Limit Surfaces of Riemann Examples

David Hoffman, Wayne Rossman

1 Introduction

The only connected minimal surfaces foliated by circles and lines are domains on one of the following surfaces: the helicoid, the catenoid, the plane, and the examples of Riemann ([Ri] p329-33, [En] p403-6, [Ni] p85-6). All these surfaces are complete and embedded. Topologically they are planar domains: the helicoid is simply-connected, the catenoid is an annulus (conformally a twice-punctured sphere), and each Riemann example (see Figure 1) is conformal to the plane minus the points $\{(n, 0), (\frac{1}{n}, 0) \,|\, n \in \mathrm{Z}\}$ [HKR]. In this section we will show that the plane, helicoid, and catenoid arise naturally as limits of well-chosen and properly normalized sequences of Riemann examples. The local behavior of domains on Riemann examples accounts for the existence of these limits, thus allowing a change of topology in the limit surfaces.

Theorem 1.1 *Any sequence of Riemann examples that converges to a surface in* R^3 *converges either to a Riemann example, a helicoid, a catenoid, an infinite set of equally spaced parallel planes, or a single plane. Furthermore, there exist sequences of Riemann examples that converge to each of these possibilities.*

In order to be precise, we state what we mean by convergence of surfaces in R^3 in this context.

Definition 1.1 *A sequence of surfaces* $\{\mathcal{S}_i\}_{i=1}^{\infty}$ converges *as* $i \to \infty$ *to a surface* $\mathcal{S}$ *in* R^3 *if, for any compact region* $B \subseteq \mathrm{R}^3$*, there exists an integer* N_B *such that for* $i > N_B$, $\mathcal{S}_i \cap B$ *is a normal graph over* $\mathcal{S}$*, and* $\{\mathcal{S}_i \cap B\}_{i=N_B}^{\infty}$ *converges to* $\mathcal{S}$ *in the* C^∞*-topology.*

One can define analogously the convergence of a one-parameter family $\mathcal{S}_t, t \in \mathrm{R}$, as $t \to t_0$. Since this convergence is in the C^∞-topology, it requires that derivatives of all orders of the coordinate functions on the graphs converge to the corresponding derivatives of the coordinate functions on the limit-surface.

We observe that catenoids, helicoids, and planes are limit-surfaces of sequences of Riemann examples. Therefore the collection of connected embedded minimal surfaces foliated by circles and lines is itself connected in the topology associated with this convergence.

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Figure 1: The Riemann Example $\mathcal{R}_\lambda$, $\lambda = 1$

2 The Family of Riemann Examples

2.1 Definition of the Riemann Examples

We now describe properties of the Riemann examples, and we give a Weierstrass representation for these surfaces. Proofs of the statements in this section may be found in [HKR].

For any Riemann example $\mathcal{R}$, there exists a translation T of R^3 so that T: leaves $\mathcal{R}$ invariant; is orientation-preserving on $\mathcal{R}$; generates the full cyclic orientation-preserving translation symmetry group of $\mathcal{R}$. The quotient-surface $\mathcal{R}/T$ is a twice-punctured rectangular torus. Thus, Riemann examples can be parametrized via a Weierstrass representation on such a torus. Fix $\lambda > 0$. We first define $\bar{M}_\lambda$:

$$\bar{M}_\lambda = \{\ (z, w) \in (\mathrm{C} \cup \{\infty\})^2 : w^2 = z(z - \lambda)(z + \frac{1}{\lambda})\ \}\ . \tag{2.1}$$

Let M_λ be the twice-punctured torus

$$M_\lambda = \bar{M}_\lambda \setminus \{(0, 0),\ (\infty, \infty)\}\ . \tag{2.2}$$

With the Weierstrass data

$$g(z, w) = z\ ,\ \eta(z, w) = \frac{dz}{zw}\ , \tag{2.3}$$

the Weierstrass representation for the Riemann examples $\mathcal{R}_\lambda$ is

$$R_\lambda(p) := Re \int_{p_0}^{p} \Phi_{(g,\eta)}\, dz\ ,\quad \Phi_{(g,\eta)} := \begin{pmatrix} \frac{(1-z^2)dz}{zw} \\ \frac{i(1+z^2)dz}{zw} \\ \frac{2dz}{w} \end{pmatrix}\ ,\quad p \in M_\lambda\ . \tag{2.4}$$

The family $\mathcal{R}_\lambda$ of Riemann examples is parametrized by $\lambda \in (0, \infty)$.

The surface M_λ can be considered to be a double covering of the punctured z-plane, branched at λ and $\frac{-1}{\lambda}$. Consider a circle α in the z-plane with center $\frac{-1}{2\lambda}$ and radius $\frac{1}{2}(\lambda + \frac{1}{\lambda})$, and lift α to a closed curve $\hat{\alpha}$ in M_λ. The Weierstrass integral about $\hat{\alpha}$ results in a nonzero period-vector, and this is the translation T.

Letting $\acute{z} = -z$ and $\acute{w} = iw$, we have

$$\acute{w}^2 = \acute{z}(\acute{z} - \frac{1}{\lambda})(\acute{z} + \lambda) \ .$$

Thus we may consider the map $(z, w) \to (\acute{z}, \acute{w})$ to be a conformal diffeomorphism between M_λ and $M_{\frac{1}{\lambda}}$.

Because the metric on $\mathcal{R}_\lambda$ ([Os] p65, [HKR]) is

$$ds^2 = \frac{(1 + |z|^2)^2}{4|z|^2|w|^2}|dz|^2 \ ,$$

the maps $(z, w) \to (\bar{z}, \bar{w})$ and $(z, w) \to (\bar{z}, -\bar{w})$ both restrict to isometries of M_λ in the induced metric. The set $\mathcal{A} = \{(t, w) \in M_\lambda \,|\, t \in \mathrm{R}\}$, which consists of the union of their fixed-point sets, is mapped by the Weierstrass integral to geodesics in $\mathcal{R}_\lambda$. It is clear from the Weierstrass integral that the intervals in $\mathcal{A}$ of the form $t \leq \frac{-1}{\lambda}$ or $0 < t \leq \lambda$ represent lines in $\mathcal{R}_\lambda$, while intervals in $\mathcal{A}$ of the form $\frac{-1}{\lambda} \leq t < 0$ or $t \geq \lambda$ represent planar geodesics in $\mathcal{R}_\lambda$. Planar geodesics must be principal curves.

The surfaces $\mathcal{R}_\lambda$ have the following additional properties:

1) $\mathcal{R}_\lambda$ is foliated by circles and lines in horizontal planes. All the lines in $\mathcal{R}_\lambda$ are the image under the Weierstrass representation of points in $\mathcal{A}$ such that $t \leq \frac{-1}{\lambda}$ or $0 < t \leq \lambda$, which is the fixed-point set of the map $(z, w(z)) \to (\bar{z}, -\bar{w}(z))$.

2) $\mathcal{R}_\lambda$ has an infinite number of equally spaced horizontal flat ends. These ends correspond to the punctures of $\bar{M}_\lambda$ at $z = 0, \infty$.

3) $\mathcal{R}_\lambda$ is invariant under reflection in a plane parallel to the (x_1, x_3)-plane. The intersection of this plane with $\mathcal{R}_\lambda$ consists of the image under the Weierstrass representation of points in $\mathcal{A}$ such that $\frac{-1}{\lambda} \leq z < 0$ or $z \geq \lambda$, which is the fixed-point set of the map $(z, w(z)) \to (\bar{z}, \bar{w}(z))$.

4) $\mathcal{R}_\lambda$ is invariant under rotation about horizontal lines that are perpendicular to the (x_1, x_3)-plane, and meet the surface orthogonally at the points where $g(z) = \pm i$. The map $(z, w) \to (\frac{-1}{z}, \frac{w}{z^2})$ restricted to M_λ represents this rotation.

We will consider a Riemann example to be any surface that is the image of one of these surfaces $\mathcal{R}_\lambda$ under an isometry or homothety of R^3.

2.2 Conjugate Pairs of Riemann Examples

Proposition 2.1 *For any positive $\lambda \neq 1$, the Riemann examples $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ are conjugate but not congruent. $\mathcal{R}_1$ is self-conjugate.*

Proof. Choose $\lambda > 1$. The conjugate surface of $\mathcal{R}_\lambda$ has Weierstrass data

$$g = z \ ,$$
$$i\eta = \frac{idz}{zw} = \frac{dz}{(-z)(iw)} \ .$$

The Weierstrass data for the conjugate surface, expressed in terms of $(\acute{z}, \acute{w}) = (-z, iw)$ on $M_{\frac{1}{\lambda}}$, is

$$\tilde{g}(\acute{z}, \acute{w}) = -\acute{z} \ ,$$
$$\tilde{\eta}(\acute{z}, \acute{w}) = \frac{-d\acute{z}}{\acute{z}\acute{w}} \ .$$

Because

$$\Phi_{(\tilde{g},\tilde{\eta})} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi_{(g,\eta)} \ ,$$

it follows from equation (2.4) that the minimal surface in R^3 with this Weierstrass data is the image of $\mathcal{R}_{\frac{1}{\lambda}}$ under a rotation by π about a vertical axis. Hence $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ are conjugate surfaces.

We use the Weierstrass data to show that $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ cannot be congruent for $\lambda \neq 1$. Consider the points on $\mathcal{R}_\lambda$ where $g = \pm 1$. As shown in the previous Section 2.1 (Property 1), when $\lambda > 1$, these points lie on the lines in $\mathcal{R}_\lambda$. By contrast, the points of $\mathcal{R}_{\frac{1}{\lambda}}$ where $g(z) = \pm 1$ lie on the planar geodesics in the vertical plane of reflective symmetry of the surface. Suppose there exists a symmetry of R^3 taking $\mathcal{R}_\lambda$ to $\mathcal{R}_{\frac{1}{\lambda}}$, then this symmetry must take lines to lines, ends to ends, and planar geodesics to planar geodesics. Since the lines in these two surfaces are all parallel to the x_2-axis and the ends are horizontal, the symmetry of R^3 must be of the form

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} .$$

Therefore points of $\mathcal{R}_\lambda$ where $g = \pm 1$ must be mapped to points of $\mathcal{R}_{\frac{1}{\lambda}}$ where $g = \pm 1$. This implies that the points of $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ where $g = \pm 1$ are the points where the lines and planar geodesics in the surface intersect i.e. where $g = \pm \lambda^{\pm 1}$. But we are assuming $\lambda \neq 1$. Thus $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ are not congruent. □

Remark. One checks that the mapping $(z, w) \to (z', w')$ from M_λ to $M_{\frac{1}{\lambda}}$ has the property that on fundamental cycles it preserves the periods (one of which is always zero) of (2.4). Hence this mapping induces an isometry of $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$. □

Upon considering the behavior of $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ as $\lambda \to 0$, one finds that $\mathcal{R}_\lambda$ and $\mathcal{R}_{\frac{1}{\lambda}}$ do not have limits that are regular surfaces. However, if these surfaces are normalized appropriately they converge to a catenoid and its conjugate surface, the helicoid, which is consistent with Proposition 2.1. We shall prove this in the next section.

3 Proof of the Main Result

To find a sequence of Riemann examples that converges to a catenoid or helicoid, we renormalize the surfaces $\mathcal{R}_\lambda$ as follows. Begin the integration in the Weierstrass representation at the point $z_0 = 1$. If $\lambda > 1$, rescale by $\sqrt{\lambda}$; if $\lambda < 1$, rescale by $\frac{1}{\sqrt{\lambda}}$. We name these normalized Riemann examples $\acute{\mathcal{R}}_\lambda$. Note that the rescaling is accomplished by multiplying η in the Weierstrass representation by $\sqrt{\lambda}$ and $\frac{1}{\sqrt{\lambda}}$, respectively. Thus, the Weierstrass representation for $\acute{\mathcal{R}}_\lambda$ is

$$g = z \ , \ \eta = \frac{\sqrt{\lambda} dz}{zw} \ , \ \lambda \geq 1 \ ,$$

$$g = z \ , \ \eta = \frac{dz}{\sqrt{\lambda} zw} \ , \ \lambda \leq 1 \ .$$

One can check as in Proposition 2.1 that $\acute{\mathcal{R}}_\lambda$ and $\acute{\mathcal{R}}_{\frac{1}{\lambda}}$ are conjugate.

Let K denote Gauss curvature.

Lemma 3.1 *There exists a universal bound c such that $|K| \leq c$ on $\acute{\mathcal{R}}_\lambda$ for all $\lambda \in (0, \infty)$.*

Proof. With Weierstrass data given locally as g and $\eta = f dz$, the Gauss curvature of a minimal surface is ([Os], p76)

$$K = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2 \ . \tag{3.5}$$

In the case of $\acute{\mathcal{R}}_\lambda$ a computation using equations (2.1) and (3.5) yields

$$|K| = 16\lambda^\alpha \frac{|z-\lambda||z+\lambda^{-1}|}{|z|(|z|+|z|^{-1})^4} \ .$$

where $\alpha = 1$ if $\lambda \leq 1$, and $\alpha = -1$ if $\lambda \geq 1$. If $\lambda \leq 1$, then

$$|K| \leq 16 \frac{(|z|+1)\lambda(|z|+\lambda^{-1})}{|z|(|z|+|z|^{-1})^4} \ \leq \ 16 \frac{(|z|+1)^2}{|z|(|z|+|z|^{-1})^4} \ \leq 4 \ .$$

Similarly, if $\lambda \geq 1$, then

$$|K| \leq 16 \frac{\lambda^{-1}(|z|+\lambda)(|z|+1)}{|z|(|z|+|z|^{-1})^4} \ \leq \ 16 \frac{(|z|+1)^2}{|z|(|z|+|z|^{-1})^4} \ \leq 4 \ .$$

Thus, $|K|$ has a universal upper bound. $\square$

Remark. More calculation shows that $|K|$ on $\mathcal{R}_\lambda$ is maximized at a fixed point of one of the symmetries of the surface. Thus K is maximized in one of the following three places: somewhere along a straight line in $\mathcal{R}_\lambda$; somewhere along a planar geodesic in the plane of reflective symmetry parallel to $\{x_2 = 0\}$; or at the points where $g(z) = \pm i$, which are the fixed points of the normal rotation. At the points where $z = g(z) = \pm i$, the value of $|K|$ on $\mathcal{R}_\lambda$ is $\lambda + \frac{1}{\lambda}$. Hence on $\acute{\mathcal{R}}_\lambda$ the value is $1 + \frac{1}{\lambda^2}$(resp. $1 + \lambda^2$) when $\lambda > 1$(resp. when $\lambda < 1$).

The following conjecture has been verified numerically: The absolute value of the Gaussian curvature on $\mathcal{R}_\lambda$ is maximized at the points where $z = g(z) = \pm i$. This implies that the optimal value in Lemma 3.1 is $c = 2$. $\square$

Proposition 3.1 *There exist sequences of Riemann examples that converge to a helicoid and sequences that converge to a catenoid.*

Proof. We will prove this proposition by showing that the limit, as $\lambda \to 0$, of $\acute{\mathcal{R}}_\lambda$ is a catenoid, and the limit, as $\lambda \to \infty$, of $\acute{\mathcal{R}}_\lambda$ is a helicoid (see Figures 2 through 4). We consider first the case $\lambda \to 0$.

For a large positive number L, let A_L be the annular ring $\{z \in \mathrm{C} : \frac{1}{L} < |z| < L\}$. One can think of z in (2.1) as a map from M_λ to $\mathrm{C} \setminus \{0\}$. Then for λ sufficiently small, $z^{-1}(A_L)$ consists of two disjoint sets in M_λ. Choose either one and name it $\acute{A}_{L,\lambda}$. Since z parametrizes $\acute{A}_{L,\lambda}$, we shall refer to points in $\acute{A}_{L,\lambda}$ by their z coordinates.

Let B_r be a ball centered at the origin in R^3 with radius r. Let S_r be the horizontal slab $\{(x_1, x_2, x_3) \in \mathrm{R}^3 \mid -r \le x_3 \le r\}$, with boundary planes $x_3 = \pm r$.

Let $\mathcal{C}$ be the catenoid obtained by using the Weierstrass data

$$g(z) = z \ , \ \eta(z) = \frac{dz}{z^2} \ ,$$

and integrating over $\mathrm{C} \setminus \{0\}$ with base point $z_0 = 1$. The Weierstrass representation for the catenoid $\mathcal{C}$ is

$$C(p) := Re \int_1^p \begin{pmatrix} \frac{(1-z^2)dz}{z^2} \\ \frac{i(1+z^2)dz}{z^2} \\ \frac{2dz}{z} \end{pmatrix} \ , \ p \in \mathrm{C} \setminus \{0\} \ .$$

Let $\hat{\mathcal{C}}_L$ be the image of the restriction of $C(p)$ to A_L. By examination of the third coordinate of the Weierstrass representation, we see that $\hat{\mathcal{C}}_L = \mathcal{C} \cap S_{2\ln L}$. Choose a large value for r, and choose $L > e^r$ so that $\mathcal{C} \cap S_{2r} \subseteq \hat{\mathcal{C}}_L$.

For $\lambda < 1$, the Weierstrass representation for $\acute{\mathcal{R}}_\lambda$ is

$$\acute{R}_\lambda(p) := Re \int_1^p \begin{pmatrix} \frac{(1-z^2)dz}{\sqrt{\lambda}zw} \\ \frac{i(1+z^2)dz}{\sqrt{\lambda}zw} \\ \frac{2dz}{\sqrt{\lambda}w} \end{pmatrix} = C(p) + Re \int_1^p \begin{pmatrix} f_{0,\lambda}(z)\frac{(1-z^2)dz}{z^2} \\ f_{0,\lambda}(z)\frac{i(1+z^2)dz}{z^2} \\ f_{0,\lambda}(z)\frac{2dz}{z} \end{pmatrix} \ , \qquad (3.6)$$

Figure 2: Portions of Riemann Examples $\acute{\mathcal{R}}_\lambda$, $\lambda = 0.1$ (left), $\lambda = 0.5$ (right)

Figure 3: Portions of Riemann Examples $\acute{\mathcal{R}}_\lambda$, $\lambda = 1.0$ (left), $\lambda = 3.0$ (right)

Figure 4: Portions of Riemann Examples $\acute{\mathcal{R}}_\lambda$, $\lambda = 5.0$ (left), $\lambda = 10.0$ (right)

where

$$f_{0,\lambda}(z) = \frac{\sqrt{z}}{\sqrt{(z-\lambda)(\lambda z+1)}} - 1 \ .$$

If λ is close to zero, then the Weierstrass integral $\acute{R}_\lambda$ along any closed curve in $\acute{A}_{L,\lambda}$ is zero. Therefore the Weierstrass integral depends only on the endpoints of a path in $\acute{A}_{L,\lambda}$. All paths from $z_0 = 1$ to $p \in A_L$ may be chosen to be of length less than cL, where c is some fixed constant.

Note that the functions

$$\frac{1-z^2}{z^2} \ , \ \frac{1+z^2}{z^2} \ , \ \frac{2}{z}$$

are bounded on A_L. Also, for all $z \in A_L$,

$$\lim_{\lambda \to 0} |f_{0,\lambda}(z)| = 0 \ .$$

It follows that the last integral in equation (3.6) becomes arbitrarily small, uniformly on $\acute{A}_{L,\lambda}$, as $\lambda \to 0$. In particular, for all $p \in A_L$, $\lim_{\lambda\to 0} \acute{R}_\lambda(p) = C(p)$.

Let $\hat{\mathcal{R}}_\lambda$ be the image of $\acute{A}_{L,\lambda}$ under the map $\acute{R}_\lambda(p)$. It follows that $\hat{\mathcal{R}}_\lambda$ is a small perturbation of $\hat{\mathcal{C}}_L$ for λ close to zero. We conclude that $\acute{\mathcal{R}}_\lambda \cap B_r \subseteq \hat{\mathcal{R}}_\lambda$. Thus to examine the behavior of $\acute{\mathcal{R}}_\lambda$, as $\lambda \to 0$, in arbitrary compact regions of R^3, it is enough to examine $\hat{\mathcal{R}}_\lambda$ in B_r for arbitrarily large fixed r.

We claim that the surface $\acute{\mathcal{R}}_\lambda \cap B_r$ is a graph over a subset of the catenoid $\mathcal{C}$, for λ close to 0. By Lemma 3.1, the Gaussian curvature K is uniformly bounded for all $\acute{\mathcal{R}}_\lambda$. This implies that the normal curvatures are also uniformly founded. This in turn implies $\acute{\mathcal{R}}_\lambda \cap B_r$ is a union of graphs over a subset of the catenoid $\mathcal{C}$. Since the surface $\acute{\mathcal{R}}_\lambda$ is foliated by lines and circles in horizontal planes, there is a single graph.

We now show that $\acute{\mathcal{R}}_\lambda \cap B_r$ converges to $\mathcal{C} \cap B_r$, in the C^∞-topology, as $\lambda \to 0$. Because the curvature of $\acute{\mathcal{R}}_\lambda$ is bounded uniformly in λ, it follows from standard elliptic theory that the convergence is C^∞. (Curvature bounds give C^1 estimates.) However, we can give in our specific case a direct proof. Let $p \in \mathcal{C} \cap B_r$. Let $z_p \in A_L$ be the point such that $C(z_p) = p$, and let $B_h(z_p) \subseteq A_L$ be a ball of radius h about z_p. Note that $\acute{R}_\lambda(B_h(z_p))$ consists of disks in R^3 converging in the C^0-norm, as $\lambda \to 0$, to a disk containing p and lying on the catenoid $\mathcal{C}$. Let $x_{i,\lambda}(z)$ be the i'th coordinate function of $\acute{\mathcal{R}}_\lambda$, let $x_i(z)$ be the i'th coordinate function on $\mathcal{C}$, and let $f_\lambda(z) = x_{i,\lambda}(z) - x_i(z)$. Note that

$$\max_{w \in B_h(z_p)} |f_\lambda(w)| \to 0 \, , \ \text{as } \lambda \to 0 \ .$$

Since $f_\lambda(z)$ is harmonic, it is the real part of a holomorphic function $F_\lambda(z)$, and $F_\lambda(z)$ can be chosen so that

$$\max_{w \in B_h(z_p)} |F_\lambda(w)| \to 0 \, , \ \text{as } \lambda \to 0 \ .$$

Assuming $z \in B_{\frac{h}{2}}(z_p)$, and using the Cauchy integral formula, we have

$$\left|f_\lambda^{(k)}(z)\right| \leq \left|F_\lambda^{(k)}(z)\right| = \left|\frac{k!}{2\pi i}\int_{\partial B_h(z_p)} \frac{F(\xi)}{(\xi - z)^{k+1}} d\xi\right|$$

$$\leq \frac{k!}{2\pi}\left(\frac{2}{h}\right)^{k+1} 2\pi h \max_{w \in B_h(z_p)} |F_\lambda(w)| \ ,$$

and so we have that $|f_\lambda^{(k)}(z)| \to 0 \ \forall k, \ \forall z \in B_{\frac{h}{2}}(z_p)$, as $\lambda \to 0$. Therefore $|f_\lambda^{(k)}(z)| \to 0 \ \forall k, \ \forall z \in A_L$, and the convergence is in the C^∞-topology in B_r.

Since r is arbitrary, we have completed the proof that $\acute{\mathcal{R}}_\lambda$ converges to the catenoid $\mathcal{C}$ as $\lambda \to 0$.

It is now intuitively clear that the surfaces $\acute{\mathcal{R}}_\lambda$ converge to a helicoid for the following reason: as $\lambda \to \infty$, their conjugate surfaces $\acute{\mathcal{R}}_{\frac{1}{\lambda}}$ converge to a catenoid. One can give an explicit proof of this, similar to the proof just given for the case $\lambda \to 0$. However, there are some differences in the proof of the case $\lambda \to \infty$. For λ close to ∞, the representation for $\acute{\mathcal{R}}_\lambda$ is

$$\acute{R}_\lambda(p) := Re \int_1^p \begin{pmatrix} \frac{\sqrt{\lambda}(1-z^2)dz}{zw} \\ \frac{i\sqrt{\lambda}(1+z^2)dz}{zw} \\ \frac{2\sqrt{\lambda}dz}{w} \end{pmatrix} =$$

$$- Re \int_1^p \begin{pmatrix} \frac{i(1-z^2)dz}{z^2} \\ \frac{-(1+z^2)dz}{z^2} \\ \frac{2idz}{z} \end{pmatrix} + Re \int_1^p \begin{pmatrix} f_{\infty,\lambda}(z)\frac{i(1-z^2)dz}{z^2} \\ f_{\infty,\lambda}(z)\frac{-(1+z^2)dz}{z^2} \\ f_{\infty,\lambda}(z)\frac{2idz}{z} \end{pmatrix} , \tag{3.7}$$

where

$$f_{\infty,\lambda}(z) = 1 - \frac{\sqrt{z}}{\sqrt{(1-\frac{z}{\lambda})(z+\frac{1}{\lambda})}} \ .$$

Note that the first integral in the above sum in equation (3.7) is the Weierstrass integral for a helicoid, since Weierstrass data for a helicoid is

$$g(z) = z \ , \ \eta(z) = \frac{idz}{z^2} \ .$$

If λ is large, then the vertical distance between adjacent planar ends of $\acute{\mathcal{R}}_\lambda$ is approximately 2π. This can be verified by integrating the third function in the integrand of the Weierstrass representation from $+1$ to -1 along the half of the unit circle lying in the upper half of the complex plane. This curve is $z = e^{it}, 0 \leq t \leq \pi$. The distance between adjacent planar ends for large λ is

$$Re \int_0^\pi \frac{2\sqrt{\lambda}ie^{it}dt}{\sqrt{e^{it}(e^{it}-\lambda)(e^{it}+\frac{1}{\lambda})}} \approx Re \int_0^\pi \frac{2\sqrt{\lambda}idt}{\sqrt{-\lambda}} = 2\pi \ .$$

The major difference from the proof in the case $\lambda \to 0$ is this: When λ is close to ∞, then a homologically nontrivial loop in $\acute{A}_{L,\lambda}$ has a nonzero real period with respect to the

Weierstrass integral. Now one may only assume that a path from $z = 1$ to $z = p$ in A_L has length less than cnL, where n is the number of times that the path wraps about the origin, and c is some fixed constant. Since the distance between adjacent ends is approximately 2π, it follows that if the wrapping number n about the origin of a path from 1 to p is sufficiently large, then $z = p$ will be mapped by the Weierstrass integral to a point outside B_r. Thus, we may assume there is an upper bound N for $|n|$, depending only on r. Hence, for large λ, we may assume that any path from 1 to p has length less than $\acute{c}L$, where $\acute{c} = cN$.

The other parts of the proof of the case $\lambda \to \infty$ transfer directly from the proof of the case $\lambda \to 0$. $\square$

Corollary 3.1 *As $\lambda \to \infty$, the radius of any level-circle of $\acute{\mathcal{R}}_\lambda$ diverges to infinity.*

Corollary 3.2 *The planar curve described by the set of centers of the horizontal circles that foliate $\acute{\mathcal{R}}_\lambda$ has the property that its maximum curvature approaches zero as $\lambda \to \infty$.*

It remains to determine what other surfaces are limits of Riemann examples. Three possibilities are clear: a single flat plane; an infinite number of equally-spaced flat planes; and another Riemann example. The following proposition states that these are the only other possibilities, completing the proof of Theorem 1.1.

Proposition 3.2 *Any convergent sequence of Riemann examples converges to one of the following surfaces:*

a Riemann example;
a helicoid;
a catenoid;
a single flat plane;
an infinite number of equally-spaced flat planes.

Proof. Let $\{\mathcal{R}_j\}_{j=1}^{\infty}$ be a sequence of Riemann examples converging to a surface $\mathcal{S}$. From the definition of convergence, $\mathcal{S}$ is clearly minimal, properly embedded, and complete. Without loss of generality we may assume that the limiting normals $\vec{v_j}$ at the ends of the surfaces $\mathcal{R}_j$ converge in S^2 to a vertical vector. With this normalization, the surfaces $\mathcal{R}_j$ are foliated by circles and lines lying in planes that are becoming horizontal as $j \to \infty$. These almost-horizontal planes intersect $\mathcal{R}_j$ in connected sets, and these connected sets each consist of a single circle or a single line. Thus every curve $\mathcal{S} \cap \{x_3 = c\}$ is the limit set of a sequence of circles and/or lines. It follows that $\mathcal{S} \cap \{x_3 = c\}$ is either a circle, a line, the empty set or the plane $\{x_3 = c\}$. Recall from the introduction that this implies that each component of $\mathcal{S}$ is either a helicoid, a catenoid, a Riemann example or a plane. Moreover: the helicoid and catenoid must have vertical axes; the plane and the ends of the Riemann example must be horizontal. If $\mathcal{S}$ has more than one component, they must all be horizontal planes (because

$\mathcal{S}$ is fibred by circles or lines, one in each horizontal plane). Suppose $\mathcal{S}$ consists of two or more horizontal planes. Since the points on the surfaces $\mathcal{R}_j$ where the Gauss map is vertical must approach the ends of $\mathcal{R}_j$ as $j \to \infty$, it is the equally spaced-ends of the surfaces $\mathcal{R}_j$ that converge to $\mathcal{S}$. Therefore $\mathcal{S}$ is an *infinite* collection of equally-spaced planes. $\square$

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