Torus Knots Extremizing the Conformal Energy

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1. Introduction.

Recently Freedman, He and Wang [FHW], following work of O'Hara [O], introduced an energy $E(\Gamma)$ for a simple closed curve $\Gamma \subset \mathbb{R}^3$. The functional E is continuous on each isotopy class of curves, and tends to infinity as Γ nears self-intersection. Moreover, E is "proper" on the set of all isotopy classes, in the sense that there are only finitely many knot types below a given energy level.

A useful geometric property of E is conformal invariance: if μ is a Möbius transformation of $\mathbb{R}^3 \cup \infty$, and $\mu(\Gamma) \subset \mathbb{R}^3$, then $E(\mu(\Gamma)) = E(\Gamma)$. This can be used to prove [FHW] that each *prime* knot class has an energy minimizing representative (of differentiability class $C^{1,1}$), and that the round circle is the unique energy minimizer among all curves ($E = 4$). It is also a non-trivial result of [FHW] that, at least for $C^{1,1}$ curves, the functional E is sufficiently smooth to have a "gradient" dE (see §3). Thus it becomes an interesting problem to find E-critical curves, that is, solutions to $dE = 0$.

In this note we construct the first explicit examples of knotted curves which are critical for E. Our basic observation (§2) is that the conformally invariant energy E extends naturally to simple closed curves $\Gamma \subset \mathbb{R}^m$. In particular, if $\Gamma \subset S^3 \subset \mathbb{R}^4$ and $\sigma : S^3 \longrightarrow \mathbb{R}^3 \cup \infty$ is stereographic projection, then $E(\sigma(\Gamma)) = E(\Gamma)$ provided $\sigma(\Gamma) \subset \mathbb{R}^3$. Then we use the principle of symmetric criticality (§3) to show that for each relatively prime pair of integers (p, q) there is a (p, q)-torus knot $\Gamma_{p,q} \subset S^3$ which is critical for E. This curve $\Gamma_{p,q}$ is a principal orbit for an isometric S^1-action on S^3. In the same way, we construct critical orbital links with more than one component.

The energies $E(\Gamma_{p,q})$ can be computed explicitly in terms of rational trigonometric integrals. This makes them ideal benchmarks to check the accuracy of computer experiments with various discretized energies. We mention one such discrete model in §4. The authors have implemented several programs using this model, run a number of experiments, and tabulated some of the results (§4), but clearly more work needs to be done. In this regard, we should mention that several Japanese mathematicians – and independently, S. Bryson – as noted in [BFHW], have also conducted computer experiments to seek nontrivial extrema.

It is well-known (see, for example, [S], p. 250) that torus knots are prime, and it is tempting to conjecture that the stereographic images $\sigma(\Gamma_{p,q})$ are the energy minimizers guaranteed to exist by [FHW]. However, we expect this conjecture is false when both p and q are large, for the following reason: we can view $\Gamma_{p,q}$ as a

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p- or q-strand braid lying on the surface of a torus, and if both p and q are at least 4, energy can be saved when "extra strands" of $\Gamma_{p,q}$ depart from the surface of the torus and push into the interiors of the complementary solid tori. Our most recent experiments (§5) suggest that $\Gamma_{p,q}$ is a stable local minimum for E when p or q is less than 4, but not a minimizer otherwise, lending evidence for our conjecture (as qualified) above. The general E-stability question will be explored elsewhere [KS].

2. Conformal Invariance.

Suppose $\Gamma \subset \mathbb{R}^m$ is a $C^{1,1}$ simple closed curve, meaning Γ admits a parametrization $\Gamma : S^1 \hookrightarrow \mathbb{R}^m$ whose first derivative $\Gamma'(z)$ is Lipschitz. (Weaker differentiability assumptions are possible – see §1 of [FHW] – but we find $C^{1,1}$ suitable for our purposes.) Let

$$D(x,y) = \min\left\{ \int_x^y |\Gamma'(z)|\, dz\,, \quad \int_y^x |\Gamma'(z)|\, dz \right\}$$

denote the distance between $\Gamma(x)$ and $\Gamma(y)$ *within* Γ.

Definition/Proposition *Given* Γ *as above, its* (conformal) energy

$$E(\Gamma) = \iint\limits_{S^1 \times S^1} \left[\frac{1}{|\Gamma(x) - \Gamma(y)|^2} \;-\; \frac{1}{D^2(x,y)} \right] |\Gamma'(x)|\, |\Gamma'(y)|\, dxdy$$

is finite and independent of the parametrization of Γ. *Moreover, if* μ *is a Möbius transformation of* $\mathbb{R}^m \cup \infty$, *then* $E(\mu(\Gamma)) = E(\Gamma)$ *provided* $\mu(\Gamma) \subset \mathbb{R}^m$.

Proof: The arguments, with only formal modifications to move from 3 to m dimensions, are essentially in §1 and §2 of [FHW]. In particular, concerning the conformal invariance, the first term in the integrand is *pointwise* invariant, whereas the second term in the integral is "intrinsic", in the sense that, for any $\varepsilon > 0$, the asymptotic expansion

$$\iint\limits_{|x-y| \geq \varepsilon} \frac{|\Gamma'(x)|\, |\Gamma'(y)|}{D^2(x,y)} dxdy = \frac{4\pi}{\varepsilon} - 4 + \mathcal{O}(\varepsilon)$$

is invariant under all $C^{1,1}$ diffeomorphisms of $\mathbb{R}^m$, not only Möbius transformations. (Note that the $4\pi/\varepsilon$ term here precisely cancels a corresponding singular term from the first integral. This is the "regularization" of E introduced in [O].) □

Corollary. *If* $\Gamma \subset S^m \subset \mathbb{R}^{m+1}$ *and* $\sigma : S^m \longrightarrow \mathbb{R}^m \cup \infty$ *is stereographic projection, then* $E(\sigma(\Gamma)) = E(\Gamma)$ *provided* $\sigma(\Gamma) \subset \mathbb{R}^m$.

Proof: Stereographic projection always extends to a Möbius transformation of $\mathbb{R}^{m+1} \cup \infty$. □

Examples. Consider the (p,q)-torus knot $\Gamma^r_{p,q} \subset S^3\left(\sqrt{1+r^2}\right) \subset \mathbb{R}^4 = \mathbb{C}^2$ defined by

$$\Gamma^r_{p,q}(x) = \begin{pmatrix} r\, \cos\, px \\ r\, \sin\, px \\ \cos\, qx \\ \sin\, qx \end{pmatrix} \quad = \quad \begin{pmatrix} r\, e^{ipx} \\ e^{iqx} \end{pmatrix}.$$

Then the conformal energy of its stereographic image $\sigma(\Gamma^r_{p,q}) \subset \mathbb{R}^3$ is

$$\begin{aligned} E\left(\sigma(\Gamma^r_{p,q})\right) &= E(\Gamma^r_{p,q}) \\ &= (r^2p^2+q^2)\int_0^{2\pi} dy \int_{y-\pi}^{y+\pi} dx \left[\frac{1}{r^2|e^{ipx}-e^{ipy}|^2+|e^{iqx}-e^{iqy}|^2} - \frac{1}{D^2(x,y)}\right] \\ &= 2\pi(r^2p^2+q^2)\int_{-\pi}^{\pi} d\theta \left[\frac{1}{r^2|e^{ip\theta}-1|^2+|e^{iq\theta}-1|^2} - \frac{1}{(r^2p^2+q^2)\theta^2}\right] \\ &= 2\pi(r^2p^2+q^2)\int_0^{\pi/2} d\varphi \left[\frac{1}{r^2\sin^2 p\varphi+\sin^2 q\varphi} - \frac{1}{(r^2p^2+q^2)\varphi^2}\right]. \end{aligned}$$

3. Symmetric Criticality.

For each pair (p,q) of integers, there is an isometric action $\alpha_{p,q}$ of S^1 on the unit sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ defined by

$$\alpha_{p,q}\left(e^{i\theta}\right) * \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} e^{ip\theta} & z \\ e^{iq\theta} & w \end{pmatrix}.$$

This action is effective when p and q are relatively prime. The case $(p,q) = (1,1)$ is the familiar *Hopf action*, whose orbits are great circles in S^3. In general, the $\alpha_{p,q}$-orbits of the points $\binom{1}{0}$ or $\binom{0}{1}$ are p- or q-fold coverings of the z- or w-axis circles S^1_z or S^1_w, respectively, while the principal $\alpha_{p,q}$-orbit which passes through the point

$$\begin{pmatrix} e^{it}\frac{r}{\sqrt{1+r^2}} \\ e^{it}\frac{1}{\sqrt{1+r^2}} \end{pmatrix}$$

for $r > 0$, is simply a rotation and rescaling of the (p,q)-torus knot $\Gamma^r_{p,q}$ defined in §2. We can think of (r,t) as polar coordinates for the 2-dimensional orbit space $\mathcal{S}_{p,q} = S^3/\alpha_{p,q}(S^1)$ of this action.

Lemma. *The knot $\Gamma^r_{p,q}$ is critical for E if and only if $\frac{\partial}{\partial r} E(\Gamma^r_{p,q}) = 0$ for some $r > 0$.*

Proof: The *only if* direction is clear, and the *if* direction is a consequence of the "principle of symmetric criticality" (see, for example, [P] for a general discussion of this principle) as follows: Consider a $C^{1,1}$ vector field $V : S^1 \longrightarrow \mathbb{R}^4$ and a variation $\Gamma_\varepsilon = \Gamma + \varepsilon V$ of $\Gamma = \Gamma^r_{p,q}$. According to [FHW] (Lemma 6.1), there is a unique linear functional dE such that

$$dE(V) = \frac{d}{d\varepsilon} E(\Gamma_\varepsilon)|_{\varepsilon=0},$$

for any V as above. (More formally, the Banach space of $C^{1,1}$ vector fields along Γ is the *tangent space* to the Banach manifold of $C^{1,1}$ curves, and dE is the *gradient one-form* of E, at Γ. Several explicit formulas for dE are given in §6 of [FHW], but its *linearity* is the only property we need here.) Suppose to the contrary that $dE(V) \neq 0$. Since Γ is an orbit of the S^1-action $\alpha = \alpha_{p,q}$, we can average the

push-forward $(\alpha(e^{i\theta}))_* V$ over S^1 to obtain an α-invariant vector field $\widetilde{V}$ for which $dE(\widetilde{V}) \neq 0$ as well.

Now $\widetilde{V}$ induces a variation through α-orbits, so we can view $\widetilde{V}$ as tangent to the orbit space $\mathcal{S} = \mathcal{S}_{p,q}$ at Γ. But

$$dE|_{\mathcal{S}} = \frac{\partial E}{\partial r}\, dr \,+\, \frac{\partial E}{\partial t}\, dt \,=\, \frac{\partial E}{\partial r}\, dr,$$

because rotation of $\mathcal{S}$ by e^{it} leaves E unchanged, and thus $dE|_{\mathcal{S}}(\widetilde{V}) = dE(\widetilde{V}) \neq 0$ implies $\frac{\partial E}{\partial r} \neq 0$ as required. □

We come next to our main result.

Theorem 1. *For each relatively prime pair of integers p and q greater than 1, there is a (p,q)-torus knot $\Gamma_{p,q} \subset S^3$ which is critical for the conformal energy E. The knot $\Gamma_{p,q}$ is a principal orbit of the S^1-action $\alpha_{p,q}$ defined above.*

Proof: The energy $E(r) = E(\Gamma^r_{p,q}) = E\left(\frac{1}{\sqrt{1+r^2}}\Gamma^r_{p,q}\right)$ is a smooth function of r. Because the orbit $\frac{1}{\sqrt{1+r^2}}\Gamma^r_{p,q}$ converges to either a p- or q-fold covering of a round circle as r approaches ∞ or 0, and since $p, q \geq 2$, we have $\lim\limits_{r\downarrow 0} E(r) = \lim\limits_{r\uparrow\infty} E(r) = +\infty$. Thus there is a finite value $r_0 > 0$ minimizing $E(r)$. By the previous lemma, the principal orbit $\Gamma_{p,q} = \frac{1}{\sqrt{1+r_0^2}}\Gamma^{r_0}_{p,q}$ is critical for E. □

In [FHW] the conformal energy of a k-component link $(\Gamma_1, \ldots, \Gamma_k)$ is defined as a natural extension of the conformal energy of a knot:

$$E(\Gamma_1, \ldots, \Gamma_k) = \sum_{1 \leq i,j \leq k} E(\Gamma_i, \Gamma_j)\,,$$

where $E(\Gamma_i, \Gamma_i) = E(\Gamma_i)$ as before, and the cross-terms $(i \neq j)$ are

$$E(\Gamma_i, \Gamma_j) = \iint\limits_{S^1 \times S^1} \frac{|\Gamma_i'(x)|\;|\Gamma_j'(y)|}{|\Gamma_i(x) - \Gamma_j(y)|^2}\; dxdy\,.$$

If we consider links whose k components are each orbits of the S^1-action $\alpha_{p,q}$, the principle of symmetric criticality applies: the orbital link $(\Gamma_1, \ldots, \Gamma_k)$ is critical for E among all links if and only if it corresponds to an E-critical configuration of k distinct regular points in the orbit space $\mathcal{S} = \mathcal{S}_{p,q}$, that is, to a critical point of E restricted to $\mathcal{S}^{k*} = \mathcal{S}^k \setminus \mathcal{D}$ (here $\mathcal{D}$ is the *big diagonal* Δ of $\mathcal{S}^k$ union the singular orbits). But E is a positive proper function on $\mathcal{S}^{k*}$, so it has a minimum, implying:

Theorem 2. *There are infinitely many E-critical links with any number of components.* □

We refer to a link L in $S^3 \subset \mathbb{C}^2$ whose component curves are orbits for the Hopf action $\alpha_{1,1}$ as a *Hopf link*. Note that L corresponds to a collection of points on $\mathcal{S}_{1,1} = S^2$, which we take to be the 2-sphere of radius 1. By a direct integration, as in the example of §2, one finds:

Proposition. *For a Hopf link L, the energy cross-term $E(\Gamma_i, \Gamma_j) = 2\pi^2 \csc \sigma$, where σ is the the angle between 2-planes in $\mathbb{R}^4$ meeting S^3 in the Hopf orbits Γ_i and Γ_j, or equivalently, where $s = 2\sigma$ is the angular distance between the corresponding points in S^2.* □

Remark. We observed with John Sullivan that this means the total energy of a k-component Hopf link is, up to a constant, simply the "Coulomb" energy for k point charges in $\mathbb{R}^3$ constrained to lie on S^2.

Corollary. *The Hopf link $L_2 = S^1_z \cup S^1_w$ with $E = 8 + 4\pi^2$ is critical for E.*

Proof: Since L_2 corresponds to a pair of antipodal points on S^2, it is the E-minimizer among 2-component Hopf links. (In fact, L_2 minimizes among all non-trivial 2-component links with at least one component a round circle [FHW], §7.) Therefore dE restricted to $\mathcal{S}^{2*}_{1,1} = S^2 \times S^2 \setminus \Delta$ vanishes at L_2, so the principle of symmetric criticality gives the result. □

Remark. One can also show that the minimum energy Hopf link L_k with $k = 3$ or 4 components corresponds to the obvious equilateral configuration of k points on S^2. (Their energies are $E(L_3) = 12 + 8\sqrt{3}\pi^2 \approx 148.7572$ and $E(L_4) = 16 + 12\sqrt{6}\pi^2 \approx 306.1059$, respectively.) On the other hand, there is a 4-component critical link $L_{4,e}$ (with $E = 16 + 8(1 + 2\sqrt{2})\pi^2 \approx 318.2805$), corresponding to 4 points arranged in an "equatorial square" on S^2, which is *not* a local minimum for E. Observe that $L_{4,e}$ lies on the surface of a torus (since, in the orbit space S^2, these 4 points lie on a circle), but that energy is lowered as a component leaves the surface (in S^2, as the 4 points move to the regular tetrahedral configuration). If we think of $L_{4,e}$ as a "$(4,4)$-torus link", this phenomenon illustrates the conjectured behavior (mentioned in the Introduction) of an E-minimizing (p,q)-torus knot for p and q at least 4. (Further results on Hopf links are given in [KS].)

4. Discretization and Computer Experiments.

Given a $C^{1,1}$ knot $\Gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^m$, and a sufficiently large integer n, the polygon γ with vertices $\gamma(h) = \Gamma(2\pi h/n)$, $1 \le h \le n$, is a knot isotopic to Γ, which we call a *polygonal approximation* to Γ. One possible discretized energy is simply $E(\gamma)$, which unfortunately is infinite! It can be regularized, by deleting the contributions of adjacent edges, to an energy $E^*(\gamma)$ which has the advantage of an infinite barrier to changing isotopy type (but the disadvantage of being difficult to compute). Another discrete energy $e(\gamma)$, defined below, is much easier to compute. It can be shown that both $E^*(\gamma)$ and $e(\gamma)$ converge to $E(\Gamma)$ as γ approaches Γ for n very large.

Definition. *The* discrete energy *of a polygon with vertices $\gamma : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{R}^m$ is*

$$e(\gamma) = \sum_{1 \le i \ne j \le n} \sum \left[\frac{1}{|\gamma(i) - \gamma(j)|^2} - \frac{1}{d^2(i,j)} \right] |\gamma'(i)|\ |\gamma'(j)|,$$

where

$$d(i,j) = \min\left\{ \sum_{i \le h < j} |\gamma(h) - \gamma(h+1)|\,, \quad \sum_{j \le h < i} |\gamma(h) - \gamma(h+1)| \right\}$$

and

$$|\gamma'(h)| = \frac{|\gamma(h-1)-\gamma(h)| \ + \ |\gamma(h)-\gamma(h+1)|}{2}.$$

Observe that if $\gamma = \gamma_{p,q}$ is a polygonal approximation to the orbit $\Gamma_{p,q} \subset S^3$, then (by setting $h = i - j$) the double-sum reduces to

$$e(\gamma) = n \ |\gamma'(1)|^2 \sum_{1<h\leq n} \left[\frac{1}{|\gamma(1)-\gamma(h)|^2} \ - \ \frac{1}{d^2(1,h)} \right].$$

We used this in a simple search routine called *Torus Knot Energy* to produce the following table of approximate values for the conformal energies of our critical (p,q)-torus knots. (Our program, written in `FORTRAN`, is available from `polyphemus.gang.umass.edu` by anonymous ftp in the directory `pub/knot/`.) The third column gives the corresponding critical radius (r_0). Note, by comparing the first six lines of the table with the exact value $E = 4$ for the round circle, that the approximate turning angle (d°, in degrees) at each vertex gives a rough percentage bound for the underestimation by $e = e(\gamma_{p,q})$ of the true energy $E(\Gamma_{p,q})$. The number of vertices (n) in the polygonal approximation $\gamma_{p,q}$ is given in the far right column. Although not recorded in our table, we checked our computed values by interchanging p and q: to the number of digits given, this led to the same e and reciprocal r_0, as expected. (Bryson reports that his experiments found a trefoil – that is, a $(2,3)$-torus knot – with energy about 74; compare with lines 7 through 10 of the table. The sharpest value we have computed for $e(\gamma_{2,3})$ is 74.4111 with r_0 around 1.8572078 at $n = 92160$. From the geometric progression of the values of e as we double n, it appears $74.4121 < E(\Gamma_{2,3}) < 74.4122$.)

5. Further Experiments.

Very recently, the second author and John Sullivan computed the gradient *de* of the discretized energy e, and had Ken Brakke incorporate this into his *Surface Evolver* computer program (*Version 1.89*). Because e and *de* are each *quadratic* in n, convergence of the negative gradient flow to a critical knot or link is considerably slower than in our program *Torus Knot Energy*, which searches only among orbits; the practical size of n in the *Surface Evolver* is limited accordingly. Nevertheless, we were able to test the stability of $\Gamma_{2,q}$ and $\Gamma_{3,q}$ for q small, as well as find a non-orbital $(4,5)$-torus knot with energy $E \approx 388$, considerably less than $E(\Gamma_{4,5}) \approx 406$ (see table). We hope to report further on this in future joint work with Brakke, Sullivan, and a group of students at the Five Colleges Geometry Institute.

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p	q	r_0	e	d°	n
1	1	——	3.9608	1	360
1	1	——	3.9804	.5	720
1	1	——	3.9902	.25	1140
1	1	——	3.9951	.12	2880
1	1	——	3.9975	.06	5760
1	1	——	3.9987	.03	11520
2	3	1.85...	74.353	.68	1440
2	3	1.857..	74.383	.34	2880
2	3	1.8572.	74.397	.17	5760
2	3	1.85720	74.405	.09	11520
2	5	3.4384.	130.69	.34	5760
2	5	3.43849	130.70	.17	11520
2	7	4.9479.	184.89	.42	5760
2	7	4.94799	184.92	.21	11520
2	9	6.4322.	238.43	.55	5760
2	9	6.43225	238.47	.27	11520
3	4	1.4868.	204.19	.21	5760
3	4	1.48687	204.20	.11	11520
3	5	1.9677.	264.20	.27	5760
3	5	1.96779	265.26	.14	11520
3	7	2.9015.	383.48	.48	5760
3	7	2.90157	383.52	.24	11520
3	8	3.3588.	441.48	.47	5760
3	8	3.35885	441.62	.23	11520
4	5	1.3409.	406.11	.29	5760
4	5	1.34094	406.14	.14	11520
4	7	2.0210.	599.52	.40	5760
4	7	2.02106	599.56	.20	11520
4	9	2.6867.	787.80	.52	5760
4	9	2.68677	787.85	.26	11520
5	6	1.2623.	686.65	.35	5760
5	6	1.26234	686.69	.18	11520
5	7	1.5265.	823.83	.40	5760
5	7	1.52659	823.88	.20	11520
5	8	1.7899.	959.15	.46	5760
5	8	1.78991	959.20	.23	11520
5	9	2.0513.	1093.0	.52	5760
5	9	2.05137	1093.1	.26	11520
6	7	1.2131.	1050.9	.41	5760
6	7	1.21313	1050.9	.21	11520
7	8	1.1793.	1503.0	.47	5760
7	8	1.17939	1503.1	.24	11520
7	9	1.3604.	1726.4	.52	5760
7	9	1.36044	1726.5	.26	11520
8	9	1.1548.	2046.6	.54	5760
8	9	1.15483	2046.7	.27	11520

References

[BFHW] S. Bryson, M.H. Freedman, Z.X. He and Z. Wang, *Mobius Invariance of Knot Energy*. Bull. Amer. Math. Soc. **28**, (1993) 99–103.

[FHW] M.H. Freedman, Z.X. He and Z. Wang, *On the Energy of Knots and Unknots*. Geometry Center Research Report GCG40.

[KS] R. Kusner and J. Sullivan, *A Conformally Invariant, Nonlocal Energy for Embedded Submanifolds*. In preparation.

[O] J. O'Hara, *Energy of a Knot*. Topology **30**, (1991) 241–247.

[P] R. Palais, *The Principle of Symmetric Criticality*. Comm. Math. Physics **69**, (1979) 19–30.

[S] H. Schubert, *Knoten und Vollringe*. Acta Math. **90**, (1953) 131–286.