The Geometry of Periodic Minimal Surfaces

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1 Introduction.

In this paper we shall demonstrate a surprising relationship between the topology of a properly embedded periodic minimal surface in R^3 and its global geometry. We shall call a minimal surface *periodic* if it is connected and invariant under a group G of isometries that acts freely on R^3. We will analyze these surfaces by studying their quotient in R^3/G. We have already carried out this study for doubly-periodic minimal surfaces [16].

Recall that a surface has *finite topology* if it is homeomorphic to a closed surface with a finite number of points removed. Our main theorem is:

Theorem 1 *A properly embedded minimal surface in a complete nonsimply connected flat three-manifold has finite total curvature if and only if it has finite topology.*

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When the flat manifold is R^3, the existence of the helicoid (which has finite topology and infinite total curvature) demonstrates that the condition that N be nonsimply connected is a necessary one.

Theorem 1 has important topological and analytical consequences. One topological consequence is that a properly embedded orientable minimal surface of finite topology in an orientable flat nonsimply connected three-manifold always has an even number of ends or it is a plane (see Theorem 9 in Section 9).

A theorem of Huber [10] states that a complete Riemannian surface with nonpositive Gaussian curvature whose total curvature is finite must be conformally diffeomorphic to a closed Riemann surface punctured in a finite number of points. We will prove that a complete minimal surface of finite total curvature in a flat three-manifold can be described in terms of meromorphic data on its conformal compactification. We shall exploit these analytic conditions to prove the following uniqueness theorem.

Theorem 2 *The plane and the helicoid are the only properly embedded simply connected minimal surfaces in* R^3 *with infinite symmetry group.*

In [16] we proved Theorem 1 in the case where the flat three-manifold N was isometric to the product $\mathrm{T} \times \mathrm{R}$ where T is some flat torus. In fact we proved that a properly embedded minimal surface M in $\mathrm{T} \times \mathrm{R}$ has finite total curvature $C(M) = 2\pi\chi(M)$. It follows from the classification of flat three-manifolds that a flat, noncompact, nonsimply-connected three-manifold is finitely covered by $\mathrm{T} \times \mathrm{R}$ or by R^3/S_θ where S_θ is the right hand screw motion obtained by rotation around the positive x_3-axis by θ followed by a nontrivial translation along the x_3-axis. Thus, to prove Theorem 1, it remains to consider only the case where the manifold N is isometric to R^3/S_θ for some θ, $0 \leq \theta \leq \pi$. However, our proof of Theorem 1 will not actually depend on our previous theorem in the special case of $\mathrm{T} \times \mathrm{R}$.

We have the following classification of the annular ends of the surfaces described in Theorem 1. As shown by work in [2, 3, 11, 12, 13], every R^3/S_θ has many examples with each possible end type.

Theorem 3 *An annular end of a properly embedded minimal surface of finite topology in R^3/S_θ is asymptotic to a plane, a flat vertical annulus, or to an end of a helicoid (with horizontal limit tangent plane). If θ is nonzero and the end is asymptotic to a plane, then the plane is horizontal. If θ is irrational, then the end is not asymptotic to a flat vertical annulus.*

The total curvature of minimal surfaces of finite topology in $N = \mathrm{R}^3/S_\theta$ can be computed in terms of the winding numbers of its annular ends. Suppose A is the image of a proper embedding of the punctured disk D^* in N. Let γ be the geodesic representing the image of the x_3-axis in N. After removing a compact neighborhood of ∂A, we may assume that A is disjoint from the ε-tubular neighborhood T of γ with boundary torus ∂T. The torus is obtained as a quotient by S_θ of the flat cylinder C of distance ε from the x_3-axis. A basis for $\pi_1(\partial T)$ is obtained from the quotient α of the oriented circle $C \cap \mathrm{R}^2$ and the quotient β of the oriented right handed helical arc of least-length on C joining a point p with $S_\theta(p)$. The boundary curve of A is homotopic in $N - \gamma$ to a unique element of $\pi_1(\partial T)$. Suppose ∂A is homotopic to $n\alpha + m\beta$. The *winding number* of the end of A is then defined to be $\frac{1}{2\pi}|2\pi \cdot n + m \cdot \theta|$. If M is a complete embedded minimal surface of finite total curvature in R^3/S_θ, then define the *total winding number of M* to be the sum of the winding numbers of the ends of M. We let $W(M)$ denote the total winding number of M.

Theorem 4 *If M is a properly embedded minimal surface of finite topological type in R^3/S_θ, then the total curvature of M is*

$$C(M) = 2\pi(\chi(M) - W(M)).$$

When the ends are asymptotic to flat vertical annuli, this formula yields $C(M) = 2\pi\chi(M)$. When there are k planar ends, $C(M) = 2\pi(\chi(M) - k)$.

The paper is organized as follows. In Section 2 we develop the analytic theory of complete minimal surfaces of finite total curvature in R^3/S_θ when $\theta = 0$ and in Section 3 we consider the case when $\theta \neq 0$. The main theorem of Section 3 is a Weierstrass-type analytic representation for a complete minimal surface M of finite total curvature in R^3/S_θ. In particular we show that these minimal surfaces are conformally equivalent to a closed Riemann surface $\overline{M}$ punctured in a finite number of points and that the coordinates of M can be recovered from two meromorphic one-forms on $\overline{M}$. In Section 4 we characterize the asymptotic behavior of properly embedded minimal annuli of finite total curvature in R^3/S_θ and prove some global results on their geometry including the main reduction of the proof of Theorem 2 from Theorem 1. In Section 5 we prove a multi-valued version of Picard's theorem that was used in the earlier Section 3. In Section 6 we prove that an annular end A of a properly embedded minimal surface in R^3/S_θ is trapped between two embedded minimal annuli of finite total curvature. In Sections 7 and 8 we use the result of Section 6 to show that A must have finite total curvature. The proof that A has finite total curvature breaks up into two cases depending on the asymptotic behavior of the finite total curvature annuli that trap it. This result on A proves Theorem 1. The remaining theorems are proved in Section 9.

We refer the reader to [3] and to [8] for related theoretical results.

2 Finite total curvature annular ends in $\mathbf{R}^3/T$.

In this section we will analytically parametrize embedded finite total curvature ends A in $N = \mathrm{R}^3/T$, where T is the group generated by translation by

$v \in \mathrm{R}^3$.

Let E denote a connected lifting of A to R^3. The Weierstrass data of E (Gauss map and holomorphic one-form) are invariant by T, hence, pass to a Gauss map on A and holomorphic one-form ω. As usual g denotes the composition of the Gauss map with stereographic projection to $\mathrm{C} \cup \{\infty\}$. Since A has finite total curvature, A is conformally the punctured disk $D^* = \{z \in \mathrm{C} \mid 0 < |z| \leq 1\}$ and (g, ω) extend to meromorphic data at the origin. This last fact is well known, however in Section 3, we will prove a more general result.

Now, after a rotation of E in R^3, we can assume $g(0) = 0$, and after a conformal reparametrization (a subend of) of A we have:

$$g(z) = z^p, \ \omega(z) = (\frac{c_{-q}}{z^q} + \frac{c_{-q+1}}{z^{q-1}} + \cdots)\, dz\,, \tag{2.1}$$

where $p \geq 1$.

The period vector is $v = \mathrm{Re} \int_{S^1} \phi$, where

$$\phi = (\phi_1, \phi_2, \phi_3) = (\frac{(1-g^2)}{2}, \frac{i(1+g^2)}{2}, g)\omega.$$

A (multi-valued) parametrization of E is given by $\widetilde{x}(z) = \mathrm{Re} \int^z \phi$. We have $\widetilde{x}(r, \theta + 2n\pi) = \widetilde{x}(r, \theta) + nv$, $z = re^{i\theta}$.

The case $v = 0$ is well known [23]. A is then an embedded finite total curvature annulus in R^3 and is asymptotic to a plane or catenoid. Henceforth, assume $v \neq 0$. The analysis of the type of A is determined by the order q of the pole of ω and the coefficient c_{-q}.

Theorem 5 *Let (g, w) be as in equation 2.1 and assume $v \neq 0$. The nature of the end A is determined by p and q as follows.*

1. *If $q = 1$, A is a Scherk-type end (this is made explicit shortly);*

2. if $q > 1$, then A is embedded only if $q = p+1$. When $q > p+1$, the trace of A on a large cylinder is not embedded (this is Toubiana's lemma). If $q < p+1$, a translation by a large horizontal period will give a self-intersection point.

Proof. First suppose $q = 1$. We will see that A is a Scherk type end (e.g. the ends of Scherk's surface $g(z) = z$, $\omega = i\,dz/(z^4-1)$, on the sphere punctured at the four roots of unity) and converges to a flat annulus. Let $a = c_{-q}$. Then

$$g(z) = z^p,\ \omega = (\frac{a}{z} + f(z))\,dz, \quad \text{and}$$
$$\phi_3(z) = (az^{p-1} + h(z))\,dz,\ f,\ h \ \text{ holomorphic in } \ z \in D.$$

Hence ϕ_3 is holomorphic at 0, v is a horizontal vector, and $x_3(z)$ converges to a constant (which we take to be 0) as $z \to 0$.

After a rotation about the x_3-axis, we can assume $v = (-2\pi \ \mathrm{Im}\ a, 0, 0)$, $a \in i\,\mathrm{R}$. Then

$$x_1 = -\frac{1}{2}a_0 \ \arg(z) + \mathcal{O}(1)\,,$$
$$x_2 = -\frac{1}{2}a_0 \ \ln\,|z| \ + \mathcal{O}(1)\,,$$
$$x_3 = \frac{1}{p}\ \mathrm{Re}\,(az^p) + \mathcal{O}(1),\ a_0 = \mathrm{Im}\,(a)\,,$$
$$x_1 - i\,x_2 = \frac{a}{2}\ \log(z) + \mathcal{O}(1)\,.$$

Here $\mathcal{O}(1)$ denotes a function continuous at $z = 0$. Henceforth, we will let $\mathcal{O}_0(1)$ denote such a function that vanishes at 0.

The image by $\tilde{x}$ of the line $\theta = 0$, $0 < r \le 1$, is a curve asymptotic to a line parallel to the x_2-axis: $x_3 = 0$, $x_1 = c_1 = \lim_{r\to 0} x_1(r, 0)$. The image of the line $\theta = 2\pi$, $0 < r \le 1$, is the same curve translated by v. So a fundamental domain in R^3 is a half band bounded by these two curves. The surface is asymptotic to the flat annulus $x_3 = 0,\ x_2 > 0$. If we think of v as vertical, then A is asymptotic to a vertical flat annulus.

Now suppose $q > 1$. E. Toubiana has proved that A embedded implies $q \le p + 1$ [26]. He proves this by showing the trace of A on a large cylinder centered at the x_3-axis, is not embedded when $q > p + 1$.

We now show that $q = p + 1$. Assume the contrary: $q < p + 1$. Then $\phi_3(z)$ is holomorphic at 0 and $x_3(z)$ tends to a constant as $z \to 0$, which we take to be zero. Notice that x_3 changes sign on every circle $|z| = r > 0$, since it is harmonic on the disk. Also the period vector v is horizontal since the residue of ϕ_3 is zero at 0. We have

$$(x_1 + i\,x_2)(z) = \int \overline{\omega} - \int g^2\omega = \int \overline{\omega} + \mathcal{O}(1) = \frac{1}{\overline{z}^{q-1}}\,(\frac{\overline{a}}{q-1} + \mathcal{O}_0(1))\,,$$

$$a = c_{-q},\ R = \sqrt{x_1^2 + x_2^2} = \frac{1}{r^{q-1}}\,(\frac{|a|}{q-1} + \mathcal{O}_0(1))\,.$$

As $z \to 0$ along a ray $\arg(z) =$ constant, $(x_1 + i\,x_2)(z)$ is asymptotic to a straight line and $\widetilde{X}(z)$ is an embedded curve in R^3 whose projection on the (x_1, x_2)-plane is $(x_1 + i\,x_2)(z)$ and whose x_3-coordinate tends to zero. For r fixed, $r \ne 0$, the total change of the argument of $(x_1, +i\,x_2)(z)$, as z traverses once the circle $|z| = r$ is $(q-1)2\pi + \mathcal{O}_0(1)$, by the above formula for $(x_1, +ix_2)(z)$.

Now consider the surface $M_0 = \widetilde{X}\,(0 \le \arg(z) \le 4\pi,\ 0 < r \le 1)$. Let T_R denote the vertical cylinder of radius R, centered at the x_3-axis. For $R_\circ$ large, M_0 projects surjectively onto the complement of the disk $D_{R_\circ} = \{x_1^2 + x_2^2 \le R_\circ^2\}$. (In fact, 4π can be replaced by $2\pi + \varepsilon$ for any $\varepsilon > 0$ for this projection to be surjective.)

Choose (r, θ) so that $\delta = x_3(\widetilde{X}(r, \theta)) > 0$; for convenience, take $\theta = 0$. Let α be the arc $\widetilde{X}(r, \theta)$, $0 \le \theta \le 2\pi$; notice that the endpoints of α differ by v and so have the same x_3-coordinate. There are points on α above and below the (x_1, x_2)-plane, since x_3 changes sign on $|z| = r$. Let $\varepsilon = 1/2 \min\{\delta, |\min x_3|\alpha|\}$.

Now consider the surfaces $M_k = \widetilde{X}\,(2k\pi \leq \arg(z) \leq (2k{+}4)\pi,\ 0 < r \leq 1)$. M_k is obtained from M_0 by horizontal translation by $2kv$. Let $R_1 > R_\circ$ be chosen so that on the complement of T_{R_1}, M_0 is at most a distance ε from the (x_1, x_2)-plane.

Choose k so that $kv > R_1$. The arc α on M_0 is translated horizontally to an arc $\widetilde{\alpha}$ on M_k by the translation by $2kv$. Outside of T_{R_1}, M_0 is at a height at most ε and M_0 projects surjectively onto the complement of D_{R_1}, so M_0 must intersect $\widetilde{\alpha}$. This contradicts A is embedded.

Thus, $q = p + 1$ and $x_3 = \alpha_0 \ln|z| + \beta \arg(z) + \mathcal{O}(1)$, $\alpha_0, \beta \in \mathrm{R}$. As before, we have $R = \sqrt{x_1^2 + x_2^2} = 1/r^{q-1}\ (|a|/(q-1) + \mathcal{O}_0(1))$ so $x_3 = \alpha \ln R + \beta \arg(z) + \mathcal{O}(1)$, $\alpha \in \mathrm{R}$. (Note that $\alpha \neq \alpha_0$.)

The trace of E on the cylinder S_R is converging to the helix $\alpha \cdot \ln R + \beta \cdot \arg(z)$, for $0 < \arg(z) < 2\pi$, hence E is embedded for R sufficiently large. Notice that the period vector v need not be vertical: e.g. $g(z) = z$, $\omega = (i/z^2 + 1/z)\,dz$. This is an embedded helicoidal type end with a non vertical axis v. If both α and β are non zero, then the helicoidal end has a logarithmic growth as $R \to \infty$, given by α; just as the usual catenoid where $\beta = 0$. We shall see later that if the end A is part of a properly embedded surface M of finite topology and $\beta \neq 0$, then there is no logarithmic growth ($\alpha = 0$), v is vertical, and so A is asymptotic to a helicoid (see Theorem 3 in the Introduction).

In summary, the embedded ends in R^3/T are planar type (asymptotic to flat annuli), catenoid type, or helicoidal type. The latter ends are helicoidal ends which may have logarithmic growth and possibly an axis not orthogonal to the end. $\square$

3 Finite total curvature annular ends in $\mathbf{R}^3/\mathbf{S}_\theta$ and their Weierstrass Representation.

Let A be a finite total curvature minimal annulus, embedded in $N = \mathrm{R}^3/S_\theta$ where $0 < \theta < 2\pi$. In this section we will derive meromorphic data on the disk that parametrizes A, and describes its aymptotic behavior at ∞.

We take S_θ to be a translation along the x_3-axis followed by rotation by θ about the x_3-axis. Since A has finite total curvature, A is conformally the punctured disk D^*. We no longer have a single valued Gauss map g on A; g is a multi-valued meromorphic map on D^* whose values differ by multiplication by λ^m, $\lambda = e^{2\pi i\theta}$. To see this, let E be a connected lifting of the universal covering space of A to R^3. The Gaussian image of the normal vector to E at $p \in E$ and the image of the normal vector to E at $S_\theta(p)$, differ by rotation about the x_3-axis by θ. Hence, the stereographic projections of these vectors on the sphere, differ by rotation by θ in C, i.e., by multiplication by $\lambda = e^{2\pi i\theta}$.

Lifting g to the Riemann surface of g (i.e., the covering Riemann surface where g is defined), we have a well defined meromorphic map $\widetilde{g}$, on the half plane $H = \{x \leq 0\}$, satisfying $\widetilde{g}(z + 2\pi mi) = \lambda^m \widetilde{g}(z)$, for $z \in H$. Then $g = \widetilde{g}\,(\exp^{-1})$.

We wish to show that A has a limiting tangent plane at ∞, i.e., g extends continuously to 0 (even though g is multi-valued). This will follow from the fact that the area of the spherical image of g (i.e. a single valued branch of g on the slit punctured disk D') is finite (see Theorem 6 below). We are grateful to Dennis Sullivan for explaining the length-area inequality of conformal maps which is used repeatedly in the proof of Theorem 6.

Theorem 6 *Let g be a multi-valued meromorphic map on D^*, $g = \widetilde{g}\,(\exp^{-1})$, with $\widetilde{g}\,(z + 2\pi i) = \lambda \widetilde{g}\,(z)$, for $z \in H$, and some λ, $|\lambda| = 1$. If $Area(g(D'))$*

is finite, then g extends continuously to 0.

The proof of Theorem 6 will be postponed to Section 5.

Now we shall use Theorem 6 to obtain a Weierstrass representation on the disk D, for finite total curvature annuli A in $N = \mathrm{R}^3/S_\theta$. We use the notation of Section 2. By Theorem 6, the multi-valued g extends continuously to 0 and since the limiting value is fixed by multiplication by λ and $\lambda \neq 1$, the limiting value is 0 or ∞; so we can assume $g(0) = 0$. Write $\lambda = e^{2\pi i a}$ with $0 < a < 1$. Since $\tilde{g}(z + 2\pi i) = \lambda \tilde{g}(z)$, the map $z^{-a} g(z)$ is indeed single valued on D^*. Furthermore, $z^{1-a} g(z)$ is bounded in a neighborhood of 0, hence $g(z) = z^{a-1} h(z)$ where h is holomorphic in a neighborhood of 0. Hence, dg/g is a well defined meromorphic one-form on D^*, and 0 is a removable singularity. The multi-valued g on D, is obtained from this form by $g(z) = \exp(\int dg/g)$.

Next notice that ϕ_3 is a well defined holomorphic form on A. To see this let $x(u, v)$ be local conformal coordinates about a point $p \in E$. Then $\tilde{x}(u, v) = S_\theta(x(u, v))$ are local coordinates about $S_\theta(p)$ and $\tilde{x}_3(u, v) = x_3(u, v) + t_0$, t_0 the vertical translation component of S_θ. Hence,

$$\tilde{\phi}_3(u, v) = \frac{\partial \tilde{x}_3}{\partial u} - i\, \frac{\partial \tilde{x}_3}{\partial v} = \frac{\partial x_3}{\partial u} - i\, \frac{\partial x_3}{\partial v} = \phi_3(u, v).$$

Denote ϕ_3 by η. We claim 0 is a removable singularity of η: the metric on A is given by

$$ds = \frac{1}{2}(|g| + \frac{1}{|g|})\ |\eta|,$$

($|g|$ is well defined on A), and since the metric is regular and complete, $\eta \neq 0$ on A, and for γ a path on A tending to 0, we have $\int_\gamma ds = \infty$. Since $g(0) = 0$ and g is continuous at 0, this implies $\int_\gamma |\eta|/|g| = \infty$. Now there is an integer m such that $|g(z)| > |z|^m$ for $|z|$ small ($|g(z)|$ is of the order $|z|^{a+n}$ for $0 < a < 1$ and $n \geq 1$ an integer, so $m = 2n$ works). Then $|\eta|/|g| < |\eta|/|z^m|$, hence $\int_\gamma |\eta|/|z^m| = \infty$ for every path γ in D^* tending to 0. This implies 0 is a removable singularity of η/z^m, hence of η too, [19].

We take as Weierstrass data on A the pair $(dg/g, \eta)$; these forms are meromorphic at the puncture and A is obtained from this data by the formula $g = \exp\left(\int dg/g\right)$,

$$x(z) = \mathrm{Re} \int \left(\frac{1}{2g} - \frac{g}{2}, \frac{i}{2g} + \frac{ig}{2}, 1\right) \eta \,.$$

In particular, we have proved:

Theorem 7 *Let M be a complete finite total curvature minimal surface in R^3/S_θ. Then there exists a conformal compactification $\overline{M}$ of M, and meromorphic forms $(dg/g, \eta)$ on $\overline{M}$, such that M is parametrized by*

$$x(z) = \mathrm{Re} \int \left(\frac{1}{2g} - \frac{g}{2}, \frac{i}{2g} + \frac{ig}{2}, 1\right) \eta$$

$$\textit{where} \;\; g = \exp\left(\int \frac{dg}{g}\right).$$

Remark 3.1 *H. Karcher has given many new examples of such M with this data [11].*

4 Some global properties of finite total curvature M in $\mathbf{N} = \mathbf{R}^3/\mathbf{S}_\theta$.

Let M be a properly embedded minimal surface in N of finite total curvature. Since the lift of M to R^3 is orientable (since it is embedded) and M is invariant under S_θ, S_θ^2 acts on the lifted surface in an orientation preserving manner. Hence, after lifting to a two-sheeted covering space, we can assume M is orientable. We know M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a finite number of points. A neighborhood of a puncture in M is an embedded annulus of finite total curvature, hence the previous section applies and $(dg/g, \eta)$ is meromorphic on $\overline{M}$; in particular, each of the ends of M has a limiting normal vector.

Proposition 4.1 *Let A be a finite total curvature embedded minimal annulus (homeomorphic to $S^1 \times [0, \infty)$) in R^3/S_θ, with limiting normal vector $g(0)$. Then A is asymptotic to a plane, a catenoid, a flat annulus, or a helicoidal-catenoid type end. This means:*

i) *If $\theta \neq 0$, then $g(0)$ is parallel to the axis of translation of S_θ (the x_3-axis) and there are real numbers α, β such that (we assume A parametrized by D^*)*

$$x_3(z) = \alpha \ln R + \beta \arg(z) + \mathcal{O}(1),$$

$$\textit{where} \quad R = \frac{1}{|z|^{q+a}}\,(c + \mathcal{O}(1)),$$

q an integer ≥ 1, $2\pi a = \theta$, and c a real constant. If $\alpha = \beta = 0$, this is a planar end; if $\beta = 0$ and $\alpha \neq 0$, a catenoid type end; if $\beta \neq 0$, $\alpha = 0$, a helicoidal type end. And if $\alpha \neq 0$, $\beta \neq 0$, we call this a helicoidal-catenoid type end.

ii) *If $\theta = 0$ and $g(0)$ is not orthogonal to the axis of S_θ, then the same statement for $x_3(z)$ as in (i) holds, where x_3 is the coordinate parallel to $g(0)$. If $g(0)$ is orthogonal to the axis of S_θ, then A is a Scherk type end, asymptotic to a flat annulus.*

Definition 4.1 *An annular end as in Proposition* 4.1 *will be called a* standard end.

Proof of Proposition 4.1. Assume first, that $\theta \neq 0$, so that the limiting normal vector to A at infinity is parallel to the axis of translation of S_θ, i.e. the x_3-axis. Then we can assume $g(0) = 0$, A is parametrized by the punctured disk D^*. Let T_R be the torus tubular neighborhood of radius R, the radius R cylinder centered at the x_3-axis modulo S_θ. For R large, A intersects ∂T_R transversally in a simple closed curve $A(R)$. We have $x_3 = \operatorname{Re} \int \eta$ and Toubiana's lemma (more precisely: the proof of Toubiana's lemma [26]) implies that η has at most a pole of order one at 0, since $A(R)$ is embedded.

First suppose η is holomorphic at 0. Then $x_3(z)$ tends to a constant as $z \to 0$, x_3 is a well defined function on A, and A lifts to an embedded annular end in R^3. A is then a planar end, asymptotic to a horizontal plane.

Now suppose η has a pole of order one. Then

$$x_1 - i\,x_2 = \int \frac{\eta}{g} + \mathcal{O}(1),$$

$$R = \frac{1}{r^{q+a}}\,(c + \mathcal{O}(1)),$$

where c is a real constant, q an integer greater than or equal to one and $a = \frac{\theta}{2\pi}$. Thus,

$$x_3 = \alpha\,\log\,R + \beta\,\arg\,(z) + \mathcal{O}(1),$$

for some real constants α, β. For R large, $A(R)$ is approximately the curve $\alpha \cdot \log\,R + \beta \cdot \arg\,(z)$. This is a horizontal circle for $\beta = 0$ and a helix for $\beta \neq 0$. When $\beta = 0$, A is a catenoid type end, $x_3 = \alpha \cdot \log R + \mathcal{O}(1)$. When $\beta \neq 0$, A is a helicoid type end with a logarithmic growth rate term; a (rather clumsy) appropriate name for these ends is *helicoidal–catenoid* type end.

When $\theta = 0$, so S_θ is translation by a vector v, the asymptotic behavior of A was analyzed in Section 2. We chose as x_3-axis the limiting normal to the end and we found A was also a planar, catenoid or helicoidal–catenoid type end. We remark, that one also had $x_3 = \alpha \cdot \log R + \beta \cdot \arg(z) + \mathcal{O}(1)$ except when the translation vector v was orthogonal to $g(0)$ (this was the case $q = 1$, a Scherk type end). $\square$

Now we return to our globally embedded M in R^3/S_θ. We will now prove that all the ends of M are of the same type, with the same coefficients α and

$|\beta|$ of $\ln R$ and $\arg(z)$. First assume that the translation vector v is vertical. If the α's of two ends of M were distinct, then the distance between the lifted trace curves $A(R)$ to R^3, would tend to ∞ as $R \to \infty$. So for some value of R, they would intersect on the torus ∂T_R. Hence M would not be embedded. If two $|\beta|$'s were not the same, then one would have helices of different slope on ∂T_R and they would intersect. Notice the β's can be of opposite sign and equal (as for the standard helicoid). As θ increases from 0 to 2π, $\beta > 0$ yields a helix with x_3-coordinate increasing and $\beta < 0$ a helix going down, i.e., x_3 decreasing. If v is not vertical, let T_R denote the torus tubular neighborhood of radius R around a translation axis of M. Note that if $\alpha \neq 0$, v is not horizontal. The above discussion shows that if the $\alpha's$ of two ends of M were distinct, then the trace curves $A(R)$ on ∂T_R would intersect for some large values of R. If two $|\beta|'$s were not the same, then the trace curves are not homotopic on $\partial T_R/S_\theta$ and so must intersect.

Proposition 4.2 *Let $A_1, \ldots, A_n$ be the ends of M and $(\alpha,\, \beta_i)$ the coefficients of $\log R$ and $\arg(z)$ at each end. Then $\sum_{i=1}^n \beta_i = 0$ and $\alpha = 0$. In particular, there are an even number of ends when $\beta_i \neq 0$ for all i.*

Proof. Let $M_0 = M - \cup_{i=1}^n \operatorname{Int}(A_i)$; M_0 is a compact surface with one boundary component ∂A_i coming from each A_i. Consider the holomorphic form η on M_0. We have:

$$0 = \int_{M_0} d\eta = \int_{\partial M_0} \eta = \sum_{i=1}^n \int_{\partial A_i} \eta = 2\pi i \sum_{i-1}^n c_i,$$

where at the end A_i, $\eta(z) = (c_i/z + \mathcal{O}(1))\, dz$. Since $x_3 = \operatorname{Re} \int \eta$, we have $\alpha = \operatorname{Re}(c_i)$ and $\beta_i = -\operatorname{Im}(c_i)$. Hence, $\alpha = 0$ and $\sum_{i=1}^n \beta_i = 0$. ☐

E. Toubiana proved that an embedded minimal two punctured sphere in R^3/T, T a translation, is a helicoid, provided the total curvature is finite

[26]. The question naturally arises whether this remains true in screw motion spaces. The answer is affirmative.

Theorem 8 (Toubiana's theorem in R^3/S_θ.) *Let M be a complete embedded minimal annulus of finite, nonzero, total curvature in R^3/S_θ. Then M is a helicoid.*

Proof. M is conformally $\mathrm{C}-\{0\}$ and of finite total curvature. We have the meromorphic data $(dg/g, \eta)$ on the Riemann sphere, that parametrizes M. We write $\eta = g\,\omega$,

$$g(z)z^{-a} = \frac{P(z)}{Q(z)}, \quad \omega(z)z^{a} = c\,\frac{Q^2(z)}{z^n}\,dz,$$

were $0 < a < 1$, P and Q polynomials, relatively prime. Parametrize M so that the normal vectors are vertical at the punctures. We can assume $g(0) = 0$, so $Q(0) \neq 0$, (we know the limiting normal vectors are vertical).

By Corollary 1 in [9], a nontotally geodesic properly embedded orientable surface in a flat orientable three-manifold N must separate N, so since $g(0) = 0$, we have $g(\infty) = \infty$. That is, $M \cap \partial T_R$ consists of two embedded curves that separate ∂T_R into 2 components, and the normal vector to M in N always points into the same component of $N - M$. Now Toubiana's lemma [26] implies that if $P(z) = z^m\,P_1(z)$, $m \geq 0$, then $n \leq m+1 \leq p+1$, $p = \deg P$.

Consider the end of M, where $g(\infty) = \infty$. We rotate R^3 by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

so that the $(\tilde{g},\, \tilde{\omega})$ at the rotated end satisfy:

$$\tilde{g} = \frac{\widetilde{\phi_3}}{\tilde{\omega}} = -\frac{1}{g}, \quad \tilde{\omega} = \widetilde{\phi_1} - i\,\widetilde{\phi_2} = g^2\omega.$$

Then

$$\widetilde{g}(z)\, z^a = \frac{Q(z)}{P(z)}, \;\; \widetilde{\omega}(z)\, z^{-a} = c\, \frac{P^2(z)}{z^n}\, dz,$$

and ∞ is a zero of order $\deg P - \deg Q$ of $\widetilde{g}$. Also $(P^2(z)/z^n)\, dz$ has a pole at ∞ of order $2 \cdot \deg P + 2 - n$, so by Toubiana's lemma:

$$2 \cdot \deg P + 2 - n \leq \deg P - \deg Q + 1.$$

Thus,

$$\deg P + \deg Q + 1 \leq n \leq m + 1 \leq \deg P + 1.$$

Hence $\deg Q = 0$ and $P(z) = c_1 z^m,\;\; m = n-1,\; c_1 \in$ C. This gives

$$z^{-a} g(z) = z^{n-1}, \;\; z^a \omega = \frac{dz}{z^n},$$

which is a helicoid. □

5 A multi-valued Picard's Theorem.

We now prove the Theorem 6 stated in Section 3. The length-area inequality will be applied to on the one hand to the circles of an annulus and also to the radial lines of an annulus.

Let D' be the punctured disk D^*, slit along $\theta = 0$. Suppose g is meromorphic on D' and $g(r, 0) = \lambda\, g(r, 2\pi)$ for $0 < r \leq 1$. Let $C(r) = \{z \in D' \mid |z| = r\}$, and $\ell(r) =$ the length of $g(C(r))$. Then

$$\ell(r) = \int_0^{2\pi} \mid dg(T_r) \mid d\theta = r \int_0^{2\pi} \sqrt{J(g(z))}\, d\theta,$$

where $T_r = r(\cos\theta, \sin\theta)$, and $J(g)$ is the Jacobian of g.

We have: $\ell(r)^2/r^2 \leq \; (\int_0^{2\pi} J(g)\, d\theta) \cdot (2\pi)$, so integrating with respect to $r\, dr$ we obtain:

$$\int_0^r \frac{\ell(r)^2}{r}\, dr \leq 2\pi \int_0^r \int_0^{2\pi} J(g)\, r\, dr\, d\theta \leq 2\pi\, (Area(g(D'))) < \infty.$$

Hence there exists a sequence $r_n \to 0$ such that $\ell(r_n) \to 0$. Let $B(n)$ be the curve $g(C(r_n))$. First consider the case $\lambda = 1$ (this means g is single valued on D^*) so each $B(n)$ is a closed curve, and $\ell(B(n)) \to 0$ as $n \to \infty$.

We show that all the $B(n)$ accumulate at the same point. This shows g extends continuously to zero since by the open mapping property of g, the annulus between $C(r_n)$ and $C(r_{n+1})$ gets sent close to this accumulation point as well (otherwise the image of this annulus would cover almost the whole sphere).

Assume, on the contrary, that $B(n_{2i})$ accumulates at p and $B(n_{2i+1})$ accumulates at q, with $p \neq q$. Then the annulus between $C(r(n_{2i}))$ and $C(r(n_{2i+1}))$ gets sent to the region of the sphere between $B(n_{2i})$ and $B(n_{2i+1})$, by the open mapping property of g. It follows that the spherical image of this annulus has area at least 2π. Since this holds for an infinite sequence of annuli, tending to 0, the area of the image of g would be infinite. Thus $p = q$ and the theorem is proved when $\lambda = 1$.

Now suppose $\lambda \neq 1$. The endpoints of $B(n)$ differ by multiplication by λ and since the lengths of the $B(n)$ tend to zero, the only possible accumulation points of the $B(n)$ are 0 and ∞.

First suppose the $B(n)$ accumulate at both 0 and ∞. We will show this is impossible by showing the image of D' by g would have infinite area. So assume the sequence $r_1 > r_2 > \cdots > r_n > \cdots$, satisfies: $B(n_{2i})$ tends to 0, $B(n_{2i+1})$ tends to ∞ and $r_i \to 0$. We will derive a contradiction by showing the image by g of the annular region between $C(r_{2i})$ and $C(r_{2i+1})$ has definite spherical area, thus the area of the image of g would be infinite.

For notational convenience, let $r_1 = r_{2i}$, $r_2 = r_{2i+1}$, $C_1 = C(r_{2i})$, $C_2 = C(r_{2i+1})$, and let $B_1 = g(C_1)$ be a short curve near 0 and $B_2 = g(C_2)$ be a short curve near ∞. Let F be the annulus on the Riemann sphere S^2 bounded by the tropic of cancer ($= F_2$) and the tropic of capricorn ($= F_1$).

B_1 is in the disk on S^2 below F_1 and B_2 in the disk above F_2.

For each θ, $0 \le \theta \le 2\pi$, let $r_1(\theta)$, $r_2(\theta)$ be chosen in $[r_1, r_2]$ so that for $r_1(\theta) \le r \le r_2(\theta)$, $g(r, \theta) \in F$, and $g(r_1(\theta), \theta) \in F_1$, $g(r_2(\theta), \theta) \in F_2$. This is possible since $g(r_1, \theta) \in B_1$ and $g(r_2, \theta) \in B_2$.

Let $\alpha_\theta(r) = r(\cos\theta, \sin\theta)$ and $L_\theta = \text{length } (g(\alpha_\theta))$ for $r_1(\theta) \le r \le r_2(\theta)$. We have $L_\theta \ge \pi/4$ for each θ.

Now do a length-area calculation:

$$L_\theta = \int_{r_1(\theta)}^{r_2(\theta)} |dg(\alpha_\theta^{'})|\, dr = \int_{r_1(\theta)}^{r_2(\theta)} \sqrt{J(g)}\, dr = \int_{r_1(\theta)}^{r_2(\theta)} \sqrt{r}\sqrt{J}\ \frac{1}{\sqrt{r}}\, dr,$$

$$L_\theta^2 \le (\int_{r_1(\theta)}^{r_2(\theta)} J(g) r\, dr)(\int_{r_1(\theta)}^{r_2(\theta)} \frac{dr}{r}) \le (\int_{r_1(\theta)}^{r_2(\theta)} J(g) r\, dr) \times K,$$

$$K = \sup_\theta \ln(\frac{r_2(\theta)}{r_1(\theta)}) \text{ (the sup exists since } \ln(\frac{r_2(\theta)}{r_1(\theta)}) \le \ln(\frac{r_2}{r_1})).$$

Now integrate with respect to θ:

$$\int_0^{2\pi} L_\theta^2\, d\theta \le K \int_0^{2\pi} \int_{r_1(\theta)}^{r_2(\theta)} J(g) r\, dr\, d\theta$$

$$\le\ K Area\, (g(D_0^{'})),$$

where $D_0^{'}$ is the slit annulus between $C(r_1)$ and $C(r_2)$. Since $L_\theta \ge \frac{\pi}{4}$, this yields

$$\frac{\pi^3}{8K} \le Area(g(D_0^{'})).$$

Now our previous length-area calculation for $\ell_r = length(g(|z| = r))$ yielded:

$$(\frac{\ell_r}{r})^2 \le 2\pi \int_0^{2\pi} J(g)\, d\theta.$$

Hence for θ fixed:

$$\int_{r_1(\theta)}^{r_2(\theta)} (\frac{\ell_r}{r})^2 r\, dr \le 2\pi \int_{r_1(\theta)}^{r_2(\theta)} \int_0^{2\pi} J(g) r\, dr\, d\theta \le 2\pi \cdot Area\, (g(D_0^{'})).$$

Now, for $r_1(\theta) \le r \le r_2(\theta)$, $g(r,\theta) \in F$, so there exists a constant c such that $l_r \ge c > 0$ (the endpoints of $g(|z| = r)$ differ by multiplication by λ, so in F their distance is uniformly bounded from below). Thus the last integral inequality yields

$$c^2 \ln\left(\frac{r_2(\theta)}{r_1(\theta)}\right) \le 2\pi \cdot Area\,(g(D_0^{'})).$$

This holds for all θ so:

$$c^2 K \le 2\pi \cdot Area\,(g(D_0^{'})).$$

Multiply this with the inequality $\pi^3/8K \le \; Area\,(g(D_0^{'}))$, to get:

$$Area(g(D_0^{'})) \ge \frac{\pi \cdot c}{4},$$

and this contradicts our finite area hypothesis and completes the proof of Theorem 1 in the case that the $B(n)$ accumulate at both 0 and ∞.

It remains to consider the case when all the $B(n)$ accumulate at one of the points, 0 say. If g is not continuous at 0, then there is a sequence $x_n \to 0$ with $g(x_n) \to q$, and $q \ne 0$. Each x_n is in an annulus A_n, bounded by circles $C(r_n)$, $C(r_{n+1})$, which get sent by g to short curves near 0. We can suppose q is on the boundary of a disk E (which we take to be of radius one for convenience) centered at 0, and all the circles in ∂A_n get sent by g into the disk centered at 0 of radius $\frac{1}{4}$. Let F be the annulus in E bounded by the circle F_1, of radius $\frac{1}{2}$, and the circle F_2 of radius $\frac{3}{4}$.

Fix an annulus A_n and for notational convenience let r_1 be its inner radius and r_2 its outer radius, $C_1 = C(r_1)$, $C_2 = C(r_2)$, $B_1 = g(C_1)$, $B_2 = g(C_2)$. Let g be well defined on A_n slit along $\theta = \theta_0$. Notice that if $g(|z| = r)$ is contained in a disk of radius R centered at 0, then any other determination of the multi-valued g has the same property, since two determinations differ by multiplication by λ^m, for some m, and $|\lambda| = 1$.

Now $x_n \in A_n$ has polar coordinates $(r(x_n), \theta(x_n))$. Recall that $g(x_n)$ converges to q, so we can suppose $g(x_n)$ is not in the disk $D(\frac{3}{4})$, of radius $\frac{3}{4}$, centered at 0. Consider the image by g of the radial segment $\alpha(\theta(x_n))$ in A_n joining $(r_1, \theta(x_n))$ to $(r_2, \theta(x_n))$. Since the extremities of this segment get sent to points in $D(\frac{1}{4})$, and $g(x_n) \notin D(\frac{3}{4})$, there are $r_1(\theta(x_n))$, $r_2(\theta(x_n))$, such that:

$$r_1 < r_1(\theta(x_n)) < r_2(\theta(x_n)) < r_2,$$

$$g(r_2(\theta_n), \theta(x_n)) \in F_2, \; g(r_1(\theta(x_n)), \, \theta(x_n)) \in F_1,$$

and

$$g(r, \theta(x_n)) \in F, \text{ for } \; r_1(\theta(x_n)) \leq r \leq r_2(\theta(x_n)).$$

Observe now, that for any θ between θ_0 and $\theta_0 + 2\pi$, the same property holds, i.e., there are $r_1(\theta)$, $r_2(\theta)$ such that:

$$r_1 < r_1(\theta) < r_2(\theta) < r_2, \quad g(r_2(\theta), \theta) \in F_2, \quad g(r_1(\theta), \theta) \in F_1,$$

and

$$g(r, \theta) \in F \text{ for } \; r_1(\theta) \leq r \leq r_2(\theta).$$

For if this failed to hold, then for some θ, the image by g of the radial segment $\alpha_\theta(r) = r(\cos\theta, \sin\theta)$, $r_1 \leq r \leq r_2$, would be contained entirely in $D(\frac{3}{4})$. Then let D_1 be the disk obtained from A_n by cutting A_n along α_θ, and let g_1 be a single-valued branch of g on D_1. The boundary of D_1 gets sent into $D(\frac{3}{4})$ by hypothesis, so D_1 gets sent into $D(\frac{3}{4})$ or $S^2 - D(\frac{3}{4})$ is in the image of D_1 by the open mapping theorem. In the latter case, the image has area at least 2π. In the former case, we conclude all the determinations of g on A_n get sent into $D(\frac{3}{4})$ and this contradicts $g(x_n) \notin D(\frac{3}{4})$. So for each θ, we have $r_1(\theta)$, $r_2(\theta)$ as desired.

Now do the two length-area calculations, just as in the case when 0 and ∞ were accumulation points of the $B(n)$. The same reasoning shows there is a constant $C > 0$ such that $Area(g(A_n)) \geq C$ for all n. This completes the proof of Theorem 6. $\Box$

6 The trapping lemma for embedded minimal annuli.

Before stating the Trapping Lemma, we prove a topological property for properly embedded surfaces in $N = \mathrm{R}^3/S_\theta$.

Lemma 6.1 *Let M be a properly embedded surface in N that separates N and has at least one annular end. Then there is a finite covering $p\colon \widetilde{N} \to N$ such that $p^{-1}(M)$ has more than one end.*

Remark 6.1 *If M is an orientable minimal surface, not a plane, then M separates N (cf. Corollary 1 in [9]).*

Proof. Let A be an annular end of M. If $\pi_1(A) \to \pi_1(N)$ is not an isomorphism, then one can clearly lift M to a covering space so that the lifted surface has more than one end. So suppose it is an isomorphism. Choose R so that ∂T_R intersects A transversally; ∂T_R is the torus in N which is all points a distance R from the x_3-axis. Also suppose $\partial A \subset T_R$. Since $\partial A \subset T_R$ and the end of A lies outside T_R, $A \cap \partial T_R$ contains an odd number of simple closed curves, each a generator of $\pi_1(\partial T_R)$, and perhaps some null homotopic cycles. In particular, there is a cycle β on ∂T_R whose intersection number with A is odd. Since M separates N, the intersection number of β and M is zero. Thus M must have other ends. $\Box$

Lemma 6.2 (The Trapping Lemma) *Let T_R denote the image in $N = \mathrm{R}^3/S_\theta$ of the solid vertical cylinder of radius R in R^3 around the x_3-axis.*

Suppose A is an annular end of a properly embedded minimal surface in N with more than one end. Then for some $R > 0$, there exist two disjoint standard ends E_1, E_2 (hence of the same type, see Definition 4.1) that satisfy:

1. $(E_1 \cup E_2) \cap T_R = \partial E_1 \cup \partial E_2 \subset \partial T_R$ *;*
2. $E_1 \cup E_2$ *separates* $N - T_R$ *into* 2 *components* C_1, C_2 *;*
3. *A has an annular end $A' \subset A$ with $A' \subset C_1$ or $A' \subset C_2$.*

Proof. If A has finite total curvature, we have shown in Proposition 4.1 that A is asymptotic to a standard end E. A vertical translation is well defined in N and so E_1 and E_2 can be obtained by small vertical (up and down) translations of E. Assume now that A has infinite total curvature. By Remark 6.1, M separates N into two components whose closures we denote by C and C'. Since ∂A is not homologous to zero in M, it can not be homologous to zero (with Z_2-coefficients) in both C and C', (since $H_2(N) = 0$). Assume that ∂A represents a nontrivial class in C.

By [6]), a stable minimal surface with compact boundary in a flat orientable three-manifold has finite total curvature. In particular A contains a compact subdomain that is unstable. After replacing A by a subend that is disjoint from this unstable compact domain, we may assume that ∂A disconnects ∂C into two unstable minimal surfaces.

Choose an exhaustion $F_1 \subset F_2 \ldots$ of A by smooth compact subannuli with $\partial A \subset \partial F_1$. Since ∂C has nonnegative mean curvature, every smooth 1-cycle Γ in C that bounds in C is the boundary of an embedded least-area surface in C (see Theorem 1 in [18] and also [24]). In particular ∂F_i is the boundary of a smooth embedded surface Σ_i in C that is least area and Z_2-homologous to F_i $rel(\partial F_i)$. Since C is orientable and $F_i \cup \Sigma_i$ is a Z_2-boundary in C, Σ_i is orientable. The usual compactness and regularity theorems for least-area surfaces (see [24]) imply that a subsequence of the Σ_i converge to a least-area orientable surface $\Sigma \subset C$ with $\partial\Sigma = \partial A$. Since Σ is stable and both components of $\partial C - \partial A$ are unstable, all three are different so the maximum principle implies that $\Sigma \cap \partial C = \partial\Sigma$.

The surface Σ separates C into two components; let $\widetilde{C}$ be the component containing A. Let $\widetilde{A}$ be a proper annular subend of A. As for A in C solve the Plateau problem for $\partial\widetilde{A}$ in $\widetilde{C}$ to obtain a stable minimal surface $\widetilde{\Sigma}$ in $\widetilde{C}$ with boundary $\partial\widetilde{A}$. Clearly $\widetilde{\Sigma}$ is disjoint from Σ. Since Σ and $\widetilde{\Sigma}$ are stable, they have finite total curvature. Let E_1', E_2' be annular ends of $\Sigma, \widetilde{\Sigma}$, respectively. Since E_1' and E_2' are standard and disjoint, Proposition 4.1 implies that for R large, $E_i = E_i' \cap (N - \mathrm{Int}(T_R))$ for $i = 1, 2$, are disjoint standard ends. It follows directly from the asymptotic properties of standard ends in N that $E_1 \cup E_2$ separates $N - T_R$ into two components C_1 and C_2. Since A is proper and disjoint from $E_1 \cup E_2$, it has an annular end representative A' with $A' \subset C_1$ or $A' \subset C_2$. This completes the proof of the lemma. $\square$

7 Trapped minimal annuli in a wedge.

Throughout this section A will denote a properly embedded minimal annulus in $N = \mathrm{R}^3/T$ where T is a vertical translation. Let S be the flat vertical annulus with boundary whose inverse image in R^3 is a vertical half plane with boundary the x_3-axis. Let $\gamma = \partial S$ be the quotient of the x_3-axis in N. A *wedge* is a region between two such vertical annuli and whose interior angle is less than π.

Lemma 7.1 *If A is contained in a wedge, then A has finite total curvature.*

Proof. Let Δ denote a wedge and suppose $A \subset \Delta$. We will prove that the Gauss map on a subend of A misses a curve of values.

Consider a family F_t $0 < t < \infty$ of parallel vertical flat annuli of distance t from γ (the image of the x_3-axis in N) and whose intersection with Δ gives rise to a foliation of $\Delta - \gamma$ by parallel compact flat annuli. Let $h\colon \Delta \to \mathrm{R}^+$ denote the level set function of this foliation and note that $h|A$ is a proper harmonic function on A. It is known (see for example Lemma 1 in [16]) that A contains an end A' that can be parametrized by $D^* = \{z \in \mathrm{C} \mid 0 < |z| \leq 1\}$ and that $h|D^* = K \ln|z| + K'$ for some constants K and K'. In particular $h|D^*$ has no critical points, so the foliation F_t is always transverse to A'. Hence the normal vector field to F_t is never normal to A'.

Note that the Gauss map of A' and its stereographic projection to $\mathrm{C}\cup\{\infty\}$ gives rise to a holomorphic map $g : D^* \to \mathrm{C} \cup \{\infty\}$ that misses the 2 normal vectors of F_t. Since the integral of the Gaussian curvature on A' equals the negative of the area (counted with multiplicity) of the Gauss map of A', Picard's theorem shows that at most 2 values of g can be taken finitely often. However, by changing the angle of F_t slightly, the above argument shows that A' has an end A'' such that $g|A''$ omits 2 new values. As we already observed this possibility contradicts Picard's theorem and the infinite total curvature assumption on A. $\square$

Lemma 7.2 *If S is a vertical annulus in N and $A \cap S = \emptyset$, then A has finite total curvature.*

Proof. Choose S_1, S_2 so that $S \cup S_1, S \cup S_2, S_1 \cup S_2$ are congruent wedges (cut a pie in three equal pieces). Since A is disjoint from S, we can translate A away from γ in the direction parallel to S, until ∂A is contained in the interior of the wedge W whose boundary is $S_1 \cup S_2$. We can also assume that A intersects ∂W transversally.

We will show that $A \cap S_1$ or $A \cap S_2$ has a noncompact component. If A is disjoint from ∂W, then $A \subset W$ and Lemma 7.1 shows A has finite total curvature. Hence A must intersect one of the vertical faces, S_1 or S_2, of ∂W. Suppose $A \cap S_1$ is nonempty. If $S_1 \cap A$ has more than one component that is compact, there would be a compact domain $\Sigma \subset A$ with $\partial\Sigma \subset S_1$. But then there would be an interior point on Σ of maximal distance from the complete vertical annulus containing S_1. The existence of such a point on Σ contradicts the maximum principle. Thus, $S_1 \cap A$ consists of at most one closed curve and this curve does not bound a disk on A. On the other hand if $S_1 \cap A$ consists of a single compact component that is a homotopically nontrivial curve, then the end of A with boundary this curve is contained in one of the convex wedges which is impossible by Lemma 7.1. Hence we are left with the possibility that $A \cap S_1$ has a noncompact component.

Consider any noncompact proper curve in $A \cap S_1$. This curve separates A into two components, C_1, C_2, where C_1 is simply connected. Our earlier remarks show that $C_1 \cap (S_1 \cup S_2)$ contains no compact components. We now check that C_1 intersects W or one of the adjacent wedges W_1 in a component C that is simply connected and has its entire boundary in S_1 or in S_2. This statement is clear if $C_1 \cap S_2 = \emptyset$. But if $C_1 \cap S_2 \neq \emptyset$, then C_1 intersects the other wedge W_2. In this case any component C of $C_1 \cap W_2$ suffices.

The following assertion proves that C can not exist; a contradiction from which the lemma follows.

Assertion 7.1 *Suppose X is a wedge in N. If C is a properly embedded simply connected minimal surface in X with boundary in the interior of one of the faces of ∂X, then C is contained in ∂X.*

Remark 7.1 *One should note that a properly embedded simply connected minimal surface C in a wedge $\widetilde{X}$ in R^3 rather than in N whose boundary is in the interior of one of the faces of $\partial\widetilde{X}$, is not in general contained in $\partial\widetilde{X}$. For example, an end of one of Scherk's surfaces, asymptotic to a half plane, can be chosen in a wedge with its boundary in the boundary of the wedge.*

Proof of Assertion 7.1. Let F_1 and F_2 be the faces of X and assume $F_1 \cap F_2 = \gamma$. Suppose $\partial C \subset F_1$ and C is not contained in F_1. If F_3 is a vertical annulus in X with $\partial F_3 = \gamma$ and $F_3 \cap C = \emptyset$, then we can replace X by the smaller wedge with faces F_3, F_1. This replacement shows that we may assume that X is minimal in the sense that if F is a flat annulus in X with $\partial F = \gamma$ and $F \cap C = \emptyset$, then $F = F_2$. If the angle between F_1 and F_2 is greater than $\frac{\pi}{4}$, then choose an F_3 in X that is transverse to C and that makes an angle at most $\frac{\pi}{4}$ with F_2. In this case replace C by a simply connected component in the subwedge of X bounded by $F_2 \cup F_3$. Clearly to derive a contradiction it suffices to prove that this component is contained in the flat annulus F_3, since F_3 was choosen to be transverse to C. Thus we may assume that the angle between F_1 and F_2 is less than $\frac{\pi}{4}$.

For visual convenience we now change coordinates so that γ corresponds to the image of the x_1-axis, F_1 is horizontal with nonnegative x_2-coordinate and F_2 is a graph over F_1. If C were a graph over F_1, then it is not difficult to prove that C is contained in F_1. (When C is a graph we shall find a curve on C, which is a graph over part of γ of slope less than one and rising arbitrarily high. Since the length of γ is bounded, this is impossible.) We essentially reduce the general case to this graph case. (See Figure 1.)

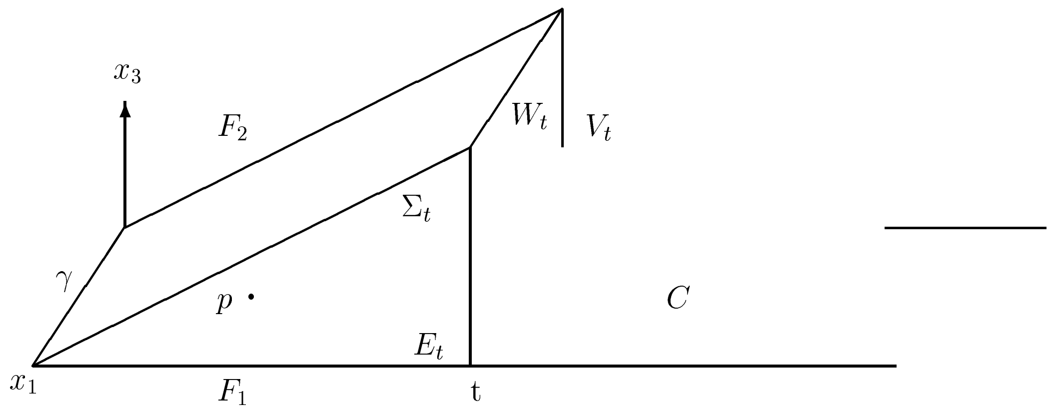

Figure 1:

Arbitrarily choose a point $p \in \partial C$. Let V_t denote the compact vertical annulus in X whose x_2-coordinate is t. Suppose t is large enough so that the x_2-coordinate of p is less than t and suppose that V_t is transverse to C. If $V_t \cap C$ contained a simple closed curve, then this curve would bound a disk in C. Then this disk would be contained in V_t by the maximum principle, an impossibility. Hence $V_t \cap C$ consists of arcs whose boundary points lie in F_1. Let $E_t \subset C$ denote the compact disk component of $C - V_t$ that contains p. E_t separates the compact region of X bounded by V_t into two components where we denote by W_t the closure of the component containing F_2. The Geometric Dehn's Lemma in [18] implies that ∂E_t is the boundary of an embedded disk $\Sigma_t \subset W_t$ of least area in W_t. (If a Jordan curve on the boundary of a mean-convex domain is homotopically trivial in the domain, then it spans an embedded minimal disk.)

Remark 7.2 *The maximum principle implies that either Σ_t equals E_t or $\Sigma_t \cap \partial W_t = \partial E_t$. If $\Sigma_t \not\subset \partial W_t$, then $\Sigma_t \cup E_t$ is an embedded sphere in W_t which must bound a ball in W_t. In particular, any arc in W that joins a point of E_t to F_2 must intersect Σ_t. We will have further use of this remark.*

Choose $t_\circ$ such that $V_{2t_\circ}$ is transverse to C and V_t is transverse to $\Sigma_{2t_\circ}$ where t is approximately $t_\circ$. Let $D_t = \Sigma_{2t_\circ} \cap W_t$. In this way, for most values of t, we produce a collection of disks D_t that are stable in W_t.

Let θ denote the angle between F_1 and F_2. For η, $0 < \eta \leq \theta$, let F_η denote the flat annulus in W with $\partial F_\eta = \gamma$ and such that the angle between F_2 and F_η is η. By our earlier choice of F_2, $F_\eta \cap C \neq \emptyset$ for $\eta \leq \theta$. Let $W(\eta)$ denote the wedge between F_η and F_2.

We now apply the curvature estimates of Schoen [22]: the Gaussian curvature at a point on a stable orientable minimal surface Σ in a flat three-manifold is bounded from above in absolute value by c/d^2 where c is a constant independent of Σ and d is the intrinsic distance from the point to $\partial\Sigma$. We will now show that these curvature estimates imply: *For all $\delta > 0$ there exists a positive $\varepsilon(\delta)$ such that if $\eta < \varepsilon(\delta)$ and Σ is a stable orientable minimal surface in W with boundary in F_1, then the normal line L_q to Σ at $q \in \Sigma \cap W(\eta)$ makes an angle less than δ from the normal line to F_η.* (This means that $\Sigma \cap W(\eta)$ is almost parallel to F_η.)

We now give the proof of the above implication. Suppose that the implication were false. Then there exists a $\delta > 0$, a positive sequence of numbers $\eta_i \to 0$, and sequence of stable orientable minimal surfaces Σ_i in W, $\partial\Sigma_i \subset F_1$ with points $q_i \in \Sigma_i \cap W(\eta_i)$ such that the angle L_{q_i} makes with the normal to F_{η_i} is always greater than δ. Similar statements hold by lifting everything to R^3 so that the inverse image of W is a wedge between two half planes. We will use the same notation for the lifted surfaces and subsets in R^3. Since we now consider $W(\eta)$ to be contained in R^3, it is invariant under homothety.

After a homothety, we may assume that $q_i \in \Sigma_i$ has x_2-coordinate equal to 1. Since $\eta_i \to 0$, the distance between q_i and F_2 goes to zero as $i \to \infty$ and distance of q_i to $\partial\Sigma_i$ is greater than a fixed constant. The second fundamental form of Σ_i is uniformly bounded in geodesic coordinate systems of some fixed radius by Schoen's estimates, and we can choose these coordinate systems to be graphs. Since q_i is converging to F_2 and Σ_i is disjoint from F_2, it is evident that the angle between L_{q_i} and the normal vector to F_2 is converging to zero. But the normal vector of F_{η_i} is converging to the normal vector of F_2. This contradicts the assumption that the angle between L_{q_i} and the normal line of F_{η_i} is greater than δ for all i. This contradiction proves the implication. Henceforth, we will work in N instead of R^3.

Since $\partial\Sigma_{2t_\circ} \subset F_1 \cup V_{2t_\circ}$ and t is approximately $t_\circ$, these same curvature estimates imply that for all $\delta > 0$ there exists a positive $\varepsilon(\delta)$ such that if $\eta < \varepsilon(\delta)$ and $q \in D_t \cap W(\eta)$, then the angle between L_q and the normal line to F_η is less than δ.

Now choose $\delta = \frac{\pi}{4}$ and fix η, $\eta < \varepsilon(\delta)$. Recall $C \cap F_\tau \neq \emptyset$ for all $\tau < \theta$. If $C \cap W(\eta)$ stays a bounded distance from F_η, then $C \cap W(\frac{\eta}{2})$ is compact which is impossible by the maximum principle. Hence we can pick a point $q \in C \cap W(\eta)$ such that the distance of q to F_η is greater than the length of γ. Choose t large enough so that $q \in E_t$. In particular $q \in E_{2t}$. Let ℓ denote the line segment joining q to F_2 and that is orthogonal to F_η. By Remark 7.2, ℓ must intersect D_t in a point q_1 whose distance from F_η is greater than the length of γ.

Let H be the flat annulus in $W(\eta)$ containing ℓ and whose boundary consists of a circle on F_2 parallel to γ and a circle on F_η. We can assume that H is transverse to D_t. Let α be the component of $H \cap D_t$ that contains the point q_1. Change coordinates in $W(\eta)$ (by rotation around γ) so that F_η is horizontal and H is vertical. Since $\delta < \frac{\pi}{4}$, the normal line to D_t along α

makes an angle of less than $\frac{\pi}{4}$ with the (new) vertical. Hence the slope of the tangent line along α is always less than one. If α were a closed curve, then it would bound a disk on D_t and this disk would be contained in H by the maximum principle, an impossibility. Hence, α is an arc with two boundary points on F_η. Since α is embedded, it is a graph over the circle $H \cap F_\eta$. Since the slope of α is less than 1, its maximum height can be at most the length of $H \cap F_\eta$ which equals the length of γ. But $q_1 \in \alpha$ has height greater than the length of γ. This contradiction completes the proof of Assertion 7.1. As remarked before, the assertion proves Lemma 7.2. $\square$

Lemma 7.3 *If A is trapped between standard ends that are Scherk type ends, then A has finite total curvature.*

Proof. By Lemma 7.2 we need only show that A is disjoint from some vertical flat annulus S. Suppose that A is trapped between standard ends that are Scherk type ends E_1, E_2. In this case E_1 is asymptotic to a flat vertical annulus S_1 and E_2 is asymptotic to a flat vertical annulus S_2. If S_1 and S_2 are disjoint, then we can clearly find the desired annulus S, so we must show that S_1 and S_2 are disjoint. Assume on the contrary that $S_1 \cap S_2 \neq \emptyset$. Since vertical flat annuli in N intersect in a compact set or the intersection contains a noncompact subannulus, we can take $S_1 = S_2$.

At this point, we could appeal to the maximum principle at infinity given in [17], which implies E_1 and E_2 can not be asymptotic at infinity and be disjoint. However, a direct proof is rather easy here so we proceed with the proof.

Since E_1, E_2 are standard ends, we can choose them, by replacing them by subends, so that they can be expressed as graphs over S_1 tending to zero. Without loss of generality we may assume that S_1 is the flat vertical annulus whose inverse image in $\mathbb{R}^3$ is the half plane with boundary the x_3-axis and containing the positive x_2-axis. Assume that the x_1-coordinate of E_1 is greater than the x_1-coordinate of E_2. Choose a small $\varepsilon > 0$ and $\varepsilon < dist(\partial E_1, \partial E_2)$, and so that $E_1' = E_1 + (-\varepsilon, 0, 0)$ intersects E_2 transversely. By the classical maximum principle, $E_1' \cap E_2$ does not contain a component that bounds a compact subdomain of E_1' or of E_2. Hence there are annular subends $\widetilde{E}_1 \subset E_1' - E_2$ and $\widetilde{E}_2 \subset E_2 - E_1'$ with common boundary curve γ. Let $\vec{n}_i$ denote the inward pointing unit conormal vector field along the boundary of the surface $\widetilde{E}_i$. Since the x_1-coordinate of $\widetilde{E}_1$ is less than the x_1-coordinate of $\widetilde{E}_2$, $grad(x_1|\partial\widetilde{E}_1) \cdot \vec{n}_1 < grad(x_1|\partial\widetilde{E}_2) \cdot \vec{n}_2$. Integrating this inequality, we obtain inequalities on the fluxes of $grad(x_1)$ across the common boundary of $\widetilde{E}_1$ and $\widetilde{E}_2$:

$$a_1 = \int_{\partial\widetilde{E}_1} grad(x_1|\widetilde{E}_1) \cdot \vec{n}_1 \;<\; a_2 = \int_{\partial\widetilde{E}_2} grad(x_1|\widetilde{E}_2) \cdot \vec{n}_2.$$

Since the coordinate functions of a minimal surface in $\mathbb{R}^3/T$ are harmonic, the divergence theorem implies that the flux of a harmonic function is constant on homologous cycles. Now by Proposition 4.1, $grad(x_1|\widetilde{E}_1)$ converges uniformly to zero as x_2 tends to infinity. The number a_1 equals the flux of x_1 across a cycle on $\widetilde{E}_1$ defined by x_2 equals constant, the constant arbitrarily large. Since these cycles are of bounded length, it follows that a_1 (and a_2 by the same reasoning) is zero, a contradiction. $\square$

8 The proof of Theorem 1.

Suppose M is a properly embedded minimal surface in a complete flat non-simply connected three-manifold N and suppose that M has finite topology. We shall prove that M has finite total curvature. Since N is finitely covered by a flat three-torus, $\mathrm{T} \times \mathrm{R}$, or by $\mathbb{R}^3/S_\theta$, we may assume, after lifting M

to a covering space, that N is $\mathrm{T} \times \mathrm{R}$ or R^3/S_θ where $\theta = 0$ or θ is irrational (clearly M is compact if one is in a flat three-torus). Now it suffices to prove each annular end A has finite total curvature. The theorem in $\mathrm{T} \times \mathrm{R}$ can be reduced to the theorem in R^3/S_θ since if A is an annular end in $\mathrm{T} \times \mathrm{R}$, then $\pi_1(A)$ is contained in an infinite cyclic subgroup of $\pi_1(\mathrm{T})$. Hence, one can lift A to a covering space $(S^1 \times \mathrm{R}) \times \mathrm{R}$ of $\mathrm{T} \times \mathrm{R}$ so that the lifted A is an annular end of a properly embedded minimal surface $\widetilde{M}$ that covers M.

By Remark 6.1, Lemma 6.1, and the Trapping Lemma, we can assume (after passing to a finite covering) that each annular end of M (or $\widetilde{M}$) is trapped between two standard ends in N. The Trapping Lemma (Lemma 6.2) shows that we may assume that A is trapped between 2 standard ends that are helicoidal, planar or are Scherk type ends. Note that Scherk ends can only occur for θ rational. Lemma 7.3 shows that Theorem 1 is true when A is trapped between Scherk type ends.

To complete the proof of Theorem 1, we must prove that an embedded annular end A that is trapped between two helicoidal (perhaps with logarithmic growth) or planar standard ends, E and D, is of finite total curvature.

Assume the limiting normal vector to E is vertical and E is never vertical. We will show that there exists an annular subend A' of A that never has a vertical tangent plane. Then a lifting of A' to R^3 has the same property so the lifting is stable (its Gaussian image is contained in a hemisphere) Hence A is stable also and thus has finite total curvature. The construction of A' will involve interesting geometric constructions and occupy all of Section 8.

Let $B = \{(x_1, x_2) \mid x_1^2 + x_2^2 \geq 1\}$ and $i\colon B \to E$ be a parametrization of E, sending the circles of radius R in B to $E \cap \partial T_R$, i.e., the helices of E for large R. We work in the manifold $W = B \times \mathrm{R}$ with the flat metric induced by the submersion: $(x_1, x_2, t) \mapsto i(x_1, x_2) + (0, 0, t) \in N$.

Observe that this metric on W is asymptotic to the product flat metric (a flat metric on B with R) since the metric on the end E is asymptotically flat. Notice that the region trapping A in N, i.e., the region bounded by E, D and a large compact cylinder (parallel to the period vector v) lifts isometrically to W and A is in this region as well. Clearly vertical lines in N correspond to vertical lines in W, hence it suffices to find a never vertical subannulus in the lifting of A to W. Henceforth, we shall work in W. For simplicity, assume that the trapping region lifts to a subdomain of $B \times [0,1]$ with E lifting to $B \times \{0\}$. Let $B_t \subset W$ be $B \times \{t\}$.

Proposition 8.1 *There exists a compact subset K_1 of $W(1) = B \times [0,1]$ such that for any other compact subset K_2 containing K_1, the following statement holds: For every $x \in W(1)$, sufficiently far from K_2, and for every vertical plane P_x at x, there is a foliation $\mathcal{F}$ of a neighborhood of $W(1) - K_1$, such that:*

1. *the leaves of $\mathcal{F}$ are compact minimal annuli F_t, $0 \leq t < \infty$, with one boundary component in B_0, and the other boundary component in B_1;*
2. *$x \in F_1$ and the tangent plane of F_1 at x is P_x;*
3. *$F_1 \subset W(1) - K_2$;*
4. *$F_0 \subset K_1$.*

Remark 8.1 *In fact K_1 will be the compact region of $W(1)$ bounded by a stable minimal annulus whose boundary consists of a circle of radius R in B_0 and its parallel translate to B_1, for some large R. Outside of some larger compact set, the leaves of $\mathcal{F}$ will be compact annuli with boundary; circles of larger radius, one in B_0 and the other its parallel translate to B_1.*

Before proving Proposition 8.1, we show why it implies A contains a subannulus that is never vertical.

Let $h: W(1) \to [0, \infty)$ be a proper function. Let T_1 be a regular value of h such that $K_1 \cup \partial A \subset h^{-1}[0, T_1]$. Choose $T_2 > T_1$ a regular value, such that the component of $A \cap h^{-1}[0, T_2]$ that contains ∂A, also contains $A \cap h^{-1}[0, T_1]$. Let $K_2 = h^{-1}[0, T_2]$. Note that any compact subdomain of A whose boundary is contained in $W(1) - K_2$, is disjoint from K_1.

Now apply Proposition 8.1 to K_1 and K_2. If A were vertical at points arbitrarily far from ∂A, then there is an $x \in A$ sufficiently far from ∂A to which we apply the proposition. Let $\mathcal{F}$ be the foliation, F_1 the minimal annulus in $W(1) - K_2$ such that F_1 is tangent to A at x. Then $A \cap F_1$ is a compact singular one cycle in A (a singularity at x) and $A - F_1$ contains a component Δ with compact closure, with boundary in $F_1 \subset W(1) - K_2$. By our choice of K_1, K_2, we have Δ disjoint from K_1. Now there is a largest (or smallest) value t such that $\Delta \cap F_t$ is nonempty. At such a point Δ is on one side of F_t and this contradicts the maximum principle.

Hence it remains to prove Proposition 8.1. This will be carried out in a series of lemmas.

Let C be a catenoid in R^3 with waist circle at height $t = \frac{1}{2}$ and let S_0, S_1 be the circles of C at heights 0 and 1 respectively; each of radius $R_\circ$. Assume $R_\circ$ is sufficiently large so that the angle that the normal vector to C, along $S_0 \cup S_1$, makes with the horizontal is less than $\pi/8$. We will work in regions of W where the vertical cylinder of height three and radius $3R_\circ$ isometrically embeds in R^3. Then the catenoid C isometrically embeds in this region of W as a vertical stable catenoid.

Let L_0 be a smooth simple closed curve in B_0 and L_1 the vertical translate of L_0 to B_1. For each $x \in L_0$, we consider the two catenoids C_x, parallel translations of C, which contain x and $x + (0, 0, 1)$ in their boundary and whose tangent line γ_x to ∂C_x at x is the horizontal projection of the tangent line of L_0.

Figure 2:

Let ℓ_x be the horizontal line of length $2R_0$, centered at x and normal to γ_x at x. Let W_x be the vertical strip over ℓ_x of height one; W_x is a rectangle of base ℓ_x and side one.

Let $\beta_x = W_x \cap C_x$; β_x is a Jordan curve, smooth except at x and $x + (0,0,1)$. β_x consists of two meridian curves on C_x (one on each catenoid of C_x) joining x to $x + (0,0,1)$. Clearly β_x bounds a disk in W_x.

Now define the torus barrier $T = T(L_0)$ to be $\bigcup_{x \in L_0} \beta_x$. In general, T is neither embedded nor a barrier, however, if it is, then we have the following lemma.

Lemma 8.1 *Suppose the torus barrier of L_0 is embedded and mean convex (i.e. the mean curvature vector of $T - (L_0 \cup L_1)$ points into the solid torus S bounded by T). Then $L_0 \cup L_1$ is the boundary of a stable embedded minimal annulus in S, and any embedded minimal annulus in S with boundary $L_0 \cup L_1$ is stable.*

Remark 8.2 *We will use this lemma to construct the foliation of Proposition 8.1. We will construct a foliation of B_0 by simple closed curves $L_0(s)$, $1 \leq s < \infty$, and the vertical translation $L_1(s)$ will foliate B_1. Lemma 8.1 will* *be used to show that $L_0(s) \cup L_1(s)$ bounds a unique stable minimal annulus and these annuli will foliate a region of W. The difficulty in this construction is to appropriately construct the foliation $L_0(s)$ in order to obtain a vertical tangent plane of the annulus A as defined in Proposition 8.1.*

Proof. The angle of $\partial S - T$ along $L_0 \cup L_1$ is always less than π so by [18], ∂S is an appropriate barrier for solving the Plateau problem in S. Let Σ be a least-area annulus with $\partial\Sigma = L_0 \cup L_1$($L_0$ is homotopic to L_1 in S and L_0 is not null homotopic in S). By the Geometric Dehn's Lemma [18], Σ is embedded.

It remains now to show an embedded minimal annulus Σ in S with boundary $L_0 \cup L_1$ is stable. Since the vertical makes sense in W, the angle the normal vector to Σ makes with (0,0,1) is a well defined function on Σ, hence $\log|g|$ is a well defined harmonic map on Σ (where Σ is not horizontal); g the Gauss map. Notice that $|g|$ is well defined even though g is multi-valued.

Now if Σ is never horizontal, then the angle between the normal lines to Σ and the vertical vary between $3\pi/8$ and $5\pi/8$. To see this, notice that the normal lines along $\partial\Sigma$ have this property because Σ is between two catenoids with normal lines making angles with the horizontal at most $\pi/8$. Also $\log|g|$ is harmonic on Σ, hence the maximum and minimum values are assumed on $\partial\Sigma$.

Let $\tilde{\Sigma}$ be a connected lifting of Σ to R^3 and let $D \subset \tilde{\Sigma}$ be a compact domain. The Gauss map of $\tilde{\Sigma}$ (hence of D as well) takes its values in a band about the equator whose maximum angle with the equator is $\pi/8$. In particular, the area of the spherical image of D (not counted with multiplicity) is less than 2π. By the theorem of Barbosa-Do Carmo, D is stable, hence $\tilde{\Sigma}$ too [1]. Since $\tilde{\Sigma}$ covers Σ, Σ is stable as well (see [4] or [7]).

It remains to show Σ is never horizontal. The proof will use a winding number argument. Let $\tilde{S}$ be the universal covering space of S and $\tilde{\Sigma} \subset \tilde{S}$ the lifting of Σ. On $\widetilde{\Sigma}$ we have a well defined meromorphic Gauss map g. Let α be an embedded arc on $\widetilde{\Sigma}$ joining a point $x \in \widetilde{L}_0$ to $x+(0,0,1) \in \widetilde{L}_1$, disjoint from the zeros and poles of g. Let σ be a nontrivial covering transformation of $\widetilde{\Sigma}$. Now consider the disk D bounded by α, $\sigma(\alpha)$, an arc ℓ_0 on $\widetilde{L}_0$ and its' parallel arc ℓ_1 on $\widetilde{L}_1$. We claim that for any $y \in \widetilde{L}_0$, $g(y)$ and $g(y+(0,0,1))$ have arguments whose difference is less than π. To see this, observe that $\widetilde{\Sigma}$ separates $\widetilde{S}$ into two components and the normal vector of $\widetilde{\Sigma}$ points into the same component. Call this component B and let $M = \partial B - \widetilde{\Sigma}$. At $y \in \widetilde{L}_0$, we have a horizontal unit vector $v(y)$ that is normal to $\widetilde{L}_0$ at y and the scalar product of $v(y)$ with the exterior normal (to B) of M at y is positive. Notice that $v(y+(0,0,1))$ is the parallel translation of $v(y)$ to $y+(0,0,1)$.

The angle between $\widetilde{\Sigma}$ and M at y is less than $\pi/8$ and $g(y)$ and $v(y)$ are both orthogonal to $\widetilde{L}_0$, so the scalar product of $g(y)$ and $v(y)$ is positive. Similarly $g(y+(0,0,1))$ and $v(y)$ have a positive scalar product. Hence $g(y)$ and $g(y+(0,0,1))$ lie in the same open hemisphere so their arguments (thought of as complex numbers after stereographic projection) differ by at most π.

The Gauss map g is never 0 or ∞ on ∂D. Observe that g restricted to ∂D has degree zero, thought of as a map into $\mathrm{C}^* = \mathrm{C} - \{0\}$. To see this one calculates the winding number. For $z \in \alpha$, $g(z)$ and $g(\sigma(z))$ differ by rotation by a fixed θ_0 (θ_0 is a multiple of the angle of the flat structure, R^3/S_θ, on W). So the total change of the argument arising by traversing α and then $-\sigma(\alpha)$ is zero. For $y \in \ell_0$, the argument of $g(y)$ and $g(y+(0,0,1))$ differ by less than π. Hence the total change of argument as one traverses ℓ_0 and then $-\ell_1$ is less than π. Consequently the total winding number (in absolute value) is less than π, hence zero. Now g is conformal so every value has positive degree. Hence g misses 0 and ∞ and Σ is never horizontal. This completes the proof of Lemma 8.1. $\square$

In the following lemma we will give a natural condition on a curve $L_0 \subset B_0$ which ensures that the torus barrier is, in fact, embedded.

Note that for each $x \in W$ sufficiently far from ∂W, the vertical cylinder V_x of radius $3R_\circ$, centered at the vertical line through x, is embedded in W.

Let $D_x \subset V_x$ be the horizontal disk of radius $3R_\circ$ centered at x. (Note that D_x is not part of a B_t.)

Lemma 8.2 *Let $L_0 \subset B_0$ be a smooth simple closed curve. For $x \in L_0$, suppose the vertical projection Γ_x of $L_0 \cap V_x$ to D_x satisfies: the disks bounded by the circles in D_x, of radius $R_\circ$ and tangent to Γ_x at x (one on each side of Γ_x), intersect Γ_x at x only. Then the torus barrier of L_0 is embedded.*

Proof. For each $x \in L_0$, let ℓ_x denote the normal line to Γ_x at x, centered at x and of length $2R_\circ$. Observe that if $x \neq y$, $x, y \in L_0$, then the vertical strips W_x and W_y over ℓ_x and ℓ_y are disjoint. For if they intersect, then $\ell_y \subset V_x$ and the vertical projection $\widetilde{\ell}_y$ of ℓ_y onto D_x, must intersect ℓ_x at a point z. Let a and b be the points of ℓ_x and $\widetilde{\ell}_y$ in Figure 3.

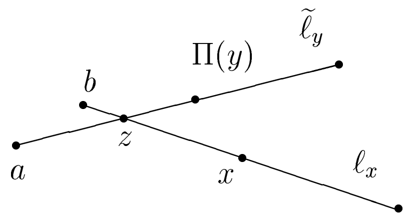

Figure 3:

Let Π denote orthogonal projection onto D_x. Assuming $d(z,b) < d(z,a)$, we have: $d(a,z) + d(z,\Pi(y)) = R_\circ$, $d(b,\Pi(y)) \leq d(b,z) + d(z,\Pi(y))$, hence

$d(b, \Pi(y)) < R_\circ$. But then $\Pi(y)$ is in the disk of radius $R_\circ$ centered at b and $\Pi(y) \in \Gamma_x$. This contradicts our hypothesis and hence the vertical strips over the ℓ_x are pairwise disjoint.

Now consider the construction of our torus barrier using the catenoids C_x, for $x \in L_0$. Each C_x (recall there are two catenoids in C_x) intersects the vertical strip over ℓ_x in an embedded curve β_x, that bounds a disk in this strip. The union of these curves is the embedded torus barrier. $\square$

Remark 8.3 *We claim that if L_0 is close to a horizontal plane P and if the injectivity radius of L_0 is large, then the hypothesis of lemma 8.2 are satisfied hence the torus barrier of L_0 is embedded. More precisely there is a $C > 0$ and $\epsilon > 0$ such that if the injectivity radius of L_0 is greater than C and if L_0 is $\epsilon - C^2$ close to P (i.e. the distance of L_0 to P is less than ϵ and the curvature and torsion of L_0 are less than ϵ) then the hypothesis of lemma 8.2 are satisfied.*

To see this, for $x \in L_0$, consider the solid vertical cylinders V_x, of radius R_0, tangent to Γ_x at x (there are two of these cylinders). We choose ϵ small enough so that the osculating plane of L_0 is always within $\pi/4$ of the horizontal. L_0 does not enter V_x at x, otherwise the curvature of L_0 at x would be greater than the curvature of the helice on ∂V_x, making a constant angle $\pi/4$ with the horizontal. Since the curvature of this helice only depends on R_0, ϵ can be chosen so this is impossible.

Now choose C large enough so that the tubular neighborhood of L_0, of radius $2R_0$ along a fixed (small) arc on L_0, centered at x, contains V_x. We need work with V_x of height R_0 (since L_0 is within R_0 of P) so such a choice of C is possible. Then the only point of L_0 in V_x is x and the hypothesis of lemma 8.2 are satisfied.

Now when will the torus barrier T of L_0 be mean convex. We claim that if L_0 is close to a plane curve (in the C^2-topology) and if the curvature of

L_0 is small enough then T is mean convex. Consider first, a plane curve L_0. For $x \in L_0$, γ_x has two smooth arcs, each joining x to $x+(0,0,1)$. One is an outside arc $a(x)$ and the other an inside arc $b(x)$; i.e., $b(x)$ is on the side of L_0 to which L_0 is curving at x. Clearly, along the outer arc $a(x)$, the mean curvature vector is pointing inside T; T is even locally convex along $a(x)$. At a point y on $b(x)$, T will be mean convex if one can find two orthogonal directions such that the sum of the normal curvatures in these directions has the right sign, the same sign as that of the normal curvature of $b(x)$ at y. Clearly if the curvature of L_0 at x is sufficiently smaller than the curvature of $b(x)$ at y (both in absolute value) then this will be satisfied. Also one can bound the mean curvature away from zero by choosing the curvature of L_0 small.

Now if $\widetilde{L}_0$ is C^2-close to a horizontal curve L_0 and if the curvature of $\widetilde{L}_0$ is sufficiently small, then $T(\widetilde{L}_0)$ will be mean convex as well. We saw in the last paragraph that bounding the mean curvature of $T(L_0)$ away from zero only depended on the curvature of L_0 being small.

We shall say a curve L_0 is R_0-admissible, if $T(L_0)$ is embedded and mean convex. In the sequel we shall work in regions of W in the complement of the tubular neighborhood of radius R about the period vector. As $R \to \infty$, the metric in this exterior domain converges to the flat product metric on $B \times \mathrm{R}$. The curves L_0 we will work with will be in the exterior domain and contained in $B_0 = B \times (0)$. By choosing the injectivity radius of L_0 large, and R large, we will have L_0 a R_0-admissible curve.

Lemma 8.3 *Let $L_0(s)$ be a family of simple closed curves in B_0, $0 \leq s \leq 1$, that foliate an annulus of B_0. Let $L_1(s)$ denote the foliation in B_1 obtained by vertical translation of the curves $L_0(s)$. For each s, assume the torus barrier $T(s)$ of Lemma 8.1, defined by $L_0(s) \cup L_1(s)$, is embedded and mean convex. Let S_0 be a stable minimal annulus in $T(0)$ with boundary $L_0(0) \cup L_1(0)$.*

Then there is a foliation $S(s)$ by stable minimal annuli satisfying:

1. $S(0) = S_0$,
2. $\partial S(s) = L_0(s) \cup L_1(s)$,
3. $S(s)$ *is in* $T(s)$.

Proof. Let s be between 0 and 1. First observe that if Σ is a minimal annulus in $T(s)$ with boundary $L_0(s) \cup L_1(s)$, then along $L_0(s) \cup L_1(s)$, Σ makes a strictly positive angle with $T(s)$, and the interior of Σ is in the interior of $T(s)$. This follows from the boundary maximum principle and the maximum principle.

Now for s near 0, $L_0(s) \cup L_1(s)$ bounds a stable minimal annulus $S(s)$, since a stable minimal surface varies smoothly with a smooth change of boundary data. This type of result can be found in [25] or [27]. By our previous paragraph and since the $T(s)$ vary smoothly, $S(s)$ is contained in $T(s)$ for s near 0. Since the variation vector field is a Jacobi field that is never zero on the boundary $(L_0(0) \cup L_1(0))$, by stability (the index theorem) it can not vanish inside. Hence, the family of surfaces that one obtains by moving along the variation vector field at time s is indeed a foliation for s near zero.

It remains to show the set of s for which the foliation exists is closed. So assume the foliation exists and satisfies 1, 2 and 3 for $s < \tau$. We know that $T(s)$ converges to $T(\tau)$. A subsequence $S(s_n)$ converges to a minimal annulus $S(\tau)$ in $T(\tau)$. By Lemma 8.1, $S(\tau)$ is stable. By the openness property, $S(\tau)$ is part of a foliation near $S(\tau)$. Since $S(s_n)$ converge to $S(\tau)$, the maximum principle implies $S(s_n)$ must be a leaf of this foliation for n large. Hence $S(s)$ converges to $S(\tau)$ as $s \to \tau$. This proves Lemma 8.3. $\square$

Figure 4:

Before proving Proposition 8.1, we will describe the idea of the proof in the special case when E is a flat horizontal annulus. The general situation is a metric perturbation of this case near infinity but it will help the reader to consider the special case of a flat annulus first.

So assume $E = \{(x, y, z) \mid z = 0,\, x^2 + y^2 \geq 1\}$ and $W = E \times \mathrm{R}$. Suppose there exists a sequence $p_n \in E \times [0, 1]$ diverging to ∞, and a sequence of vertical planes P_n at p_n. Here is a resumé of what we shall do next. We construct an $R_\circ$-admissible curve L as follows. Let Γ be a planar convex curve as in Lemma 8.4. We take a long arc on Γ, centered at q, and join it to a convex curve as in Figure 4.

We translate L in E_0 so that q is near p_n, and then rotate about the vertical line through p_n, so that the new curve L so obtained satisfies: $L \cup L_1$ bounds a stable minimal annulus F_1 and F_1 is tangent to $P(n)$ at p_n. (In fact one will be obliged to move F_1 vertically to realize this tangency, since p_n is not necessarily at height $z = \frac{1}{2}$.)

Next construct a foliation in E_0 by $R_\circ$-admissible Jordan curves $L_0(s)$, $0 \leq s < \infty$, such that $L_0(0)$ is the circle of radius 2 centered at the origin, $L_0(1)$ is the L we constructed above and $L_0(s)$ for large s is also a circle. Now apply Lemma 8.3 where S_0 is a catenoid. Observe that F_1 is necessarily a leaf of the foliation given by Lemma 8.3. This follows from the maximum principle. This foliation contradicts our assumption so Proposition 8.1 follows.

We remark that the boundary planes we worked with in the above argument were E_0 and E_1. In fact, when we prove Proposition 8.1, we will need to work at heights, such as -1 and 2, to acquire the tangency at p_n. Also, our construction of L must be done with great care since the metric on E is not flat in the general case. Instead it is asymptotic to a flat metric which is why we will construct a sequence of foliations $\mathcal{F}(n)$, working at p_n when n is large. This is the end of the resumé.

Now we continue with the proof of the case E a flat annulus; i.e., we shall make precise the previous resumé!

We now need a technical lemma which is not difficult to prove but is necessary for our proof.

Lemma 8.4 *There is a planar curve Γ contained in the positive quadrant of the (x, y) plane having the following properties:*

1. *Γ is convex, asymptotic to the x and y axis and invariant under $(x, y) \to (y, x)$;*

2. *Γ is $R_\circ$-admissible, i.e., Γ and its parallel translate Γ_1 to E_1 define an embedded envelope $\mathcal{E}(\Gamma)$. Topologically $\mathcal{E}(\Gamma)$ is $S^1 \times \mathrm{R}$, and $\mathcal{E}(\Gamma)$ is defined as in Lemma 8.1 using the catenoids C_x;*

3. *$\Gamma \cup \Gamma_1$ bound a unique, area minimizing strip $M(\Gamma)$, contained in $\mathcal{E}(\Gamma)$;*

4. *$M(\Gamma)$ is invariant by reflection in $E_{1/2}$ and by reflection in the vertical plane $y = x$.*

Notice that Property 4 implies that $M(\Gamma)$ is vertical along $E_{1/2}$, (cf. Figure 5).

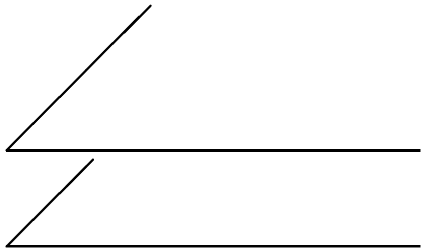

Figure 5:

Proof. Let A be the infinite strip $\{0 \leq z \leq 1,\, y = -x\}$. Γ is constructed as a graph over one of the boundary components of A so as to satisfy conditions 1 and 2. Then Γ_1 is the same graph over the other component of ∂A. It is known that every continuous function on ∂A extends to a solution of the minimal surface equation in A, so $M(\Gamma)$ is the graph of this solution [5]. Using catenoids as barriers above and below the graph of $\Gamma \cup \Gamma_1$ it's easy to see that $M(\Gamma)$ is contained in the torus barrier $T(\Gamma)$. To see that $M(\Gamma)$ is unique in $T(\Gamma)$, one reasons as follows. Let M be any other minimal surface in $T(\Gamma)$ with boundary $\Gamma \cup \Gamma_1$. A straightforward application of the Alexandrov reflection principle, using planes parallel to A, shows M is also a graph over A (see [23] for this type of argument). Now one has two minimal graphs over A, with the same boundary values and whose difference is bounded. It is known that this implies they are equal [5]. Since minimal graphs are area minimizing, we have proved Property 3. Property 4 follows from unicity. $\square$

We will need to fix a base point q on Γ. We take q to be the intersection of Γ with $y = x$.

Remark 8.4 *Lemma 8.4 also holds in a sector whose angle is almost* Π*; so the curvature of* Γ *can be as small as desired.*

Replacing p_n by a subsequence, we can assume the vertical cylinder $V_{3n}(p_n)$, of radius $3n$ and centered at the vertical line through p_n, embeds in

Figure 6:

$W(2) = E \times [-1, 2]$.

Translate $V_{3n}(p_n)$ horizontally and rotate so that the plane P_n becomes the plane $y = -x$ and p_n is on the z-axis. Let f_n denote this rigid motion of $V_{3n}(p_n)$ and let V_R be the vertical cylinder of radius R centered at the z-axis.

We construct a foliation $\mathcal{G}$ of the planes $\{z = -1\} \cup \{z = 2\}$ as follows. Foliate $z = 0$ by translating Γ along the line $y = x$. $\mathcal{G}$ is obtained by parallel translation (vertically) of this foliation to the planes $\{z = -1\} \cup \{z = 2\}$. Note that $\mathcal{G}$ is the boundary of a foliation of $\mathrm{R}^2 \times [-1, 2]$ by minimal strips $\{M(\Gamma)\}$, parallel translates of a fixed $M(\Gamma)$. This $M(\Gamma)$ is the same as in Lemma 8.4 except that the planes $z = 0$ and $z = 1$ have become $z = -1$ and $z = 2$, respectively.

We construct a foliation $\mathcal{G}(n)$ of part of $\{z = -1\} \cup \{z = 2\}$ by Jordan curves as follows. In the top and bottom of $V_{2n}(p_n)$, $\mathcal{G}(n)$ is the foliation by arcs $f_n^*(G)$. In the complement of the top and bottom of $V_{3n}(p_n)$, $\mathcal{G}(n)$ is the foliation by circles centered at the z-axis. In the top and bottom of $V_{3n}(p_n) - V_{2n}(p_n)$, $\mathcal{G}(n)$ is a foliation by arcs so that the resulting foliation is a foliation by $R_\circ$-admissible Jordan curves; Figure 6. It is not hard to show $\mathcal{G}(n)$ exists for n large. We choose the innermost circle of $\mathcal{G}(n)$ to be of radius ten; this guarantees that the stable catenoid, whose boundary is these circles of radius ten, has a waist circle of radius greater than one.

We need to choose the $R_\circ$ of the last paragraph so the stable vertical catenoid bounded by two circles of radius $R_\circ$, and of height 3, exists, and makes an angle less than $\pi/8$ with the horizontal along its boundary. Henceforth, we work with this value of $R_\circ$.

Now apply Lemma 8.3 to $\mathcal{G}(n)$; there is a foliation $\mathcal{F}(n)$ by stable minimal annuli F_t, $0 \leq t < \infty$, inducing $\mathcal{G}(n)$ on the boundary. Notice that each F_t intersects each horizontal plane in a simple closed curve, since the foliation of F_t induced by the horizontal planes can only have hyperbolic singularities (maximum principle), so it has no singularities.

We will prove the following assertion:

Assertion 8.1 *Let q_n be the vertical projection of p_n onto $E_{1/2}$. The trace of $\mathcal{F}(n)$ on the intersection of the plane P_n with the vertical cylinder $V_n(q_n)$, has a unique singularity, $\widehat{q}_n$, that is near q_n, for n large.*

Before proving this we will explain how this assertion completes the proof of Proposition 8.1 in the special case E is flat. First we arrange so that $\widehat{q}_n$ is on the same vertical line as q_n. To do this, one redoes the construction of the foliation $\mathcal{G}$ starting with a curve $\Gamma(s)$ in $z = 0$ where $\Gamma(s)$ is the curve Γ translated a distance s along the tangent line to Γ at q, $-1 \leq s \leq 1$. Then the foliation $\mathcal{F}_n(s)$ so obtained will yield a unique point $\widehat{q}_n(s)$, which is the singularity the trace foliation $\mathcal{F}_n(s)$ induces on $P_n \cap V_n(p_n)$. This point $\widehat{q}_n(s)$ is near $q_n(s)$, for n large, where $q_n(s)$ is the translation of q_n a distance s along $P_n \cap E_{1/2}$. Since $\mathcal{F}_n(s)$ varies continuously with s, the points $\widehat{q}_n(s)$ vary continuously with s. So for an appropriate choice of s, $\widehat{q}_n(s)$ and q_n will be on the same vertical line. For convenience, assume $s = 0$, and the leaf of $\widehat{q}_n(s)$ is labelled F_1.

Once $\widehat{q}_n$ and q_n are on the same vertical line, one does a vertical translation of $\mathcal{F}_n$ to make $\widehat{q}_n$ coincide with p_n. It's easy to see the translated foliation has a trace on $W(1) = E \times [0,1]$ as desired. K_1 can be chosen to be the vertical cylinder of radius ten, centered at the z-axis, intersected with $W(1)$. The points p_n diverge so one constructs the foliation of Jordan curves $\partial \mathcal{F}_n$ so that the leaves passing through the top of $V_n(p_n)$ are outside K_2 and their torus barriers are also outside of K_2. This guarantees that $F_1 \subset W(1) - K_2$.

Proof of Assertion 8.1. Let R_1 be the rectangle $P_n \cap V_{2n}(p_n)$. For points x on $R = P_n \cap V_n(p_n)$, far enough from the vertical line through q_n, the leaf $F_t(x)$ of x is in the torus barrier T of its' boundary curves. By construction T then intersects R_1 in disks D_1, D_2 each of whose boundary is smooth except along two points, one on the top of R_1, the other on the bottom. Clearly $F_t(x)$ intersects $D_1 \cup D_2$ transversally and joins the top of R_1 to the bottom of R_1. This shows the trace of $\mathcal{F}(n)$ on R has at least two nonsingular leaves as in Figure 7.

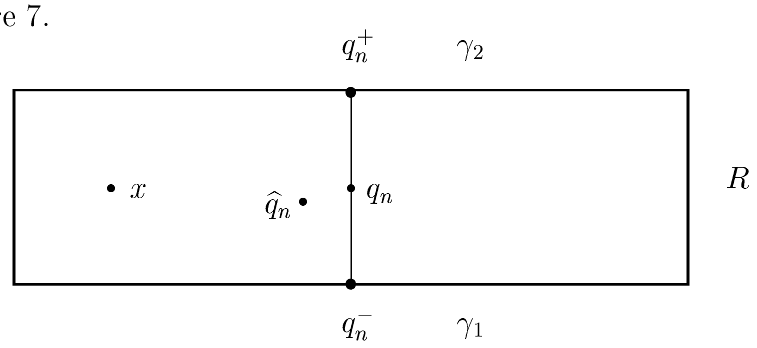

Figure 7:

The trace foliation near the segments $\gamma_1 \cup \gamma_2$ is as in Figure 7, since P_n is transverse to the foliation $\partial \mathcal{F}_n$ except at q_n^+ and q_n^-. This boundary data and the fact that minimal surfaces must have hyperbolic contact with R guarantees that there is exactly one singularity $\widehat{q}_n$ of the trace foliation. It remains to prove $\widehat{q}_n$ is near q_n for n large.

To show $\widehat{q}_n$ is near q_n we will show the foliation $H(n) \cap V_{10}$ converges to the foliation $\{M(\Gamma)\} \cap V_{10}$, where $H(n)$ is the image of $\mathcal{F}(n)$ by f_n. Clearly, the torus barriers of the upper and lower trace leaves of $H(n)$ converge to the corresponding torus barriers of the curves of $\mathcal{G}$. This will imply the leaves of $H(n)$ converge to the foliation, extending $\mathcal{G}$, whose leaves are the parallel translates of $M(\Gamma)$. Since $H(n)$ induces a foliation of a convex compact region of R^3 by compact minimal leaves, each leaf is area minimizing relative to its boundary. Hence, any subsequence of the leaves of $H(n)$ whose boundaries converge to a leaf $L_{-1} \cup L_2$ of $\mathcal{G}$ contains a convergent subsequence, that converges to a minimal surface which is contained in the torus barrier of $L_{-1} \cup L_2$. We showed earlier that there is a unique such surface in this torus barrier (our graph argument and the Alexandrov reflection principle), and it is a translate of $M(\Gamma)$. Hence, the leaves of $H(n)$ converge to parallel translates of $M(\Gamma)$. The Gauss map of $M(\Gamma)$ is injective in a neighborhood of q_n so the same is true for nearby minimal surfaces. It is now clear that $\widehat{q}_n$ is near q_n as $n \to \infty$. This proves Assertion 8.1 and completes the proof of Proposition 8.1 when E is a flat annulus. $\square$

Proof of Proposition 8.1. Now we work in $W(2) = B \times [-1, 2]$, with the flat metric induced from the map $(x, t) \mapsto i(x) + (0, 0, t)$. The torus ∂T_R (the boundary of an R-tubular neighborhood about the period vector v) intersects E in a simple closed curve $C(R)$, for R large, whose geodesic curvature tends to 0 as $R \to \infty$. By choosing a subend of E, we can assume E is foliated by the $C(R)$, $1 \leq R < \infty$, (we work from $R = 1$ for notational convenience). We can also assume that for $R \geq 10$, the vertical translates $C_{-1}(R), C_2(R)$ of $C(R)$ to B_{-1} and B_2 are $R_\circ$-admissible. Finally, assume $C(10)$ has the property that the torus barrier of $C_{-2}(10) \cup C_1(10)$ is contained in T_{20}. Then define K_1 to be $T_{20} \cap W(1)$. We will show that for every divergent sequence of points $p_n \in W(1)$ and vertical planes P_n at p_n, there is a foliation $\mathcal{F}_n$ for n large, satisfying the conclusion of 8.1.

Consider such a sequence of planes P_n and points p_n. Replacing p_n by a subsequence, we can assume the vertical cylinder $V_{3n}(p_n)$ in $W(2)$, of radius $3n$ and centered at the vertical line through p_n, embeds in $W(2)$, and isometrically embeds in R^3.

Isometrically embed $V_{3n}(p_n)$ in R^3 so that the vertical line through p_n goes to the z-axis and the origin corresponds to a point q_n on $B_{1/2}$, and P_n goes to the vertical plane through the line $y = -x$. Let V_{3n} denote the image cylinder of radius $3n$ in R^3. Consider the foliation of the (x, y)-plane by the parallel translates of Γ along $y = x$. Notice that $y = -x$ is tangent to a leaf of this foliation at $(0, 0)$. Pull back this foliation, to the top and bottom disks of V_{3n} by vertical projection onto the (x, y)-plane. Now consider the induced foliation of the top and bottom of $V_{2n}(p_n)$, back in $W(2)$.

Let D_r be the bottom disk of $B_{-1} \cap V_r(p_n)$ for $r \le 3n$. Now foliate B_{-1} as follows: in D_{2n} we take the above induced foliation of the previous paragraph. In $B_{-1} - D_{3n}$ we take the foliation by the curves $C_{-1}(R)$, $10 \le R < \infty$, and then fill in the foliation in $D_{3n} - D_{2n}$ so that each leaf L of this foliation, is a simple closed curve, and L together with its vertical translation to B_2 is $R_\circ$-admissible.

Then by Lemma 8.3, there is a foliation $\mathcal{F}(n)$ in $W(2)$ by stable minimal annuli F_t, $0 \le t < \infty$, inducing the previous foliation in $B_{-1} \cup B_2$. Notice that each F_s intersects each B_t in a simple closed curve since the induced foliation of F_s can have only hyperbolic singularities.

Now the top and bottom of $V_{2n}(p_n)$ converge, in the isometric embedding into R^3 to horizontal planes at heights 2 and -1 respectively, as $n \to \infty$. This is because the geometry of a standard end E converges to the Euclidean metric near infinity. Therefore the foliation $\mathcal{F}(n)$ in $V(n)$, when viewed in R^3 under the isometric embedding, converges to the foliation given by parallel translates by $M(\Gamma)$. The same argument as in our special case E flat, proves the above assertion.

It now follows that there is a unique point $\widehat{q}_n \in V_{2n}(p_n)$, near q_n, such that the leaf of $\mathcal{F}(n)$, through $\widehat{q}_n$, is tangent to P_n. Just as in the special case E flat, we can modify $\mathcal{F}(n)$ in $V_{2n}(p_n)$ to make $\widehat{q}_n$ move horizontally. So one can assume $\widehat{q}_n$ and p_n are on the same vertical line and have distance less than one. Vertical translation is well defined in W so one can vertically translate the foliation $\mathcal{F}(n)$, taking $\widehat{q}_n$ to p_n, and this new foliation (intersected with $W(1)$) satisfies the conclusions of Proposition 8.1. We have now completed the proof of Proposition 8.1 and hence of Theorem 1. $\square$

Remark 8.5 *In [20] Rosenberg and Toubiana constructed a properly immersed minimal annulus A in R^3 with proper x_1-coordinate function and infinite total curvature. Projecting A into R^3/S_θ yields a properly immersed minimal annulus. Thus, the embeddedness assumption in Theorem 1 is a necessary one for proving the finite total curvature property.*

A second remark is that a properly embedded minimal annulus A with compact boundary in R^3/S_θ has finite total curvature. The proof of Theorem 1 shows that this is the case if A is the end of a properly embedded minimal surface, since in this case A can be trapped between standard ends. The trapping argument can be generalized using the technical results in [17] to show that a general A can be trapped, thereby proving A has finite total curvature.

9 Applications of Theorem 1.

In this section we shall give the proofs of the remaining theorems. The proofs of these theorems are based on Theorem 1 and the results of Sections 2-4 on the geometry of properly embedded minimal surfaces of finite total curvature.

Proof of Theorem 2. Suppose M is a properly embedded simply connected minimal surface in R^3 with infinite symmetry group $\mathrm{Sym}(M) \subset \mathrm{Sym}(\mathrm{R}^3)$.

If M is not the helicoid, then its symmetry group is a discrete subgroup of $\mathrm{Sym}(\mathrm{R}^3)$. Every discrete infinite subgroup of $\mathrm{Sym}(\mathrm{R}^3)$ contains a screw motion (which may be a translation) and so, after a possible rigid motion of M, we may assume that M is invariant under a screw motion S_θ. Since S_θ acts freely and properly discontinuously on R^3, M/S_θ is a properly embedded minimal annulus in R^3/S_θ. By Theorem 1 M has finite total curvature and so Theorem 8 implies M is a helicoid or a flat plane. $\square$

Proof of Theorem 3. Let $M \subset \mathrm{R}^3/S_\theta$ be such a surface. By lifting to a 2-sheeted cover of R^3/S_θ, we may assume that M is orientable. Theorem 1 implies M has finite total curvature and Propositions 4.1 and 4.2 show that an end of M must be asymptotic to a plane, a vertical flat annulus, a helicoid or else $\theta = 0$ and the end is a nonhorizontal helicoidal type end. We now prove that the last case can not occur. For convenience rotate the surface so that the normal vector on a punctured disk neighborhood D^* of the end is vertical and let T denote the translation by the period vector v of the lifted surface in R^3. Note v has a nonzero horizontal component.

Let $\overline{g}\colon D \to \mathrm{C} \cup \{\infty\}$ be the extension of $g|D^*$ to the origin. We may assume that $g(0) = 0$. In this case we have by the work in Section 2 that

$$g(z) = z^p \, , \, \omega(z) = (\frac{c_{p+1}}{z^{p+1}} + \frac{c_p}{z^p} + \cdots)dz$$

where $c_{p+1} \in \mathrm{R}$ and $c_1 \in i\mathrm{R}$. It remains to prove that $c_1 = 0$, i.e. the translation vector v is actually vertical, a contradiction. Consider the extended map $\overline{g} : \overline{M} \to S^2 = \mathrm{C} \cup \{\infty\}$ to the conformal compactification across the puncture points $P \subset \overline{M}$, corresponding to the ends of M. Since M is embedded with an even number of helicoidal type ends, we see that $\overline{g}(P) = \{0, \infty\}$. Since M has helicoidal ends and it is embedded, the map $\widehat{g} : \mathrm{C}^* \to \mathrm{R}^3/T$, defined by $\widehat{g}(z) = \sum_{p \in g^{-1}(z)} p$, where the sum is taken in the abelian group R^3/T and taken with multiplicity, is a constant, or a complete branched minimal immersion. (See [16, 21] for these properties of $\widehat{g}$.)

We claim that $\widehat{g}$ is the helicoid. To see this, recall that the helicoidal type ends of M, with the same limiting normal value, have the same coefficients c_{p+1} in ω (Proposition 4.2). Notice that $c_{p+1} = -i\beta \neq 0, \beta$ given by Proposition 4.1. To obtain the "sum surface" $\widehat{g}$, one adds all points with the same g values. Parametrize the ends of M with the same limiting normal values by

$$g(z) = z^p,\ \omega(z) = (\frac{c}{z^{p+1}} + o(z^{-p}))dz.$$

Then the Weierstrass Representation of the sum of these ends is given by

$$\widehat{g}(\eta) = \eta,\ \widehat{\omega}(\eta) = \widehat{\omega}_1(\eta) + \ldots + \widehat{\omega}_\ell(\eta),$$

where $\widehat{\omega}_i(\eta)$ is the 1-form of the "sum surface" of the i'th end $A_i; A_1, \ldots, A_\ell$ the ends with the same limiting normal.

We calculate $\widehat{\omega}_i(\eta)$: For fixed η let z be a p'th root of η. So $z, jz, \ldots, j^{p-1}z$ are all the p'th roots of η, where j is a nontrivial p'th root of unity. Then

$$\widehat{\omega}_i(\eta) = c_{p+1}\sum_{k=0}^{p-1}(\frac{1}{(zj^k)^{p+1}} + o(z^{-p}))d(j^k z) = \frac{c_{p+1}}{z^{p+1}}(\sum_{k=0}^{p-1} j^{-pk} + o(z^{-p}))d(z) =$$

$$(\frac{(p-1)c_{p+1}}{z^{p+1}} + o(z^{-p}))dz = [(\frac{p-1}{p})(c_{p+1})\frac{1}{\eta^2} + o(\eta^{-1})]d\eta.$$

Since the $c_{p+1's}$ are the same at $A_1, \ldots, A_\ell$, this shows

$$\widehat{\omega}(\eta) = (\frac{\ell(p-1)c_{p+1}}{p})[\frac{1}{\eta^2} + o(\eta^{-1})]d\eta.$$

Now to obtain the $\widehat{\omega}$ of the sum surface, one must also add the $\omega's$ at the points of M having η as normal value, which are not on the ends of M. These $\omega's$ are holomorphic forms at these points hence their sum as well. It follows $\widehat{\omega} = (c/\eta^2 + o(\eta^{-1}))d\eta$ and the ends of $\widehat{g}$ are helicoidal, c some constant.

Now $\widehat{\omega}$ is a meromorphic form on S^2 with a double pole at zero and regular at infinity. Hence $\widehat{\omega}(\eta) = c\, d\eta/\eta^2$. Thus $\widehat{g}$ is an associate surface of a genuine helicoid. Since the coefficient of $1/\eta^2$ in $\widehat{\omega}$ is purely imaginary, the surface is a genuine helicoid. But the translation vector of the helicoid is vertical whereas the vector v has a horizontal component. This contradiction completes the proof of the first part of Theorem 3.

If $\theta \neq 0$, it is clear that the planar ends are horizontal and when θ is irrational R^3/S_θ does not contain vertical flat annuli. This completes the proof of the theorem. $\square$

Remark 9.1 *The above argument proves that if M is a properly embedded, finite topology, minimal surface in R^3/T, and M has helicoidal ends, then the sum surface $M + M$ is a genuine helicoid. It's not hard to see (by a similar argument) that if the ends are planar, then $M + M$ is a point. If M has four Scherk type ends, then $M + M$ is a Scherk surface.*

Theorem 9 *A properly embedded orientable minimal surface of finite topology in an orientable flat nonsimply connected three-manifold has an even number of ends or it is a plane.*

Proof. By Theorem 1, the minimal surface $M \subset N$ has finite total curvature. If the manifold N is isometric to $\mathrm{T} \times \mathrm{R}$, then Theorem 3.1 in [16] states that M has an even number of ends. If N is compact, then M is closed and has zero ends, an even number. The only other possibility is that N is isometric to some R^3/S_θ. If M has helicoidal type ends, then M has an even number of ends by Proposition 4.2. If M is not a plane, it must separate N by Remark 6.1. If the ends of M are planar, then the work in Section 2 shows that for large R the vertical torus ∂T_R of radius R centered along the x_3-axis intersects M in a family of parallel simple closed curves, one for each end of M. Since M separates N, the number of curves in $\partial T_R \cap M$ is even. The remaining case is when the ends of M are Scherk type ends. For large

R, $\partial T_R \cap M$ consists of parallel almost-vertical, simple, closed curves, one for each end. As in the previous case, this implies M has an even number of ends. $\square$

Proof of Theorem 4. Suppose that $M \subset N = \mathrm{R}^3/S_\theta$ is a properly embedded minimal surface of finite topology. By Theorem 1, M has finite total curvature and by Theorem 3 the ends of M are asymptotic to parallel planes, vertical flat annuli or to ends of parallel helicoids in N. We will implicity use the analytic results of Sections 2–4.

Let $M_R = T_R \cap M$ and note that for R large $\mathrm{Int}(M_R)$ is homeomorphic to M_R and $M - \mathrm{Int}(M_R)$ consists of the annular ends of M. Hence $\chi(M_R) = \chi(M)$.

By Gauss-Bonnet, the total curvature of M_R is

$$C(M_R) = \int_{M_R} K\, dA = 2\pi\chi(M) - \int_{\partial M_R} \kappa_g$$

where K is the Gaussian curvature and κ_g is the geodesic curvature of ∂M_R. Since $C(M) = \lim_{R\to\infty} C(M_R)$, the theorem will follow by showing that the total geodesic curvature of ∂M_R converges to $2\pi \cdot W(M)$ as $R \to \infty$, where $W(M)$ is the total winding number of M.

First consider the case when the ends of M are planar. In this case it is clear that the geodesic curvature of a component δ_R of ∂M_R converges to 2π. It is equally clear that δ_R is homotopically trivial in N since it lifts to R^3. Recall the definition of the curves α and β used in defining the winding number of an end of M. Define the related curves $\alpha_R = \partial T_R \cap \mathrm{R}^2$ and β_R on ∂T_R. Since δ_R is homotopically trivial in N, $\delta_R \subset \partial T_R$ is homotopic to $\alpha_R + 0 \cdot \beta_R$. Hence, the winding number of the end associated to δ_R is 1 which proves the formula when the ends of M are planar.

Suppose R is large and the ends of M are asymptotic to flat vertical annuli. Let A_R be a component of $M - M_R$. Since A_R is proper, θ must be a rational multiple of 2π. Suppose $\theta = 2\pi \cdot \frac{m}{n}$ where m and n are relatively prime. As $R \to \infty$ the curve δ_R approximates a geodesic of M_R, almost vertical in N. On the other hand it is clear that the absolute value of the intersection number of δ_R with α_R is n and with β_R is m. Since the sign of $\delta_R \cap \alpha_R$ is the negative of the sign of $\delta_R \cap \beta_R$, δ_R is homotopic to $\pm(m\alpha_R - n\beta_R)$. By definition, the winding number of A_R is $|2\pi \cdot m - n \cdot 2\pi \cdot \frac{m}{n}| = 0$. Thus, the total curvature formula holds when M has Scherk type ends.

Now consider the case where the ends of M are helicoidal. Let A_R be a component of $M - M_R$. Note that ∂A_R approximates a helix h_R on ∂T_R when R is large. Since the limiting normal vector to A_R is vertical, the geodesic curvature of ∂A_R converges to the curvature of h_R as $R \to \infty$. It remains to calculate the winding number and the total curvature of h_R as $R \to \infty$. The helix γ_R is homotopic to $n\alpha_R + m\beta_R$ for some relatively prime integers n, m. It is geometrically evident that the total curvature of γ_R converges to $|2\pi n + m \cdot \theta|$. This observation completes the proof of Theorem 4. $\square$

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