's discovery is attributed to Meusnier in 1776; together with the catenoid (Euler, circa 1744) these were the only minimal surfaces explicitly known to Eighteenth Century mathematics. From Scherk in the 1830s, came the next major discovery: multiple families of periodic minimal surfaces, including the famous families of singly- and doubly-periodic examples that bear his name [11, 15]. That the surfaces in these two families share the same normal mapping is implicit in the work of Scherk. This fundamental relationship was made explicit by Enneper, Weierstrass and Riemann. They developed an integral representation formula for minimal surfaces via contour integration of meromorphic data derived from the normal mapping, which they knew to be conformal [13]. (See (3.3) below.) Minimal surfaces were seen to be, from this point of view, the real part of null curves in $\mathbb{C}^3$. The helicoid and catenoid were recognized as, locally, the real and imaginary part of the same curve. The same is true of the two families of Scherk. Minimal surfaces related in this way are said to be *conjugate*.

For complete minimal surfaces, whose quotient by orientation-preserving symmetries has finite total curvature, the quotient is naturally a compact Riemann surface, possibly punctured in a finite number of points. Moreover, the meromorphic data is well-defined on the compact surface (Osserman [12, 13]). Translations are produced when the integral representation has periods on the Riemann surface. The classical examples mentioned above can be represented on a sphere punctured two (resp. four) times for the helicoid/catenoid pair (resp. Scherk's singly-/doubly-periodic surfaces). Moreover, the Gauss map can be taken to be the identity on S^2. All these examples are *embedded*.

The existence of higher-genus embedded examples has been an open question until recently. For complete embedded examples of finite total curvature (FEMS) in R^3, Lopez-Ros [9] showed that the plane and the catenoid are the only FEMS of genus zero. Schoen [16] proved that the catenoid was the only FEMS with two ends. The existence of examples with genus greater than zero and more than two ends is well documented [2, 3, 4, 5, 18].

The helicoid was the only known example of a complete embedded minimal surface with finite topology and infinite total curvature. It has been a longstanding open question as to whether there are others. Theorem 2 answers this question affirmatively. Also, all known properly embedded examples of infinite total curvature had infinite symmetry groups, and the quotients of these surfaces by these groups were compact (possibly punctured) Riemann surfaces whose inherited metric had finite total curvature. The surface $\mathcal{H}e_1$ of Theorem 2, is conformally a once-punctured rhombic torus with symmetry group $Z_2 \oplus Z_2$.

All classical, complete, embedded, doubly-periodic minimal surfaces (DPEMS) can be defined by meromorphic data with periods on punctured spheres. Karcher [8] and Meeks-Rosenberg [10] constructed new families of DPEMS that had genus one in the quotient. No higher-genus examples were known that were not coverings of these examples. Moreover, there were no known genus-one examples with the same end behavior as the Scherk doubly-periodic examples. In [17], Wei constructed the first DPEMS of genus equal to two in the quotient. Based on the construction strategy used in that paper, Karcher was able to modify Scherk's doubly-periodic example to produce a genus-one DPEMS that had the same end-behavior as the Scherk example. We refer to this surface as $\mathcal{K}_{\frac{\pi}{2}}$ for reasons that will be made clear below.

2 SPEMS as limits of DPEMS

The Scherk family can be considered to be the desingularization of two families of equally spaced, parallel, vertical halfplanes meeting at an angle $\theta, 0 < \theta \leq \pi/2$. In the

Figure 2: Scherk's doubly periodic surface, $\theta = \frac{\pi}{2}$ (top) and $\mathcal{K}_{\frac{\pi}{2}}$(bottom)

slab $|x_3| < \epsilon$, the surfaces look like saddles over alternating regions in a tiling of $x_3 = 0$ by rhombi. With appropriate scaling as θ goes to zero, the rhombi diagonals grow in one direction only, approaching a strip in the plane. There is a basic relationship between the Scherk family and the helicoid. Namely, if one keeps the symmetric point of a fixed saddle at the origin, the limit surface, with appropriate scaling as θ goes to zero, exists and is the helicoid (Hoffman and Wohlgemuth, [6]).

The generalization, $\mathcal{K}_{\frac{\pi}{2}}$, of Scherk's surface can be understood as Scherk's surface with a tunnel replacing every other saddle. The underlying Riemann surface is the square torus punctured in four points. We proved that this surface can be deformed in exactly the same manner as the Scherk family.

Proposition 1. *There exists a one-parameter family $\mathcal{K}_\theta$ of embedded doubly-periodic minimal surfaces, whose quotient has genus equal to one and four Scherk ends, two up and two down. Each genus-one surface is a rhombic torus. The angle θ between the up and down ends, $0 < \theta \leq \pi/2$, parametrizes the family.*

Figure 3: A surface in the Scherk family (top) and $\mathcal{K}_\theta$ (bottom), $\theta = \frac{\pi}{4}$

Each member of the family may be considered to be a desingularization of two families of parallel halfplanes. Unlike the Scherk family, these planes are not equally spaced. The interplanar distance alternates. The smaller distance between planes is spanned by tubes, while the larger one is bridged by saddles.

The singly periodic surface $\mathcal{W}_1$ of Theorem 1 has the same relationship to $\mathcal{K}_\theta$ as the helicoid has to the Scherk family. Namely, choose a distinguished point in a fundamental domain that is identifiable on each surface (for example the point on the saddle where the normal is vertical) and keep this point at the origin. Then

Theorem 3. *The limit surface as $\theta \to 0$ of the surfaces $\mathcal{K}_\theta$ exists and is equal to $\mathcal{W}_1$.*

3 The Weierstrass Representation for $\mathcal{K}_\theta$ and $\mathcal{W}_1$

We shall first derive the Weierstrass representation for the surfaces $\mathcal{K}_\theta$. When we take $\theta = 0$ in the representation, it defines the singly-periodic minimal surface $\mathcal{W}_1$.

Consider an orientable fundamental domain of Scherk's doubly-periodic surface with an angle θ between the top and bottom ends. On that surface, there are vertical lines connecting the top and bottom ends, as well as two horizontal lines meeting orthogonally at the saddle points. Moreover, there is a $180°$ rotational symmetry about a vertical line passing through the saddle point. Now imagine putting a tunnel between the two top ends over one of the saddles. We wish to preserve the $180°$ rotational symmetry on the vertical lines passing the saddle points as well as the two horizontal tangential lines which meet orthogonally at saddle points.

Assume such a surface $\mathcal{K}_\theta$ exists for the time being. Then $\mathcal{K}_\theta$ is a genus-one surface with four ends. We may assume that the normals at the saddle points are vertical and that the x_1- and x_2- directions are parallel to the two horizontal lines. Since $\mathcal{K}_\theta$ has a $180°$ rotational symmetry R around a vertical line passing the saddle points, g^2 is well defined on the quotient $\mathcal{K}_\theta/R$, where g is the stereographic projection of the Gauss map. Cutting the quotient sphere along the horizontal lines, we get a piece of $\mathcal{K}_\theta/R$ that is conformal to a half plane. Use a Möbius transformation to define a map z, which takes this region to the upper half plane and maps the three saddle points to $\infty, 0$ and a, where a is a complex number. Then compare g^2 and z on the quotient sphere $\mathcal{K}_\theta/R$. We may write

$$g^2 = \frac{(z-a)(z-\overline{a})}{z}\,. \tag{3.1}$$

By the relation between the zero and poles of g and those of the meromorphic form dh, (see [17]), we can deterimine that

$$dh = \frac{zdz}{(z-e)(z-\overline{e})}\,, \tag{3.2}$$

where dh is the complex differential of the height function of the surface in the direction parallel to the ends, and e is the *end-point* where $g(e) = e^{i\theta}/2$.

Now without assuming the existence of $\mathcal{K}_\theta$, we define a closed Riemann surface as in (3.1) and a meromorphic function g and one-form dh as in (3.1) and (3.2). Then the Weierstrass representation,

$$X(p) = Re\int_{p_0}^{p} (\frac{1}{2}(g^{-1}-g),\, \frac{i}{2}(g^{-1}+g), 1)dh \tag{3.3}$$

defines an immersed minimal surface. It is easy to check that this map is invariant under the desired symmetries, namely, a 180° rotation about a vertical line and about two horizontal lines on the surface.

To make $X\colon M \to \mathrm{R}^3$ in (3.3) a doubly-periodic minimal surface, we must satisfy the condition that the image of the homotopy cycle around a cut with end points 0 and a be closed. That can be written as

$$Re\int_0^a (g^{-1}-g)dh = 0\,,$$
$$Re(i\int_0^a (g^{-1}+g)dh) = 0\,.$$

We now consider the cases $\theta = \frac{\pi}{2}$ and $\theta = 0$. For $\theta = \frac{\pi}{2}$, the top ends are orthogonal to the bottom ones. Moreover, the surface has two reflectional symmetries. When $\theta = 0$, the number e in (3.2) must be real. From the image it is clear that the surface $\mathcal{K}_0$ is generated by reflection about the straight lines and a 180° rotation about a line connecting $X(0)$ and $X(a)$. This gives the surface $\mathcal{W}_1$.

4 Construction of $\mathcal{H}e_1$

The surface $\mathcal{W}_1$ can be described as a helicoid, into which has been sewn a handle at every other half-turn. Thus a handle has been added to the surface modulo translation. One could imagine adding a handle to every other fundamental domain, producing three half-twists between handles, and (why stop at three?) in general $2k+1$ half-twists between handles, $k \geq 0$. The quotient by orientation-preserving translations of such a surface will have genus-one. Now imagine fixing one horizontal line in a fundamental domain to be the x_2-axis and letting $k \to \infty$. The resulting surface: will have genus one; will contain the x_2-axis and x_3-axis but no other lines; will *not* be periodic; and should be asymptotic in some sense to the helicoid. In fact such a surface exists and is the surface $\mathcal{H}e_1$ of Theorem 2. Determining whether or not the imagined surfaces with $2k+1$ half-twists exist is an exercise, which we have not carried out, involving specification of Weierstrass data on a once-punctured torus, as is the case for $\mathcal{W}_1$. However, $\mathcal{H}e_1$ is *not* periodic and its Gauss map has an essential singularity at the end so it cannot be described by a meromorphic Gauss map on a punctured torus.

The key to constructing $\mathcal{H}e_1$ is to realize that, while its Gauss map g has an essential singularity at the end, its logarithmic differential, $\frac{dg}{g}$, is meromorphic. Note that, in general, at an end where g has a pole or a zero of finite order, $\frac{dg}{g}$ has a simple pole. The helicoid may be described on C by the Weierstrass data $g = e^z, dh = idz$. The single end occurs at infinity, where $\frac{dg}{g}$ has a *double* pole, as does dh. Thus, in order to expect a helicoidal end on a torus, we must look for a meromorphic differential dh with a double pole at the end and, by the Riemann relation, two zeros. We expect from our Gedankenexperiment that the zeros of dh will lie on the horizontal line, and by symmetry, there must be two simple zeros. A zero of dh occurs precisely at a point on $\mathcal{H}e_1$ where $g = 0$, or ∞, and at such a point $\frac{dg}{g}$ has a simple pole. This forces $\frac{dg}{g}$ to have precisely four zeros, corresponding to branch points of g, which we assume, for the aforementioned reason, to be simple and to lie on the lines in $\mathcal{H}e_1$. Rotation, ρ, by π about the x_1-axis is a symmetry of the surface, produced by successive reflection in the x_2- and x_3-axis. This rotation fixes three finite points on $\mathcal{H}e_1$ (one of which is the origin), and also the end. The surface $\overline{\mathcal{H}e_1}/\rho$ is the sphere $S^2 = \mathrm{C} \cup \{\infty\}$. Let z denote the projection onto $\mathrm{C} \cup \{\infty\}$ and also the variable there. We may specify that the end is over $z = \infty$, and the two other fixed points not at $\vec{0} \in \mathrm{R}^3$ lie over $z = \pm i$. The rotation ρ leaves invariant the x_2- and the x_3-axis, while fixing three points on the x_1-axis, where z is real. Hence the x_2- and x_3-axis on $\mathcal{H}e_1$ project by ρ to the real axis in the z-plane. In particular, the origin sits over a point on the real line, say λ, and $\overline{\mathcal{H}e_1}$ is conformally

$$w^2 = (z - \lambda)(z^2 + 1)\,, \tag{4.1}$$

for some $\lambda \in \mathrm{R}$. (As we already knew, because of the presence of the two lines on $\mathcal{H}e_1$ crossing at $\vec{0}$ and at the end, $\mathcal{H}e_1$ is a rhombic torus). The vertical points of g occur at two points where $z = a$, $a \in \mathrm{R}$, and the branch points of g at four points, $z = \alpha$ or $z = \beta$, $\alpha, \beta \in \mathrm{R}$. Since $\frac{dz}{w}$ is holomorphic on $\overline{\mathcal{H}e_1}$ we have

$$\frac{dg}{g} = \rho\frac{(z-\alpha)(z-\beta)}{(z-a)}\frac{dz}{w} \tag{4.2}$$

$$dh = c(z-a)\frac{dz}{w} \tag{4.3}$$

The symmetries of $\mathcal{H}e_1$ force c to be purely imaginary and, by scaling, we may assume $c = i$. A natural residue condition for g to be well-defined gives $\rho^{-1} = \pm\frac{(a-\alpha)(\alpha-\beta)}{w(a)}$, while $\frac{dg}{g}$ must have period $2\pi in$, $n \in Z$ on nontrivial cycles in M. By symmetry conditions there is only one essential cycle, which may be considered to lie in the z-plane. We choose $n = \pm 1$, which turns out to work, but is not forced. (Other values may work, too.) This gives additional constraints. To see how to make the Weierstrass integral (3.3) well-defined on the Riemann surface in (4.1), we may agrue geometrically. Beginning integration at $\lambda \in \mathrm{C}$ automatically maps this point to the origin $\vec{0} \subset \mathrm{R}^3$. We want the image of i to be a fixed-point of rotation about the x_1-axis. That is

$$Re \int_{\lambda}^{i} dh = 0\,, \tag{4.4}$$

$$Re(i \int (g^{-1} + g) dh) = 0\,. \tag{4.5}$$

Given $\lambda > 0$, i.e. knowing the conformal type of $\mathcal{H}e_1$, we can use (4.4) to write a as a function of λ, and the period condition on $\frac{dg}{g}$ to determine α and β. Thus (4.5) becomes a single real condition on $\lambda > 0$, now considered a variable. For $\lambda \cong -.32$ we get a solution.

5 Computation

To produce the pictures and find strong experimental evidence that these surfaces exist, MESH [1, 7] was used. Computational programs were used to solve the period problems inherent in these representations. A proof for the existence of $\mathcal{W}_1$ can be given using a degree theory argument for the period mapping.

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