Minimal surfaces and the affine Toda field model

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1 Introduction

Minimal immersions or, more generally, harmonic maps of a Riemann surface S into S^n, $\mathbb{C}P^n$ and other Riemannian symmetric spaces have been intensively studied over the past twenty five years.

The subject has been given considerable impetus by the interest of mathematical physicists in non-linear σ-models, which are harmonic maps of S^2 into $\mathbb{C}P^n$, and in related problems which may be handled using twistor theory and methods of complex geometry.

For example, [EW83], harmonic maps of S^2 into $\mathbb{C}P^n$ may be characterised as being elements of Frenet frames of holomorphic curves in $\mathbb{C}P^n$ or, equivalently, in twistorial terms as projections of suitable holomorphic horizontal curves in the twistor space $\boldsymbol{SU}(n+1)/T^n$ over $\mathbb{C}P^n$ (see [Bur90]). Indeed, considering $\boldsymbol{SU}(n+1)/T^n$ as the full flag manifold these holomorphic horizontal curves in $\boldsymbol{SU}(n+1)/T^n$ correspond precisely to Frenet frames of holomorphic curves in $\mathbb{C}P^n$. From an analytic viewpoint one has reduced the non-linear harmonic map equations to the Cauchy-Riemann equations.

A similar resolution of the problem may be found for harmonic maps of S^2 into S^n [Cal67], a complex Grassmannian $G_k(\mathbb{C}^n)$ [Wol88], or a compact Lie group G [Uhl89]. In the latter case, one has to consider holomorphic curves in a finite dimensional stratum of the based loop group of G.

For a (compact) Riemann surface S of higher genus the situation is more complicated. The above constructions give some but not all weakly conformal harmonic maps into S^n and $\mathbb{C}P^n$. Those which can be so constructed are the easiest to deal with. They have strong isotropy properties and are called *superminimal* or *totally isotropic* [Bry82], [EW83]. Such maps into S^2, S^3 and S^4 are particularly well-understood. Those into S^2 are $\pm$ holomorphic, those into S^3 factor through a totally geodesic S^2, and those into S^4 can be described using a Weierstrass representation [Bry82].

Considerable stimulus was given to the study of non-superminimal harmonic maps by the work of Wente [Wen86] on the existence of constant mean curvature tori in $\mathbb{R}^3$. These are precisely the immersed tori in $\mathbb{R}^3$ whose Gauss map $\phi\colon T^2 \to S^2$ is a non-superminimal harmonic map. Then in [PS89], Pinkall and Sterling showed that all non-superminimal harmonic maps $\phi\colon T^2 \to S^2$ are of finite type, that is to say are obtained from solutions to a finite dimensional completely integrable system of ordinary differential equations. In analytical terms this involves showing that every doubly periodic solution of the sinh-Gordon equation

$$w_{z\bar{z}} + \sinh 2w = 0\,, \tag{1.1}$$

is obtained by solving a finite dimensional system of ordinary differential equations. This is reminiscent of ideas in the theory of soliton equations [FT87], of which (1.1) is an example. Using an algebro-geometric integration algorithm developed for such equations, Bobenko [Bob91] gave parametrizations of all constant mean curvature tori in 3-dimensional space forms using θ-functions.

That such finite type behavior is to be expected for harmonic maps of 2-tori into higher dimensional spheres was indicated by Hitchin's study of harmonic 2-tori in S^3 [Hit90], and by the results in [FPPS92]

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where all minimal non-superminimal 2-tori in S^4 were constructed from completely integrable Hamiltonian ordinary differential equations in finite dimensional subspaces on a certain loop algebra. In the latter case the minimal surface equations turn out to be the affine Toda field equations for $\boldsymbol{SO}(5)$,

$$\begin{aligned} 2w_{z\bar{z}} - e^{2w}\cosh 2\eta + e^{2w} &= 0 \\ 2\eta_{z\bar{z}} + e^{-2w}\sinh\eta &= 0 \end{aligned} \tag{1.2}$$

which is another well-known example of a soliton system [For90]. Note that (1.1) is obtained from (1.2) by putting $\eta = 0$. Equations (1.1) and (1.2) have also been studied in [DW], [Wu93] using loop group factorization techniques.

The general theory of harmonic 2-tori in symmetric spaces was developed along similar lines in [BFPP], where, among other things, it was shown that every non-conformal harmonic map of a 2-torus into a rank 1 symmetric space is of finite type. However, the condition of non-conformality excludes the geometrically interesting case of minimal 2-tori.

By contrast, in the present paper we consider harmonic maps into S^n and $\mathbb{C}P^n$ which satisfy very strong isotropy conditions just short of superminimality. (Recall that conformality may be expressed as an isotropy condition on first derivatives.) Our approach extends that of [FPPS92] in a way which seeks to make the underlying theory more transparent.

We now describe the basic ideas for the case of harmonic maps into $\mathbb{C}P^n$, the S^n case being similar but rather more technical. Suppose that $\phi\colon S \to \mathbb{C}P^n$ is a harmonic map of a Riemann surface S into $\mathbb{C}P^n$, and without loss of generality we suppose that ϕ is linearly full, that is to say that $\phi(S)$ is not contained in a proper linear subspace of $\mathbb{C}P^n$. Then, as we recall in section 4, ϕ gives rise to a sequence $\{\phi_\ell\}$ of harmonic maps $\phi_\ell\colon S \to \mathbb{C}P^n$ with $\phi_0 = \phi$, called the *harmonic sequence* of ϕ. Adjacent elements in this sequence are orthogonal (with respect to the standard Hermitian inner product on $\mathbb{C}^{n+1}$) by construction and if some k consecutive elements are orthogonal then so are any k consecutive elements [BW86], [BW92], in which case ϕ is said to be k-isotropic. In particular, ϕ is conformal if and only if ϕ is 3-isotropic and, in all cases, ϕ is at most $(n+1)$-isotropic. There are two possible ways in which this latter case can arise. If ϕ is superminimal then the harmonic sequence has length $n+1$ and is just the Frenet frame of a holomorphic curve (the *directrix curve* of ϕ) so that in particular ϕ is $(n+1)$-isotropic. If ϕ is $(n+1)$-isotropic but not superminimal then the harmonic sequence of ϕ is, in an obvious sense, orthogonally periodic of period $n+1$ (c.f. remark 4.5) and we say that ϕ is *superconformal.*

Suppose now that $\phi\colon S \to \mathbb{C}P^n$ is a superconformal harmonic map. Then there is a map $\widetilde{\phi}\colon S \to \boldsymbol{SU}(n+1)/T^n$ of S into the manifold of full flags in $\mathbb{C}^{n+1}$ given by

$$\widetilde{\phi} = (\phi_0, \ldots, \phi_n)\,.$$

It follows that if $\pi\colon \boldsymbol{SU}(n+1)/T^n \to \mathbb{C}P^n$ is the map which assigns to a full flag its initial element, then $\phi = \pi \circ \widetilde{\phi}$. This is the analogue of the twistor lift for superminimal maps mentioned above, but in this case $\widetilde{\phi}$ is neither holomorphic nor horizontal. It is, however, τ*-primitive* (see definition 2.2) where $\tau = Ad\,\mathrm{diag}(1, \zeta, \ldots \zeta^n), \zeta = e^{2\pi/n+1}$, so that, in particular $d\widetilde{\phi}(T^{(1,0)}S)$ is contained in the ζ-eigenspace of the endomorphism of the complexified tangent bundle $T(\boldsymbol{SU}(n+1)/T^n)$ induced by τ. The significance of τ is that it is the symmetry of smallest order m which exhibits $\boldsymbol{SU}(n+1)/T^n$ as an m-symmetric space [Jim88] with $m = n+1$. Conversely, if $\psi\colon S \to \boldsymbol{SU}(n+1)/T^n$ is a τ-primitive map then $\phi = \pi \circ \psi$ is a superconformal harmonic map.

The relationship between superconformal harmonic maps and the affine Toda field model comes about from the fact that (theorem 2.5) any τ-primitive map $\psi\colon S \to \boldsymbol{SU}(n+1)/T^n$ has an essentially unique local framing $F\colon S \to \boldsymbol{SU}(n+1)$ whose Maurer-Cartan equations

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \qquad \alpha = F^{-1}dF$$

can be rewritten as the affine Toda field equations for $\boldsymbol{SU}(n+1)$(theorem 2.6). Thus τ-primitive maps, or equivalently superconformal harmonic maps, are the geometrical objects described by the affine Toda field model for $\boldsymbol{SU}(n+1)$. This, and its analogue for certain other compact simple Lie groups, is the crucial fact on which the whole paper is based.

In a similar fashion, superconformal harmonic maps of S into S^{2n-1} and S^{2n} (see section 5 for the definitions and geometrical interpretation of superconformality in these cases) correspond to τ-primitive maps into $\boldsymbol{SO}(2n+1)/T^n$ and thus give the geometric analogues of the affine Toda field models for these groups. The S^{2n}-case and its relationship to the affine Toda field model of $\boldsymbol{SO}(2n+1)$ has also been considered in [DS].

We remark that τ-primitive maps provide natural examples of equiharmonic maps, that is maps $\psi: S \to G/T$, where T is a maximal torus of a compact simple Lie group G, which are harmonic with respect to every G-invariant metric on G/T. Such maps have been introduced by Black [Bla91].

Finally, all harmonic maps into $\mathbb{C}P^1 = S^2$, all conformal harmonic maps into $\mathbb{C}P^2, S^3$ and S^4, and all almost complex curves in the nearly Kähler S^6 are either superminimal or superconformal, and as we show in section 6, the non-superminimal almost complex curves in S^6 correspond to solutions of the affine Toda field model for the exceptional group G_2.

The layout of the paper is as follows. In section 2 we study τ-primitive maps $\psi: S \to G/T$, where G is a compact simple Lie group with maximal torus T. We show (theorems 2.5 and 2.6) that such maps are related to the affine Toda field model for G and show that when S is a torus T^2, the corresponding solutions of the affine Toda field equations for G are doubly periodic (corollary 2.7). While reading this section the reader may find it helpful to refer to section 4 where the special case of $G = \boldsymbol{SU}(n+1)$ is discussed.

In section 3 we construct a large class of τ-primitive maps $\psi: \mathbb{R}^2 \to G/T$ by integrating certain Hamiltonian ODE's on finite dimensional subspaces of the τ-twisted loop algebra of the Lie algebra $\mathcal{G}$ of G. This is a standard construction which can be done by using either the r-matrix formalism [RSTS88] or the A-K-S integration scheme [AvM80]. The main result, which follows directly from the developments in [FPPS92] and [BFPP] is that if a τ-primitive map $\psi: \mathbb{R}^2 \to G/T$ factors through a 2-torus then ψ is obtained by the above construction, and hence is of finite type. In principle, this gives an explicit method of finding all doubly periodic solutions to the affine Toda field model for G, although it is not easy to write down solutions explicitly. A slightly different approach to solving the affine Toda equations using loop group splittings in the Grassmannian model appears in [McI].

Section 4 deals with superconformal harmonic maps $\psi: S \to \mathbb{C}P^n$ and the correspondence between such maps and τ-primitive maps $S \to \boldsymbol{SU}(n+1)/T^n$ is established (see theorem 4.6). As a corollary, using the results of section 3, it follows that any superconformal harmonic 2-torus $\psi: T^2 \to \mathbb{C}P^n$ is of finite type. In addition to recovering the results of [PS89] this, together with the results in [EW83], accounts for all harmonic 2-tori in $\mathbb{C}P^2$.

In section 5 we adapt the methods of section 4 to the case of superconformal harmonic maps into S^n. In particular, we recover the results of [FPPS92] on minimal tori in S^4.

Finally, in section 6 we consider almost complex curves in the nearly Kähler S^6. We recall that $S^6 = G_2/\boldsymbol{SU}(3)$ is a symmetric space for the exceptional Lie group G_2 and that G_2 is the group of automorphisms of the nearly Kähler structure. We show that non-superminimal almost complex curves in the nearly Kähler S^6 correspond to solutions of the affine Toda field model for G_2. This is derived from the results of section 5 together with the characterization (proposition 6.1) of non-superminimal almost complex curves as those superconformal harmonic maps $\psi: S \to S^6$ whose τ-primitive lifts $\widetilde{\psi}: S \to \boldsymbol{SO}(7)/T^3$ take values in G_2/T^2. Thus, in light of the results in section 3, every non-superminimal almost complex 2-torus in S^6 is of finite type. Together with the results in [BVW] this accounts for all almost complex 2-tori in S^6.

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2 Affine Toda fields and harmonic maps

In this section we give a geometric formulation of the (periodic) two dimensional affine Toda field model in terms of special harmonic maps from a connected Riemann surface S into the quotient space G/T of a compact simple Lie group G by its maximal torus T, where G/T is equipped with a G-invariant metric. It is well known that horizontal holomorphic curves into G/T (which are harmonic with respect to any G-invariant metric on G/T) give rise to the open (or non-periodic) affine Toda field equations.

We consider here a rather different class of maps $\psi: S \to G/T$, called *τ-primitive* maps (see definition 2.2), which are harmonic with respect to any G-invariant metric. These maps are neither holomorphic nor horizontal but are adapted to the canonical m-symmetric space structure on G/T given by a certain inner automorphism τ of G of order m.

We begin by recalling some basic notions about Lie algebras and the structure of G/T, and then relate these ideas to the affine Toda field equations.

Let $\mathcal{G}^{\mathbb{C}} = \mathcal{T}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathcal{G}^{\alpha}$ be the root space decomposition of the complexification $\mathcal{G}^{\mathbb{C}}$ of the Lie algebra $\mathcal{G}$ of G with maximal torus T. We fix (once and for all) a Cartan-Weyl basis $\xi_\alpha \in \mathcal{G}^\alpha, \alpha \in \Delta$, of the root spaces. In particular, we have the following relations:

$$\begin{aligned}
&\overline{\xi}_\alpha = \xi_{-\alpha}, \\
&[\xi_\alpha, \xi_{-\alpha}] = \alpha^{\#} \\
&(\xi_\alpha, \xi_\beta) = \delta_{\alpha,-\beta}
\end{aligned} \tag{2.1}$$

where $\#: \mathcal{T}^* \to \mathcal{T}$ is the isomorphism induced by the Killing form $(\,,)$ on $\mathcal{G}$ which corresponds to "raising indices". We also fix a set of simple roots $\alpha_1, \ldots, \alpha_r, r$=rank G. If $\theta = \sum_{k=1}^r m_k \alpha_k, m_k \in {}^{+}$, denotes the highest root we define α_0 by $\alpha_0 := -\theta$. Then we define

$$\mathcal{M}_1 = \bigoplus_{k=0}^{r} \mathcal{G}^{\alpha_k} \tag{2.2}$$

as the sum of the simple root spaces and the root space corresponding to α_0. Following [Kos59] we call an element $\xi \in \mathcal{M}_1$ *cyclic* if

$$\xi = \sum_{k=0}^{r} a_k \xi_{\alpha_k} \tag{2.3}$$

with all $a_k \in \mathbb{C}\backslash\{0\}$. The following facts, which are proved in [Kos59], hold for cyclic elements:

Lemma 2.1.

(i) cyclic elements are regular semisimple.

(ii) if $P_1, \ldots, P_r: \mathcal{G}^{\mathbb{C}} \to \mathbb{C}$ are homogeneous generators of the ring of $AdG^{\mathbb{C}}$-invariant polynomials with $\deg P_1 < \ldots < \deg P_r$ *then*

$$P_k \mid_{\mathcal{M}_1} = 0 \textit{ for } k = 1, \ldots, r-1,$$

and $\xi \in \mathcal{M}_1$ is cyclic if and only if $P_r(\xi) \neq 0$.

(iii) if $\xi, \widetilde{\xi} \in \mathcal{M}_1$ are cyclic then there exists $t \in T^{\mathbb{C}}$ with $Ad(t)(\xi) = \widetilde{\xi}$ if and only if $P_r(\xi) = P_r(\widetilde{\xi})$.

The simple roots and the height of the highest root θ determine, up to conjugation, an inner automorphism $\tau: G \to G$ of order $m = \sum_{k=0}^r m_k$ [Hel78], where by definition $m_0 := 1$. The differential of τ at the identity of G, which we also denote by τ, is an automorphism of $\mathcal{G}$ which acts on $\mathcal{M}_1$ via multiplication by $\zeta = e^{\frac{2\pi i}{m}}$.

Explicitly we have $\tau = Ad\exp(2\pi i Z)$ where $Z = \frac{1}{m}\sum_{k=1}^{r}\eta_k$ where each $\eta_k \in \mathcal{T}$ and $\alpha_j(\eta_k) = \delta_{jk}$. Note that the fixed point set G^τ of τ is the maximal torus T of G. In fact, τ is the automorphism of smallest order m which exhibits G/T as an m-symmetric space [Jim88]. This gives us the $_m$-grading

$$\mathcal{G} = \bigoplus_{k \in {}_m} \mathcal{M}_k \tag{2.4}$$

where $\mathcal{M}_k = \overline{\mathcal{M}_{-k}}$ is the ζ^k-eigenspace of τ on $\mathcal{G}$. The adjoint action of G on $\mathcal{G}$ thus leads to a decomposition

$$T(G/T) = \bigoplus_{k \in {}_m \setminus \{0\}} [\mathcal{M}_k] \subset G/T \times \mathcal{G} \tag{2.5}$$

of the complexified tangent bundle of G/T into a direct sum of subbundles of the trivial $\mathcal{G}$ -bundle over G/T where $[\mathcal{M}_k]_{gT} = Adg(\mathcal{M}_k) \subset \mathcal{G}$. An element of $[\mathcal{M}_1]$ will be said to be cyclic if it is in the orbit of a cyclic element of $\mathcal{M}_1$ under the adjoint action of G.

Definition 2.2. *A map $\psi\colon S \to G/T$ is called* τ-primitive *if $d\psi(T^{(1,0)}S) \subset [\mathcal{M}_1]$ and contains a cyclic element.*

Remark 2.3 (i) In the case where G/T is the 2-sphere S^2 the above definition puts no condition on ψ. In order not to exclude this case from our discussion, τ-primitive in this case will mean harmonic with $d\psi(T^{(1,0)}S)$ containing a semi-simple element.

(ii) It is shown in [Bla91] that τ-primitive maps are equiharmonic, i.e. harmonic with respect to any G-invariant metric on G/T. Moreover, they project to equiharmonic maps under every homogeneous projection $\pi\colon G/T \to G/H$ hence, in particular, to harmonic maps in the underlying symmetric spaces G/K. In sections 4 and 5 we discuss those harmonic maps into P^n and S^n which are obtained in this way.

A standard holomorphic differential argument (contained in the proof of the next lemma) shows that τ-primitive maps have cyclic $\frac{\partial}{\partial z}$-derivatives except perhaps at a discrete set of points.

Lemma 2.4. *Let $\psi\colon S \to G/T$ be τ-primitive. Then, viewing $P_r \in S^{d_r}(T(G/T))$ as a symmetric multi-linear form of degree $d_r = \deg P_r$, $P_r(d\psi)^{(d_r,0)}$ is a holomorphic -valued differential of degree d_r. In particular, $\frac{\partial\psi}{\partial z}(p) \in [\mathcal{M}_1]$ is cyclic except possibly at a discrete set of points.*

Proof. Let ∇ be the Levi-Civita connection for a G-invariant metric on G/T. Since $P_r\colon T(G/T) \to$ is the restriction of an AdG -invariant polynomial on $\mathcal{G}$, P_r is ∇-parallel. Thus

$$\frac{\partial}{\partial\bar{z}} P_r\left(\frac{\partial\psi}{\partial z}, \ldots, \frac{\partial\psi}{\partial z}\right) = d_r \cdot P_r\left(\nabla_{\frac{\partial}{\partial\bar{z}}}\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial z}, \ldots, \frac{\partial\psi}{\partial z}\right). \tag{2.6}$$

Since ψ is a harmonic map, $\nabla_{\frac{\partial}{\partial\bar{z}}}\frac{\partial\psi}{\partial z} = 0$, so that $P_r(\frac{\partial\psi}{\partial z})$ is holomorphic. Hence $P_r(\frac{\partial\psi}{\partial z})$ vanishes only on a discrete set of points. □

We now come to the main results of this section. We show that, locally, τ-primitive maps correspond to solutions of the affine Toda field equations.

By a (local) frame (or framing) of a map $\psi\colon S \to G/T$ we mean a map $F\colon U \to G$ from an open subset $U \subset S$, with $\pi \circ F = \psi$ where $\pi\colon G \to G/T$ is the homogeneous projection. Note that ψ is τ-primitive if and only if

$$F^{-1}\frac{\partial F}{\partial z} = A_0 + A_1 \in \mathcal{M}_0 \oplus \mathcal{M}_1 \tag{2.7}$$

where $A_0 \in \mathcal{M}_0$, $A_1 \in \mathcal{M}_1$ and A_1 is cyclic at some point. In this case, since F is real

$$F^{-1}\frac{\partial F}{\partial \overline{z}} = \overline{A}_0 + \overline{A}_1, \tag{2.8}$$

so that, since $\frac{\partial^2 F}{\partial z \partial \overline{z}} = \frac{\partial^2 F}{\partial \overline{z} \partial z}$, we obtain the Maurer-Cartan equations namely,

$$\frac{\partial A_0}{\partial \overline{z}} - \frac{\partial \overline{A}_0}{\partial z} = [A_1, \overline{A}_1], \tag{2.9}$$

$$\frac{\partial A_1}{\partial \overline{z}} = [A_1, \overline{A}_0]. \tag{2.10}$$

These are the integrability conditions for the existence of a real G-valued solution of (2.7).

A local frame will be called a *Toda frame* if there is a complex coordinate $z\colon U \to$ and a smooth map $\Omega\colon U \to i\mathcal{T}$ such that

$$F^{-1}\frac{\partial F}{\partial z} = \frac{\partial \Omega}{\partial z} + Ad\exp(\Omega)(B) \in \mathcal{M}_0 \oplus \mathcal{M}_1. \tag{2.11}$$

where $B = \sum_{k=0}^{r} \sqrt{m_k}\xi_{\alpha_k} \in \mathcal{M}_1$.

Theorem 2.5. *Let $\psi\colon S \to G/T$ be τ-primitive and let $p_0 \in S$ be such that $\frac{\partial \psi}{\partial z}(p_0)$ is cyclic. Then there is a local Toda frame F of ψ around p_0. Moreover, the complex coordinate z for the Toda frame is unique up to an arbitrary translation and rotation by a d_r-th root of unity, while the Toda frame F is unique up to multiplication by an element of the center $Z(G)$ of G.*

Theorem 2.6. *Let $z\colon U \to$ be a complex coordinate on a simply connected open subset U of S and let $\Omega\colon U \to i\mathcal{T}$ be smooth. Then (2.11) has a real solution F if and only if Ω satisfies the affine Toda field equations for G [For90], namely*

$$2\frac{\partial^2 \Omega}{\partial z \partial \overline{z}} + \sum_{k=0}^{r} m_k e^{2\alpha_k(\Omega)}\alpha_k^{\#} = 0. \tag{2.12}$$

In this case, $\psi = \pi \circ F\colon U \to G/T$ is τ-primitive and F is a Toda framing of ψ.

Proof of theorem 2.5. Let $F\colon U \to G$ be a framing of $\psi\colon U \to G/T$ over a simply connected domain with complex coordinate $z\colon U \to$ so that $\frac{\partial \psi}{\partial z}(p)$ is cyclic for all $p \in U$. Then, writing $F^{-1}\frac{\partial F}{\partial z} = A_0 + A_1$ as in (2.7), it follows from lemma 2.4 that $P_r(A_1)$ is holomorphic and nowhere vanishing. Hence there exists a holomorphic change of coordinate such that

$$P_r(A_1) = P_r(B)$$

on U. Such a coordinate change is necessarily unique up to an arbitrary translation and rotation by a d_r-th root of unity. Using lemma 2.1 (iii), we have a map

$$\eta\colon U \to \mathcal{M}_0 = \mathcal{T}$$

with

$$Ad\exp\eta(B) = A_1.$$

Now write $\eta = \Lambda + \Omega$ with $\Lambda = \overline{\Lambda}$ and $\Omega = -\overline{\Omega}$. Regauging F by $\exp\Lambda$, we obtain a new framing (again called F) for which

$$Ad\exp(\Omega)(B) = A_1. \tag{2.13}$$

It follows that $\frac{\partial A_1}{\partial \bar{z}} = [\frac{\partial \Omega}{\partial \bar{z}}, A_1]$, so using (2.10) we obtain

$$[A_1, \overline{A}_0 + \frac{\partial \Omega}{\partial \bar{z}}] = 0.$$

Hence, at every point $p \in U$, $\overline{A}_0 + \frac{\partial \Omega}{\partial \bar{z}}$ is in the centralizer of A_1, which is a maximal torus orthogonal to $\mathcal{M}_0$ [Kos59]. Thus, $\overline{A}_0 = -\frac{\partial \Omega}{\partial \bar{z}}$, or equivalently

$$A_0 = \frac{\partial \Omega}{\partial z}. \tag{2.14}$$

The first part of theorem 2.5 now follows from (2.7), (2.13) and (2.14), so it remains to prove the uniqueness property of the Toda frame. Let $\widetilde{F}\colon U \to G$ be another framing with

$$\widetilde{F}^{-1}\frac{\partial \widetilde{F}}{\partial z} = \frac{\partial \widetilde{\Omega}}{\partial z} + Ad\exp(\widetilde{\Omega})(B) \tag{2.15}$$

for some smooth map $\widetilde{\Omega}\colon U \to i\mathcal{T}$. Since any two framings differ by a gauge with values in T we have a map $\Lambda\colon U \to \mathcal{T}$ such that

$$\widetilde{F} = F\exp\Lambda. \tag{2.16}$$

It now follows from (2.11), (2.15) and (2.16) that

$$\frac{\partial \widetilde{\Omega}}{\partial z} + Ad\exp(\widetilde{\Omega})(B) = \frac{\partial \Omega}{\partial z} + \frac{\partial \Lambda}{\partial z} + Ad\exp(\Omega - \Lambda)(B). \tag{2.17}$$

Equating $\mathcal{M}_1$ components in (2.17), we have

$$Ad\exp(\widetilde{\Omega} - \Omega + \Lambda)(B) = B,$$

or equivalently, from the definition of B,

$$\sum_{k=0}^{r}\sqrt{m_k}e^{\alpha_k(\widetilde{\Omega}-\Omega+\Lambda)}\xi_{\alpha_k} = \sum_{k=1}^{r}\sqrt{m_k}\xi_{\alpha_k}.$$

However, $\xi_{\alpha_0}, \ldots, \xi_{\alpha_r}$ are linearly independent, so that

$$\alpha_k(\widetilde{\Omega} - \Omega + \Lambda) \in 2\pi i\ \ ,\ k = 0, \ldots, r.$$

It follows that

$$\widetilde{\Omega} = \Omega \tag{2.18}$$

and that $\exp\Lambda$ is in the center of G. The final statement of theorem 2.5 now follows from (2.16). □

Proof of theorem 2.6. It follows from (2.1), (2.9) and (2.10) that the affine Toda equations (2.12) are the integrability conditions for the existence of a real G-valued solution of (2.11). The final statement of the theorem is clear from (2.7). □

In what follows we will be concerned with τ-primitive maps from a 2-torus.

Corollary 2.7. *Let $\psi\colon T^2 \to G/T$ be a τ-primitive map of a 2-torus. Then there exists a coordinate $z\colon {}^2 \to$ on the universal cover ${}^2 \to T^2$ which is unique up to an arbitrary translation and rotation by a d_r-th root of unity and a Toda frame $F\colon \widetilde{T}^2 \to G$ on a 2-torus $\widetilde{T}^2$ covering T^2 such that (2.11) holds. Moreover, Ω factors through T^2 and so is a solution of the affine Toda field equations (2.12) defined on T^2. Conversely, every τ-primitive map $\psi\colon T^2 \to G/T$ is obtained from a solution of (2.12) on T^2.*

Proof. We use the notation in the proof of theorem 2.5 with z being the standard complex coordinate on $\;^2 = \;$. Since $P_r(d\psi)^{(d_r,0)}$ is non vanishing and holomorphic on T^2 and since $P_r(d\psi)^{(d_r,0)} = P_r(A_1)dz^{d_r}$ we must have that $P_r(A_1)$ is non-zero constant. After rescaling z by a suitable constant we may assume that $P_r(A_1) = P_r(B)$. The construction in the proof of theorem 2.5 may now be carried out on the universal cover $\;^2 \to T^2$. Since a Toda frame is unique up to the center $Z(G)$, which is finite, we have a Toda frame $F\colon \widetilde{T}^2 \to G$ defined on a torus $\widetilde{T}^2$ covering T^2. That Ω factors through T^2 follows from (2.18). □

3 Integration of the affine Toda field equations

In the previous section we have seen that τ-primitive maps $\psi\colon T^2 \to G/T$ of a 2-torus into G/T give rise to solutions of the affine Toda field equations on T^2 and that any such τ-primitive map comes from some doubly periodic solution of the affine Toda equations. We now describe how all doubly periodic solutions of the affine Toda equations may be obtained by integrating a pair of commuting Hamiltonian flows on a finite dimensional subspace of a certain loop algebra. The treatment we give here extends and complements the ideas in [FPPS92] and [BFPP] from which it derives.

Let $\psi\colon S \to G/T$ be τ-primitive and let $F\colon \widetilde{S} \to G$ be a framing of ψ on the universal cover $\widetilde{S} \to S$. Then

$$A = F^{-1}dF = A_1'' + A_0 + A_1' \tag{3.1}$$

where $A_0\colon T\widetilde{S} \to \mathcal{M}_0, A_0 = \overline{A}_0, A_1'\colon T^{(1,0)}\widetilde{S} \to \mathcal{M}_1, A_1'' = \overline{A_1'}$, and $P_r(A_1') \not\equiv 0$. The integrability condition for the existence of a frame $F\colon \widetilde{S} \to G$ satisfying (3.1) is the Maurer-Cartan equation

$$dA + \frac{1}{2}[A \wedge A] = 0 \tag{3.2}$$

which, when split into its $\mathcal{M}_0$ and $\mathcal{M}_1$ parts, becomes

$$\begin{aligned} dA_0 + \frac{1}{2}[A_0 \wedge A_0] &= -[A_1' \wedge A_1''] \\ dA_1' + [A_0 \wedge A_1'] &= 0. \end{aligned} \tag{3.3}$$

The crucial observation in dealing with these equations is that they may be reformulated (see [Uhl89], [ZS78]) using a "spectral parameter" as follows:

$$\begin{aligned} &A = A_1'' + A_0 + A_1' \text{ satisfies (3.3) if and only if} \\ &A^\lambda = \lambda^{-1}A_1'' + A_0 + \lambda A_1' \text{ satisfies (3.3) for all } \lambda \in S^1. \end{aligned} \tag{3.4}$$

Thus we obtain an S^1-family of frames $F^\lambda\colon \widetilde{S} \to G$ and, correspondingly, an S^1-family of τ-primitive maps $\psi^\lambda = \pi \circ F^\lambda\colon \widetilde{S} \to G/T$ with $\psi = \psi^1$ (if one bases $\psi^\lambda(p_0) = \psi(p_0)$ for all $\lambda \in S^1$ at some point $p_0 \in \widetilde{S}$).

These ideas may be conveniently described in terms of certain loop spaces. Let $\zeta = e^{2\pi i/m}$ where m is the order of the automorphism $\tau\colon G \to G$ which exhibits G/T as an m-symmetric space as discussed in section 2 and define the (Banach) Lie group ΛG_τ of τ-*twisted* loops in G by

$$\Lambda G_\tau = \{g : S^1 \to G \mid g \text{ is } H^1-\text{smooth}, \, g(\zeta\lambda) = \tau g(\lambda)\}. \tag{3.5}$$

The Lie algebra $\Lambda\mathcal{G}_\tau$ of ΛG_τ is then given by

$$\Lambda\mathcal{G}_\tau = \{\xi\colon S^1 \to \mathcal{G} \mid \xi \text{ is } H^1-\text{smooth}, \, \xi(\zeta\lambda) = \tau\xi(\lambda)\} \tag{3.6}$$

which admits the $Ad\,\Lambda G_\tau$-invariant inner product

$$(\xi, \eta) = \int_{S^1} (\xi(\lambda), \eta(\lambda)) d\lambda. \tag{3.7}$$

Expanding $\xi \in \Lambda\mathcal{G}_\tau$ in a Laurant series we get

$$\xi = \sum_{k\in\ } \lambda^k \xi_k,\ \xi_k \in \mathcal{M}_k,\ \overline{\xi}_k = \xi_{-k}. \tag{3.8}$$

We also have a filtration of $\Lambda\mathcal{G}_\tau$ by finite dimensional subspaces of Laurant polynomials of degree $d \in\ $,

$$\Lambda_d = \{\xi \in \Lambda\mathcal{G}_\tau \mid \xi = \sum_{|k|\le d} \lambda^k \xi_k\}. \tag{3.9}$$

Finally we note that there is a bijective correspondence between S^1-families of maps $F^\lambda : \widetilde{S} \to G$, $\lambda \in S^1$, and maps $\hat{F} : \widetilde{S} \to \Lambda G_\tau$ given by $F^\lambda(\ ) = \hat{F}(\ )(\lambda)$. The above discussion now may be summarised as follows:

Lemma 3.1. *A map $\psi : \widetilde{S} \to G/T$ is τ-primitive if and only if there exists a map $\hat{F}\colon \widetilde{S} \to \Lambda G_\tau$ with*

$$\hat{F}^{-1} d\hat{F} = \lambda^{-1} A_1'' + A_0 + \lambda A_1' \colon T\widetilde{S} \to \Lambda_1 \subset \Lambda\mathcal{G}_\tau$$

and $P_r(A_1') \not\equiv 0$, so that $\psi = \pi \circ F^1$.

We now show how to construct a large class of τ-primitive maps $\psi :\ ^2 \to G/T$ using a version of the A-K-S integration scheme or the r-matrix formalism [AvM80], [RSTS88, FPPS92]. Since the results in [BFPP, FPPS92] will imply that any doubly periodic τ-primitive map is in the above class this shows, in principle, how all τ-primitive maps defined on 2-tori may be constructed.

For each $d \equiv 1(m)$ define a pair of vector fields V, $\overline{V}$ on $\Lambda\mathcal{G}_\tau$ by

$$\begin{aligned} V(\xi) &= [\xi, i(\frac{\xi_{d-1}}{2} + \lambda\xi_d)] \\ \overline{V}(\xi) &= [\xi, (-i)(\frac{\xi_{1-d}}{2} + \lambda^{-1}\xi_{-d})]. \end{aligned} \tag{3.10}$$

It is clear that V and $\overline{V}$ are tangent to Λ_d . It follows from the basic integration lemma of [BFPP, FPPS92] that

$$[V, \overline{V}] = 0 \tag{3.11}$$

and that the system of ODE's

$$\begin{aligned} \frac{\partial\xi}{\partial z} &= V(\xi), \\ \frac{\partial\xi}{\partial\overline{z}} &= \overline{V}(\xi), \\ \xi(0) &= \overset{\circ}{\xi} \in \Lambda_d, \end{aligned} \tag{3.12}$$

has a unique, global, real solution $\xi\colon\ ^2 \to \Lambda_d$ for any given real initial condition $\overset{\circ}{\xi} \in \Lambda_d$.

Given such a solution $\xi\colon\ ^2 \to \Lambda_d$ of (3.12) the Λ_1-valued 1-form $\hat{A}$ on $\ ^2$

$$\hat{A} = i(\frac{\xi_{d-1}}{2} + \lambda\xi_d)dz - i(\frac{\xi_{1-d}}{2} + \lambda^{-1}\xi_{-d})d\overline{z} \tag{3.13}$$

satisfies the Maurer-Cartan equation and thus gives rise to a map $\hat{F}\colon\ {}^2 \to \Lambda G_\tau$ with $\hat{F}^{-1}d\hat{F} = \hat{A}$.

We note that, since P_r is AdG -invariant, it follows that along solutions of (3.12),

$$\begin{aligned} dP_r(\xi) &= d_r \cdot P_r(d\xi, \xi, \ldots, \xi) \\ &= d_r \cdot P_r([\xi, \hat{A}], \xi, \ldots, \xi) = 0, \end{aligned}$$

and hence $P_r(\xi)$ is a constant of the motion. In particular, since $\xi \in \Lambda_d$, it follows by consideration of highest powers of λ that $P_r(\xi_d)$ is also a constant of the motion so that

$$P_r(\xi_d) = P_r(\overset{\circ}{\xi}_d).$$

The following lemma is now immediate from lemma 3.1.

Lemma 3.2. *If* $\overset{\circ}{\xi}_d \in \mathcal{M}_1$ *is a cyclic element then, for each* $\lambda \in S^1$, $\psi^\lambda = \pi \circ F^\lambda\colon\ {}^2 \to G/T$ *is* τ*-primitive with framing* $F^\lambda\colon\ {}^2 \to G$.

The proof of the next lemma shows how solutions of the affine Toda field equations arise from this construction.

Lemma 3.3. *Let* $\overset{\circ}{\xi}_d \in \mathcal{M}_1$ *be the cyclic element* B *of section 2 and let* $F^\lambda\colon\ {}^2 \to G$, $\lambda \in S^1$, *be the corresponding framings constructed above. Then each* F^λ *is a Toda frame for the* τ*-primitive map* $\psi^\lambda = \pi \circ F^\lambda\colon\ {}^2 \to G/T$ *and thus there exists* $\Omega\colon\ {}^2 \to i\mathcal{T} \subset \mathcal{M}_0$ *which satisfies equation (2.11) with* $F = F^1$*. In particular,* Ω *is a solution of the affine Toda field equations (2.12) for* G.

Proof. By considering terms in λ^{d-1} and λ^d in (3.12) we have

$$\begin{aligned} \frac{\partial \xi_{d-1}}{\partial \bar{z}} &= i[\xi_d, \xi_{-d}], \\ \frac{\partial \xi_d}{\partial \bar{z}} &= -\frac{i}{2}[\xi_d, \xi_{1-d}], \\ \frac{\partial \xi_d}{\partial z} &= -\frac{i}{2}[\xi_d, \xi_{d-1}]. \end{aligned} \tag{3.14}$$

Then the $\mathcal{M}_0$ component A_0 of $(F^\lambda)^{-1}dF^\lambda$ is given by

$$A_0 = i\frac{\xi_{d-1}}{2}dz - i\frac{\xi_{1-d}}{2}d\bar{z},$$

so that

$$*A_0 = -\frac{\xi_{d-1}}{2}dz - \frac{\xi_{1-d}}{2}d\bar{z},$$

and hence, using (3.14), $d * A_0 = 0$. Let $\Omega\colon\ {}^2 \to i\mathcal{T}$ be a solution of

$$d\Omega = -i * A_0, \qquad \Omega(0) = 0.$$

Then

$$\frac{\partial \Omega}{\partial z} = i\frac{\xi_{d-1}}{2},$$

and hence $Ad \exp \Omega(B)$ and ξ_d both satisfy the same system of ODE's

$$\begin{aligned} \frac{\partial \eta}{\partial z} &= -\frac{i}{2}[\eta, \xi_{d-1}] \\ \frac{\partial \eta}{\partial \bar{z}} &= -\frac{i}{2}[\eta, \xi_{1-d}], \end{aligned} \tag{3.15}$$

with initial condition $\eta(0) = B$. Hence

$$\xi_d = Ad \exp \Omega(B).$$

Thus (2.11) holds for $F = F^\lambda$ with $\lambda = 1$. $\square$

Remark 3.4 (i) The vector fields $V, \overline{V}$ are canonical Hamiltonian vector fields on $\Lambda\mathcal{G}_\tau$ with respect to the Poisson structure defined by the r-matrix

$$R\colon \Lambda\mathcal{G}_\tau \to \Lambda\mathcal{G}_\tau\ ,\ R(\lambda^k \xi_k) = \frac{i}{2}\ \text{sign}\ (k)\lambda^k \xi_k$$

and the $Ad\,\Lambda G_\tau$-invariant Hamiltonians $f\colon \Lambda\mathcal{G}_\tau \to\ $, $f(\xi) = \frac{\lambda^{1-d}}{2}(\xi, \xi)$. In A-K-S-language this corresponds to the splitting of $\Lambda\mathcal{G}_\tau$ into two subalgebras

$$\Lambda\mathcal{G}_\tau = \Lambda\mathcal{G}_\tau \bigoplus \Lambda^-$$

where $\Lambda^- = \{\xi \in \Lambda\mathcal{G}_\tau \mid \xi = \sum_{k \leq 0} \lambda^k \xi_k,\ \xi_0 \in i\mathcal{T}\}$. (See [BP] or [DPW] for more details.)

(ii) Equations (3.12) are in Lax form and hence isospectral. It follows from standard techniques [FPPS92], [Bur] that (3.12) linearise on the Jacobian of their spectral curve. This is where periodicity conditions on the solutions can be discussed. A detailed study of this aspect will be done elsewhere.

Conforming to standard notation [BFPP] a τ-primitive map $\psi\colon\ {}^2 \to G/T$ obtained via the construction above will be called a *τ-primitive map of finite type.* The above discussion may be summarised as follows:

Theorem 3.5. *A τ-primitive map $\psi\colon\ {}^2 \to G/T$ is of finite type if and only if there exists an S^1-family $\hat{F} = F^\lambda\colon\ {}^2 \to \Lambda G_\tau$ of Toda frames with $\psi = \pi \circ F^1$ and a Laurant polynomial solution $\xi\colon\ {}^2 \to \Lambda_d$, some $d \equiv 1(m)$, with*

$$\begin{aligned} d\xi &= [\xi, \hat{F}^{-1} d\hat{F}], \\ \hat{F}^{-1}\frac{\partial \hat{F}}{\partial z} &= i(\frac{\xi_{d-1}}{2} + \lambda \xi_d). \end{aligned} \tag{3.16}$$

We call such $\xi\colon\ {}^2 \to \Lambda_d$ an *adapted polynomial Killing field* (for the Toda frame F^λ).

We now come to the main result of this section:

Theorem 3.6. *Let $\psi\colon T^2 \to G/T$ be a τ-primitive map of a 2-torus. Then ψ is of finite type.*

Combining this with corollary 2.7 we obtain the following

Corollary 3.7. *Every doubly periodic solution of the affine Toda field equation for G is obtained by integrating the finite dimensional Hamiltonian ODE's (3.12).*

Proof of theorem 3.6. All the crucial ingredients for proving theorem 3.6 are contained in [BFPP], section 2. Hence we will confine ourselves to showing how to adopt our situation to the setup therein. From corollary 2.7 we know that there exists an (essentially) unique Toda frame $F^\lambda\colon\ {}^2 \to G$ with

$$(F^\lambda)^{-1}\frac{\partial F^\lambda}{\partial z} = \frac{\partial \Omega}{\partial z} + \lambda Ad \exp \Omega(B)$$

where $\Omega\colon T^2 \to i\mathcal{T}$. To find an adapted polynomial Killing field for F^λ we have to solve

$$d\xi = [\xi, (\partial\Omega + \lambda Ad\exp\Omega(B))dz + (-\overline{\partial}\Omega + \lambda^{-1}Ad\exp(-\Omega)(\overline{B}))d\overline{z}] \tag{3.17}$$

for a Laurant polynomial $\xi\colon\ ^2 \to \Lambda_d$, some $d \equiv 1(m)$, with leading terms

$$i(\frac{1}{2}\xi_{d-1} + \lambda\xi_d) = \frac{\partial\Omega}{\partial z} + \lambda Ad\exp\Omega(B).$$

We start by regauging F^λ to a complex frame

$$\widetilde{F} = F^\lambda \exp\Omega \tag{3.18}$$

which has

$$\widetilde{F}d\widetilde{F}^{-1} = (2\partial\Omega + \lambda B)dz + \lambda^{-1}Ad\exp(-2\Omega)(\overline{B})d\overline{z}. \tag{3.19}$$

Hence (3.17) becomes

$$d\widetilde{\xi} = [\widetilde{\xi}, \widetilde{F}^{-1}d\widetilde{F}] \tag{3.20}$$

for the gauged polynomial Killing field $\widetilde{\xi} = Ad\exp(-\Omega)(\xi)\colon\ ^2 \to \Lambda_d$ which we require to satisfy

$$i(\frac{\widetilde{\xi}_{d-1}}{2} + \lambda\widetilde{\xi}_d) = 2\frac{\partial\Omega}{\partial z} + \lambda B.$$

We choose a faithful representation $G \hookrightarrow \boldsymbol{U}(N)$ of G into a unitary group and view $\mathcal{G} \subset \boldsymbol{g\ell}(N, \ )$ as a Lie subalgebra of $\boldsymbol{g\ell}(N, \ )$. We claim it suffices to find a formal solution $Y = \sum_{k\leq j}\lambda^k Y_k, j \equiv 1(m), Y_k\colon T^2 \to \boldsymbol{g\ell}(N, \ )$, satisfying (3.20) with top term $Y_j = B$. That such a solution Y exists follows verbatim from the proof of theorem 2.4 in [BFPP]: B is regular semisimple (in $\boldsymbol{g\ell}(N, \ )$) and (3.20) for Y is precisely equation (3.18) in the proof of theorem 2.4. in [BFPP]. Also note that $2\frac{\partial\Omega}{\partial z}\colon T^2 \to \mathcal{T}$ is defined on the 2-torus T^2, hence the Y_k constructed in the proof of theorem 2.4. in [BFPP] as differential polynomials in Ω, will be defined on T^2, i.e. $Y_k\colon T^2 \to \boldsymbol{g\ell}(N, \ )$. Since we have granted that we have a formal solution to (3.20) defined on T^2 we use the standard ellipticity argument in [FPPS92] or theorem 2.3 of [BFPP], to obtain a complex, polynomial solution to (3.20),

$$\widetilde{X} : T^2 \to \Lambda\boldsymbol{g\ell}(N, \ )$$

$$\widetilde{X} = \sum_{k=0}^{d}\lambda^k\widetilde{X}_k, d \equiv 1(m), \widetilde{X}_d = B.$$

Thus $X = Ad\exp\Omega(\widetilde{X}) = \sum_{k=0}^{d}\lambda^k X_k, d \equiv 1(m)$, $X_d = Ad\exp(\Omega)(B)$, is a complex polynomial solution to the real equation (3.17). Putting

$$\widetilde{\eta} = X + \overline{X}$$

we obtain a real solution of equation (3.17) with $\widetilde{\eta}_d = Ad\exp(\Omega)(B)$. Since $\mathcal{G}$ is simple we have an ad-invariant projection $\boldsymbol{g\ell}(N, \ ) \to \mathcal{G}$ which projects $\widetilde{\eta}$ into a Λ_d-valued Killing field $\eta\colon T^2 \to \Lambda_d$. Finally we average η using τ to obtain

$$\xi = (\frac{1}{m}\sum_{k=1}^{m}\tau^k)(\eta)\colon T^2 \to \Lambda_d$$

thus solving (3.17) with $\xi_d = Ad\exp(\Omega)(B)$. But the top term in (3.17) gives

$$[\xi_d, \frac{\partial\Omega}{\partial z}] + [\xi_{d-1}, \xi_d] = \frac{\partial\xi_d}{\partial z} = [\frac{\partial\Omega}{\partial z}, \xi_d]$$

and hence

$$[\xi_d, 2\frac{\partial\Omega}{\partial z} - \xi_{d-1}] = 0.$$

But ξ_d being cyclic in $\mathcal{M}_1$ implies

$$\frac{1}{2}\xi_{d-1} = \frac{\partial\Omega}{\partial z}$$

which completes the proof. $\square$

4 Superconformal harmonic surfaces in CP^n

In this section we give the τ-primitive condition a more geometric flavour by characterizing those harmonic maps into $\mathbb{C}P^n$ which arise from τ-primitive maps into $\boldsymbol{SU}(n+1)/T^n$ under homogeneous projection. The harmonic maps so obtained we call *superconformal* (definition 4.4), since they turn out to be isotropic of the highest possible order without being totally isotropic, the totally isotropic case having been dealt with by [EW83] and [Cal67], for instance.

We begin by writing down the general theory of section 2 in the special case of $G=\boldsymbol{SU}(n+1)$, and hence obtain a characterization in lemma 4.1 of those maps $\psi: S \to \boldsymbol{SU}(n+1)/T^n$ which are τ-primitive.

The group $\boldsymbol{SU}(n+1)$ has rank n and a maximal torus T^n consists of the diagonal elements of $\boldsymbol{SU}(n+1)$, so that $\mathcal{T}$ is the space of purely imaginary diagonal matrices with zero trace. The complexification of $\boldsymbol{su}(n+1)$ is $\boldsymbol{s\ell}(n+1,\mathbb{C})$, and the roots are

$$\{\sigma_i - \sigma_j : i \neq j;\ i,j = 0,\dots,n\}$$

where

$$\sigma_i(\mathrm{diag}(y_0,\dots,y_n)) = y_i,\ i = 0,\dots,n.$$

We take

$$\{\sigma_i - \sigma_j : i > j;\ i,j = 0,\dots,n\}$$

to be the positive roots, and $\alpha_j = \sigma_j - \sigma_{j-1},\ j = 1,\dots,n$, as the positive simple roots. Then $\theta = \sigma_n - \sigma_0 = \sum_{j=1}^n \alpha_j$ is the highest root, so we take $\alpha_0 = -\theta$. For each pair $(i,j), i,j \in \{0,\dots,n\}, i \neq j$, let E_{ij} be the element of $\boldsymbol{s\ell}(n+1,\mathbb{C})$ whose only non-zero entry is a 1 in the (i,j)-th place. Then E_{ij} spans the root space corresponding to $\sigma_i - \sigma_j$ and

$$\{E_{ij} : i \neq j;\ i,j = 0,\dots,n\}$$

is a Cartan-Weyl basis. Hence, in the notation of section 2, $\mathcal{M}_1$ is the vector subspace of $\boldsymbol{s\ell}(n+1,\mathbb{C})$ spanned by $E_{1,0},\dots,E_{n,n-1},E_{0,n}$. Thus the elements of $\mathcal{M}_1$ are of the form $\sum_{i\in\mathbb{Z}_{n+1}} a_{i,i-1}E_{i,i-1}$ where $a_{1,0},\dots,a_{n,n-1},a_{0,n} \in \mathbb{C}$ and $B = \sum_{i\in\mathbb{Z}_{n-1}} E_{i,i-1}$.

We are now in a position to identify τ-primitive maps $\psi: S \to \boldsymbol{SU}(n+1)/T^n$. The elements of the flag manifold $\boldsymbol{SU}(n+1)/T^n$ may be written as ordered $(n+1)$-tuples $(L_0,\dots,L_n)$ of mutually orthogonal 1-dimensional subspaces of $\mathbb{C}^{n+1}$. Then, given a map $\psi: S \to \boldsymbol{SU}(n+1)/T^n$, we write $\psi(p) = (L_0(p),\dots,L_n(p)), p \in S$. Thus ψ corresponds to a decomposition of the trivial bundle $S\times\mathbb{C}^{n+1}$ into an ordered orthogonal direct sum of line subbundles, which we also denote by $L_0,\dots,L_n$.

Lemma 4.1. *Let $\psi: S \to \boldsymbol{SU}(n+1)/T^n$ and, as above, write $\psi = (L_0,\dots,L_n)$. Then ψ is τ-primitive if and only if for each nowhere vanishing local section g_k of L_k,*

$$\frac{\partial g_k}{\partial z} \in L_k \oplus L_{k+1},\ k = 0,\dots,n-1,$$
$$\frac{\partial g_n}{\partial z} \in L_n \oplus L_0,$$

and $\frac{\partial g_k}{\partial z} \in L_k$ at only a discrete set of points for all $k = 0,\dots,n$.

We now recall some elementary facts about harmonic sequences associated to harmonic maps into $\mathbb{C}P^n$, (see for example [BW92, BW86, Wol88]) and also give the definition of superconformality for harmonic maps into $\mathbb{C}P^n$.

Let $L \to \mathbb{C}P^n$ denote the tautological line bundle whose fibre over $x \in \mathbb{C}P^n$ is the line $x \subset \mathbb{C}^{n+1}$. There is a bijective correspondence between maps $\phi: S \to \mathbb{C}P^n$ and smooth complex line subbundles of $S\times\mathbb{C}^{n+1}$ given by $\phi \leftrightarrow \phi^* L$.

Let $\phi: S \to \mathbb{C}P^n$ be smooth and let $L_0 = \phi^* L$. Then the standard identification of $T\mathbb{C}P^n$ with $\mathrm{Hom}(L, L^\perp)$ enables us to regard the derivative $d\phi$ as a map

$$d\phi: TS \otimes L_0 \to L_0^\perp,$$

where $\perp$ denotes the orthogonal complement with respect to the standard Hermitian inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^{n+1}$. Specifically, if $X \in TS$ and s is a local section of L_0 then $d\phi(X \otimes s)$ is the component of $ds(X)$ orthogonal to L_0.

Every complex vector subbundle V of $S \times \mathbb{C}^{n+1}$ inherits a holomorphic structure for which s is a local holomorphic section if and only if $\frac{\partial s}{\partial \bar{z}}$ is orthogonal to V. Then ϕ is harmonic if and only if the $(1,0)$ part ∂ of $d\phi$ is a holomorphic bundle map or equivalently, if and only if the $(0,1)$ part $\overline{\partial}$ of $d\phi$ is an antiholomorphic bundle map.

Assume now that ϕ is harmonic. If ∂(resp. $\overline{\partial}$) is not identically zero then the zeros are isolated and there are uniquely determined line bundles $L_1 \supseteq im(\partial)$(resp. $L_{-1} \supseteq im(\overline{\partial})$) extending $im(\partial)$(resp. $im(\overline{\partial})$) across the zeros. The maps $\phi_1, \phi_{-1}: S \to \mathbb{C}P^n$ corresponding to L_1, L_{-1} are again harmonic hence, proceeding inductively, we obtain a sequence of line bundles, the *harmonic sequence*,

$$\ldots L_{-2}, L_{-1}, L_0, L_1, L_2 \ldots$$

and the corresponding sequence

$$\ldots \phi_{-2}, \phi_{-1}, \phi_0 = \phi, \phi_1, \phi_2 \ldots$$

of harmonic maps from S into $\mathbb{C}P^n$. We also have a sequence of holomorphic bundle maps

$$\partial_k: T^{(1,0)}S \otimes L_k \to L_{k+1}$$

and antiholomorphic bundle maps

$$\overline{\partial}_k: T^{(0,1)}S \otimes L_k \to L_{k-1}.$$

Let z be a local complex coordinate on S and let f_0 be a nowhere zero holomorphic local section of L_0. Then for each k there is a meromorphic local section f_k of L_k such that

$$f_{k+1} = \partial_k\left(\frac{\partial}{\partial z} \otimes f_k\right). \tag{4.1}$$

Then f_{k+1} is the component of $\frac{\partial f_k}{\partial z}$ orthogonal to f_k and we have the following equations, which hold away from the singularities of f_k:

$$\frac{\partial f_k}{\partial z} = f_{k+1} + \frac{\partial}{\partial z} \log |f_k|^2 \, f_k, \tag{4.2}$$

$$\frac{\partial f_k}{\partial \bar{z}} = -\frac{|f_k|^2}{|f_{k-1}|^2} \, f_{k-1}. \tag{4.3}$$

Note that, by construction, any two adjacent line bundles in the harmonic sequence are orthogonal.

The following fact, which can be found in [BW86] or [BW92], follows immediatly from (4.2), (4.3).

Proposition 4.2. *If some k consecutive bundles in a harmonic sequence are mutually orthogonal then every k consecutive bundles of this sequence are mutually orthogonal.*

Definition 4.3. *A harmonic map $\phi: S \to \mathbb{C}P^n$ is* k-isotropic *if k consecutive bundles in the harmonic sequence of ϕ are mutually orthogonal.*

We remark that a harmonic map is conformal if and only if it is 3-isotropic [BW92]. Those harmonic maps which are given by the elements of the Frenet frame of a linearly full holomorphic curve in $\mathbb{C}P^n$ are said to be *superminimal* or *totally isotropic* [EW83]. The harmonic sequence of such a map is given by the Frenet frame, and in particular there are exactly $n+1$ elements in the harmonic sequence, all of which are mutually orthogonal. Every harmonic map of the 2-sphere into $\mathbb{C}P^n$ is totally isotropic.

If a harmonic map ϕ is not totally isotropic then each map ∂_k, $\overline{\partial}_k$ is non-zero and the harmonic sequence $\{L_k\}$ is defined for all $k \in \mathbb{Z}$. In particular, $\frac{\partial f_k}{\partial z} \in L_k$ at only a discrete set of points for each $k \in \mathbb{Z}$.

The results in [BFPP] only apply to non-conformal harmonic 2-tori in $\mathbb{C}P^n$. This excludes the geometrically interesting case of minimal tori. In this section we describe all harmonic 2-tori in $\mathbb{C}P^n$ which are isotropic of the highest possible order without being totally isotropic.

Definition 4.4. *A harmonic map $\phi: S \to \mathbb{C}P^n$ is* superconformal *if it is $(n+1)$-isotropic but not totally isotropic.*

Remark 4.5 Proposition 4.2 implies that a harmonic map $\phi: S \to \mathbb{C}P^n$ if and only if ϕ has an orthogonally periodic harmonic sequence (i.e. $L_0, \ldots, L_n$ are mutually perpendicular with $L_{n+1} = L_0$). Thus, in particular, every harmonic map $\phi: S \to \mathbb{C}P^1$ is either $\pm$ holomorphic or superconformal, and every conformal harmonic map $\phi: S \to \mathbb{C}P^2$ is either totally isotropic or superconformal.

Let $\pi: \boldsymbol{SU}(n+1)/T^n \to \mathbb{C}P^n$ be the map which assigns to the flag its first element

$$\pi(L_0, \ldots, L_n) = L_0.$$

The manifolds $\boldsymbol{SU}(n+1)/T^n$ and $\mathbb{C}P^n$ have canonical $\boldsymbol{SU}(n+1)$-invariant Kähler metrics and the map π is an $\boldsymbol{SU}(n+1)$-equivariant complex analytic Riemannian submersion. We note that every superconformal harmonic map $\phi: S \to \mathbb{C}P^n$ determines a lift $\widetilde{\phi}: S \to \boldsymbol{SU}(n+1)/T^n$ given by

$$\widetilde{\phi} = (L_0, \ldots, L_n). \tag{4.4}$$

We are now in a position to state and prove the main theorem of this section.

Theorem 4.6. *Let $\phi: S \to \mathbb{C}P^n$ be a superconformal harmonic map. Then the lift $\widetilde{\phi}: S \to \boldsymbol{SU}(n+1)/T^n$ given by (4.4) is τ-primitive. Conversely, if $\psi: S \to \boldsymbol{SU}(n+1)/T^n$ is τ-primitive then $\phi = \pi \circ \psi: S \to \mathbb{C}P^n$ is a superconformal harmonic map with lift $\widetilde{\phi}$ equal to ψ.*

Proof. Let $\phi: S \to \mathbb{C}P^n$ be a superconformal harmonic map with lift $\widetilde{\phi} = (L_0, \ldots, L_n)$. It follows from (4.2) that if g_k is a nowhere vanishing smooth local section of L_k then $\frac{\partial g_k}{\partial z} \in L_k \oplus L_{k+1}, k = 0, \ldots, n$, and, since ϕ is not totally isotropic, $\frac{\partial g_k}{\partial z} \in L_k$ at only a discrete set of points . However, since the harmonic sequence is orthogonally periodic, $L_{n+1} = L_0$, so that $\widetilde{\phi}$ is τ-primitive by lemma 4.1.

Conversely, assume that $\psi: S \to \boldsymbol{SU}(n+1)/T^n$ is τ-primitive and let $\psi = (L_0, \ldots, L_n)$. The harmonicity of $\phi = \pi \circ \psi$ follows from the general theory of Black [Bla91] but in this particular case one may give an elementary argument as follows. Let $g_0, \ldots, g_n$ be local sections of $L_0, \ldots, L_n$. Then, since ψ is τ-primitive, $\frac{\partial g_r}{\partial z} \in L_r \oplus L_{r+1}$ so that, for $r \neq s, s+1$,

$$\langle \frac{\partial g_r}{\partial \overline{z}}, g_s \rangle = -\langle g_r, \frac{\partial g_s}{\partial z} \rangle = 0. \tag{4.5}$$

Now let f_0 be a local holomorphic section of L_0 and let $f_1 = \partial_0(\frac{\partial}{\partial z} \otimes f_0)$, so that f_1 is the component of

$\frac{\partial f_0}{\partial z}$ orthogonal to L_0. It follows from (4.5) that $\frac{\partial f_0}{\partial \bar{z}} \in L_n$, and hence

$$
\begin{aligned}
\langle \frac{\partial f_1}{\partial \bar{z}}, g_1 \rangle &= \langle \frac{\partial}{\partial \bar{z}} \frac{\partial f_0}{\partial z}, g_1 \rangle \\
&= \langle \frac{\partial}{\partial z} \frac{\partial f_0}{\partial \bar{z}}, g_1 \rangle \\
&= 0.
\end{aligned}
\tag{4.6}
$$

Hence, using (4.5) and (4.6), we see that f_1 is a holomorphic section of $L^\perp$. Thus ∂_0 is a holomorphic bundle map, so that ϕ is harmonic. Furthermore, lemma 4.1 and the method of construction of the harmonic sequence show that ϕ has $L_0, \ldots, L_n$ as the first $n+1$ elements of its harmonic sequence, so that ϕ is superconformal with lift $\widetilde{\phi} = \psi$. □

Theorems 4.6 and 3.6 have the following important consequence:

Corollary 4.7. *Every superconformal harmonic 2-torus $\phi: T^2 \to \mathbb{C}P^n$ is of finite type.*

In particular, together with the results on superminimal maps [EW83] this accounts for all harmonic 2-tori in $\mathbb{C}P^1 = S^2$ and $\mathbb{C}P^2$.

Finally it is interesting to note that the geometry of the harmonic sequence of a superconformal harmonic map $\phi: S \to \mathbb{C}P^n$ may be used to produce directly the Toda framing whose existence was proved in theorem 2.5, thus giving an alternative proof of theorem 4.6 which does not use lemma 4.1. We begin by letting z be a local complex coordinate on a simply connected open subset U of S, and $\{f_k\}$ a sequence of holomorphic sections of the bundles $\{L_k\}$ satisfying the Frenet equations (4.2) and (4.3). Then, since $L_{n+1} = L_0$ from remark 4.5, we have that $f_{n+1} = \alpha f_0$ for some holomorphic non-vanishing function $\alpha: U \to \mathbb{C}$. Thus, after a holomorphic change of coordinate we may assume that

$$
f_{n+1} = f_0. \tag{4.7}
$$

Moreover, by (4.3)

$$
\frac{\partial}{\partial \bar{z}} \det(f_0, \ldots, f_n) = 0
$$

so that, replacing f_k by $\det(f_0, \ldots, f_n)^{-\frac{1}{n+1}} f_k$ we may assume that

$$
\det(f_0, \ldots, f_n) = 1. \tag{4.8}
$$

Since $f_0, \ldots, f_n$ are mutually perpendicular it follows that we may define $F = (F_0, \ldots, F_n): U \to \boldsymbol{SU}(n+1)$ by setting $F_k = e^{-w_k} f_k$, where $e^{w_k} = |f_k|$, for $k = 0, \ldots, n$. It now follows from (4.2), (4.8) and (4.7) that

$$
F^{-1} \frac{\partial F}{\partial z} = \partial \Omega + Ad \exp(\Omega)(B)
$$

where $\Omega = \mathrm{diag}(w_0, \ldots, w_n): U \to i\mathcal{T}$, and $B = \sum_{i \in \mathbb{Z}_{n+1}} E_{i,i-1}$. Hence F is the required local Toda framing and z is a corresponding complex coordinate.

5 Superconformal harmonic surfaces in S^n

Suppose $f: S \to S^n$ is a smooth map from a connected Riemann surface S, and let $\pi: S^n \to \mathbb{R}P^n$ denote the standard Riemannian double covering and $i: \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ the standard isometric totally geodesic inclusion. Then $\phi = i \circ \pi \circ f: S \to \mathbb{C}P^n$ is harmonic if and only if f is harmonic. (Also ϕ is linearly full if and only if f is linearly full.) As in section 4, we may construct the harmonic sequence of ϕ but now refer to this as the harmonic sequence of f. In this case, $f_0 = f$ is a global holomorphic section of L_0 and $L_{-k} = \overline{L}_k$ (see [BW92]).

A harmonic map $f\colon S \to S^n$ will be called *k-isotropic* if $\phi = i \circ \pi \circ f\colon S \to \ P^n$ is k-isotropic in the sense of section 4.

Lemma 5.1. *([BW92]) If, for some positive integer $k < \frac{1}{2}n + 1$, $f\colon S \to S^n$ is $(2k-1)$-isotropic then f is $2k$-isotropic. If n is even and if f is $(n+1)$-isotropic then f is totally isotropic.*

Suppose then that $f\colon S \to S^{2m}$ is a harmonic map. If f is $(2m+1)$-isotropic then by lemma 5.1 f is totally isotropic (superminimal). We define f to be *superconformal* if it is $2m$-isotropic but not totally isotropic.

Lemma 5.2. *If $f : S \to S^{2m}$ is a superconformal harmonic map then $L_1, \dots, L_{m-1}$ are mutually orthogonal lines whose span is an isotropic subspace of $\ ^{2m+1}$ orthogonal to L_0. Moreover, L_m is not an isotropic line.*

Proof. The first part follows from the fact that $L_{-(m-1)}, \dots, L_{m-1}$ are mutually orthogonal and $L_{-k} = \overline{L_k}$ for all k. For the second part, we note that if L_m were isotropic then $L_{-m}, \dots, L_m$ would be mutually orthogonal, so that f would be $(2m+1)$-isotropic which gives a contradiction to lemma 5.1. □

There are two types of superconformal harmonic maps into S^{2m}, namely those which are

(a) linearly full in S^{2m};

(b) linearly full in a totally geodesic S^{2m-1} in S^{2m}.

In case (b) the harmonic sequence is orthogonally periodic (c.f. remark 4.5) since $L_{k+2m} = L_k$ for all k, and any $2m$ consecutive line bundles $L_k, \dots, L_{k+2m-1}$ are mutually orthogonal. Moreover, $L_m = \overline{L}_m$ and the corresponding map $\widetilde{f}\colon S \to S^{2m-1}$ is the polar of S in the sense of [Law70].

Notice that any conformal harmonic (i.e. minimal) map $f\colon S \to S^3$ or S^4 is superconformal or superminimal and so is every almost complex curve $f\colon S \to S^6$ (c.f. section 6).

We now consider the theory of section 2 in the case where $G = \boldsymbol{SO}(2m+1)$. First let $\ _m$ denote the flag manifold whose elements are ordered sequences

$$X = (X_0, \dots, X_m)$$

of mutually orthogonal complex lines in $\ ^{2m+1}$ where X_0 is the complexification of a real line in $\ ^{2m+1}$ and $X_1, \dots X_m$ span a maximal isotropic subspace of $\ ^{2m+1}$. Note that each element $X = (X_0, \dots, X_m) \in \ _m$ determines a unique unit vector $u_0 = \rho(X) \in \ ^{2m+1}$ which spans X_0. For if $\frac{1}{\sqrt{2}}(u_{2k-1} - iu_{2k})$ is a unit basis vector for X_k $(k = 1, \dots, m)$ then u_0 is the unique unit vector in $\ ^{2m+1}$ for which $u_0, \dots, u_{2m}$ is a positively oriented orthonormal basis of $\ ^{2m+1}$. This defines a projection $\rho\colon \ _m \to S^{2m}$. We also note that, similarly, $X \in \ _m$ is uniquely determined by $\rho(X)$ and $X_1, \dots, X_{m-1}$.

The group $\boldsymbol{SO}(2m+1)$ acts transitively on $\ _m$ and the stabilizer of each point is a maximal torus. Indeed, for $k = 0, \dots, 2m$, let e_k be the $(k+1)$-st standard basis vector of $\ ^{2m+1}$ (having a 1 in the $(k+1)$-st place and zeros elsewhere) and let $C_0 = \ e_0, C_k = \ (e_{2k-1} - ie_{2k})$ for $k = 1, \dots, m$. Then the stabilizer of $C = (C_0, \dots, C_m) \in \ _m$ is the maximal torus

$$T^m = \{\mathrm{diag}(1, R(\theta_1), \dots, R(\theta_m)) \mid \theta_1, \dots, \theta_m \in \ \}$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus we may identify $\ _m$ with $\boldsymbol{SO}(2m+1)/T^m$ and ρ with the projection of $\boldsymbol{SO}(2m+1)/T^m$ onto S^{2m} induced by sending an element of $\boldsymbol{SO}(2m+1)$ to its first column.

Let $f\colon S \to S^{2m}$ be a superconformal harmonic map. Then, by lemma 5.2, there is a lift $\widetilde{f}\colon S \to \boldsymbol{SO}(2m+1)/T^m$ defined by

$$\widetilde{f} = (X_0, \dots, X_m) \tag{5.1}$$

where $X_k = L_k, k = 0, \dots, m-1$ and X_m is determined by the condition that $\rho \circ \widetilde{f} = f$. Note that L_m is contained in the orthogonal complement $X_m \oplus \overline{X_m}$ of $L_{-(m-1)} \oplus \cdots \oplus L_{m-1}$. This is needed in the proof of theorem 5.4.

As we will show, $\widetilde{f}$ is τ-primitive and the process of lifting gives a $1-1$ correspondence between superconformal harmonic maps $f\colon S \to S^{2m}$ and τ-primitive maps $\psi\colon S \to \boldsymbol{SO}(2m+1)/T^m$.

We first characterize τ-primitive maps using the root space decomposition for $\boldsymbol{so}(2m+1)$ as follows: let $e_0, \dots, e_{2m}$ be the standard basis of $\mathbb{R}^{2m+1}$. As a Cartan subalgebra $\mathcal{T}$ for $\boldsymbol{so}(2m+1)$, we take the subspace of $\boldsymbol{so}(2m+1)$ consisting of those $(2m+1) \times (2m+1)$ matrices for which each of the subspaces $\mathbb{R}e_0, \mathbb{R}e_{2k-1} \oplus \mathbb{R}e_{2k}$ is an invariant subspace. Thus, these are matrices of the form

$$\operatorname{diag}(0, \Theta_1, \dots, \Theta_m), \text{ where } \Theta_k = \begin{pmatrix} 0 & -\theta_k \\ \theta_k & 0 \end{pmatrix}, k = 1, \dots, m.$$

For each $k = 1, \dots, m$, let $\sigma_k\colon \mathcal{T} \to i\mathbb{R}$ be defined by $\sigma_k(\operatorname{diag}(0, \Theta_1, \dots, \Theta_m)) = i\theta_k$. Then as positive simple roots we take $\sigma_1, \sigma_2 - \sigma_1, \dots, \sigma_m - \sigma_{m-1}$. The highest root is then given by

$$\sigma_m + \sigma_{m-1} = 2(\sigma_1 + (\sigma_2 - \sigma_1) + (\sigma_3 - \sigma_2) + \cdots + (\sigma_{m-1} - \sigma_{m-2})) + \sigma_m - \sigma_{m-1}.$$

In order to describe the subspace $\mathcal{M}_1$ of $\boldsymbol{so}(2m+1)^{\mathbb{C}}$ we first define $u_0, \dots, u_m$ to be the unit vectors in $\mathbb{C}^{2m+1} = \mathbb{R}^{2m+1} \otimes \mathbb{C}$ given by

$$u_0 = e_0, \ u_k = \frac{1}{\sqrt{2}}(e_{2k-1} - ie_{2k}), \ k = 1, \dots, m.$$

Then $u_0, u_1, \dots, u_m, \overline{u_1}, \dots, \overline{u_m}$ is a basis of $\mathbb{C}^{2m+1}$. Let $V_1, \dots, V_{m+1} \in \boldsymbol{so}(2m+1)^{\mathbb{C}}$ be defined as follows:

(a) for $k = 1, \dots, m$,

$$V_k u_{k-1} = u_k, \ \ V_k u_\ell = 0, \ \ \ell \neq k-1$$
$$V_k \overline{u_k} = -\overline{u_{k-1}}, \ \ V_k \overline{u_\ell} = 0, \ \ \ell \neq k.$$

(b)

$$V_{m+1} u_k = 0, \ k = 0, \dots, m-2$$
$$V_{m+1} u_{m-1} = \overline{u_m}, \ V_{m+1} u_m = -\overline{u_{m-1}}$$
$$V_{m+1} \overline{u_k} = 0, \ k = 1, \dots, m.$$

Then V_1 spans the eigenspace $\mathcal{A}_1$ corresponding to σ_1, V_k spans the eigenspace $\mathcal{A}_k$ corresponding to $\sigma_k - \sigma_{k-1}$ for $k = 2, \dots, m$, and V_{m+1} spans the eigenspace $\mathcal{A}_{m+1}$ corresponding to $-(\sigma_m + \sigma_{m-1})$. Then

$$\mathcal{M}_1 = \bigoplus_{k=1}^{m+1} \mathcal{A}_k.$$

We now give a characterization of τ-primitive maps $\psi\colon S \to \boldsymbol{SO}(2m+1)/T^m$.

Proposition 5.3 *The map $\psi = (X_0, \dots, X_m)\colon S \to \boldsymbol{SO}(2m+1)/T^m$ is τ-primitive if and only if, for each nowhere vanishing local section g_k of X_k*

(i) $\frac{\partial g_k}{\partial z} \in X_k \oplus X_{k+1}$ *for* $k = 0, \ldots, m-2$*, and has a non-zero* X_{k+1} *component except possibly at a discrete set of points.*

(ii) $\frac{\partial g_{m-1}}{\partial z} \in X_{m-1} \oplus X_m \oplus \overline{X}_m$*, and has non-zero* X_m*,* $\overline{X_m}$ *components except possibly at a discrete set of points.*

(iii) $\frac{\partial \overline{g}_k}{\partial z} \in \overline{X}_{k-1} \oplus \overline{X}_k$ *for* $k = 1, \ldots, m-1$.

Proof. Let $g_0, \ldots, g_m$ be nowhere vanishing local sections of $X_0, \ldots, X_m$ respectively, and for $k = 0, \ldots, m$ let $G_k = g_k/|g_k|$. If (i), (ii), (iii) hold then, for suitable scalars $\lambda_k, \mu_k, \alpha_k, \beta_k, \nu$ we may write

$$\begin{aligned}
\frac{\partial G_k}{\partial z} &= \lambda_k G_k + \mu_k G_{k+1},\ k = 0, \ldots, m-2, \\
\frac{\partial G_{m-1}}{\partial z} &= \lambda_{m-1} G_{m-1} + \mu_{m-1} G_m + \nu \overline{G_m}, \\
\frac{\partial \overline{G_k}}{\partial z} &= \lambda_k \overline{G_{k-1}} + \beta_k \overline{G_k},\ k = 0, \ldots, m-1.
\end{aligned} \tag{5.2}$$

It then follows that for $k = 1, \ldots, m-1$, $\alpha_k = -\mu_{k-1}$, and for suitable β_m,

$$\frac{\partial \overline{G_m}}{\partial z} = -\mu_{m-1} \overline{G_{m-1}} + \beta_m \overline{G_m}.$$

Hence, if $\widetilde{G}$ is the $\boldsymbol{SO}(2m+1)$-valued map given by

$$\widetilde{G} = (G_0, \frac{1}{\sqrt{2}}(G_1 + \overline{G_1}), \frac{i}{\sqrt{2}}(G_1 - \overline{G_1}), \ldots, \frac{1}{\sqrt{2}}(G_m + \overline{G_m}), \frac{i}{\sqrt{2}}(G_m + \overline{G_m}))$$

then

$$\widetilde{G}^{-1} \frac{\partial \widetilde{G}}{\partial z} = A_0 + A_1 \in \mathcal{M}_0 \oplus \mathcal{M}_1$$

where

$$A_1 = \sum_{k=0}^{m-1} \mu_k V_{k+1} + \nu V_{m+1}.$$

The additional assumtions on $\frac{\partial g_k}{\partial z}$ show that each of $\mu_0, \ldots, \mu_{m-1}, \nu$ vanish only at isolated points. Hence ψ is τ-primitive. The converse is similar but easier. $\square$

We now state the main result of this section. The proof is similar to that of theorem 4.6, except that we also need the second statement of lemma 5.2.

Theorem 5.4. *Let* $f: S \to S^{2m}$ *be a superconformal harmonic map. Then the lift* $\widetilde{f}: S \to \boldsymbol{SO}(2m+1)/T^m$ *given by (5.1) is* τ*-primitive. Conversely if* $\psi: S \to \boldsymbol{SO}(2m+1)/T^m$ *is* τ*-primitive, then* $f = \rho \circ \psi: S \to S^{2m}$ *is a superconformal harmonic map with lift* ψ*.*

We recall from above that superconformal harmonic maps $f: S \to S^{2m}$ are either linearly full in S^{2m} or are linearly full in a totally geodesic $(2m-1)$-sphere $S^{2m} \cap (N)^{\perp}$ where N is a unit vector in $^{2m+1}$. These latter maps are characterized by the fact that $X_m \oplus \overline{X}_m$ contains the constant real line spanned by N (for then X_0 is orthogonal to N). The real vector in $X_m \oplus \overline{X}_m$ orthogonal to N determines the *polar* of f in the sense of [Law70].

As in the P^n case, the above theorem has the following important consequence:

Corollary 5.5. *Every superconformal harmonic 2-torus $\phi\colon T^2 \to S^{2m}$ is of finite type.*

Since every (non-super)minimal 2-torus in S^4 is superconformal we recover the results in [FPPS92].

Also, as in the P^n case, we may produce directly the corresponding Toda framing, hence proving theorem 5.4 without using proposition 5.3. Firstly, given a local complex coordinate z in some simply connected open subset of S we have the sequence of ${}^{2m+1}$-valued functions (4.1) where $f_0 = f$. Then from (4.3) and the fact that f is not superminimal it follows that (f_m, f_m) is a non-zero holomorphic function and hence we may suppose that z is chosen such that $(f_m, f_m) = 1$. (The coordinate z is thus uniquely determined up to multiplication by a $2m$-th root of unity).

Now define a real positively oriented orthonormal frame $E_0, \ldots, E_{2m}$ on U and a real valued function η on U as follows:

$$E_0 = f_0, \frac{f_k}{|f_k|} = \frac{1}{\sqrt{2}}(E_{2k-1} - iE_{2k}), \quad k = 1, \ldots, m-1,$$
$$f_m = \cosh\eta E_{2m-1} - i \sinh\eta E_{2m}.$$

We note that $E_0, \ldots, E_{2m}$ and η are uniquely determined by f_0 (up to ambiguities involving $2m$-th roots of unity in the choice of the coordinate z mentioned above).

A straightforward calculation then shows that for $E = (E_0, \ldots, E_{2m}) \in \boldsymbol{SO}(2m+1)$,

$$E^{-1}\frac{\partial E}{\partial z} = \frac{\partial \Omega}{\partial z} + Ad \exp(\Omega)(B)$$

where $i\Omega = \operatorname{diag}(0, \Theta_1, \ldots, \Theta_m)$ with $\Theta_k = \begin{pmatrix} 0 & -w_k \\ w_k & 0 \end{pmatrix}, e^{w_k} = |f_k|,\ k = 1, \ldots, m-1, \Theta_m = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}$ and $B = \sum_{k=1}^{m+1} V_k$.

We note that the case where $f\colon S \to S^{2m}$ is linearly full into a totally geodesic S^{2m-1} is characterized by $\eta \equiv 0$.

6 Almost complex curves in S^6

We recall that the vector cross-product on ^{7} (see [BVW] for details relevant to the present context and further references) derived from Cayley multiplication on ^{8} may be used to define an almost complex structure on the unit sphere S^6 in ^{7}.

For any unit vector $x \in {}^7$, the orthogonal complement $(x)^\perp$ admits an orthogonal complex structure J_x defined by

$$J_x v = x \times v, \ \ v \in (x)^\perp.$$

Since $(x)^\perp$ may be identified with the tangent space $T_x S^6$, this defines an orthogonal almost complex structure on S^6. This, together with the standard metric on S^6, defines a nearly Kähler structure whose automorphism group is the exceptional Lie group G_2.

The group G_2 is the subgroup of $\boldsymbol{SO}(7)$ consisting of those 7×7 matrices whose columns $u_1, \ldots, u_7$ form an orthonormal basis for ^{7} for which

$$u_3 = u_1 \times u_2, \ \ u_5 = u_1 \times u_4, \ \ u_6 = u_2 \times u_4, \ \ u_7 = u_3 \times u_4.$$

Such a basis is called a *G_2-basis.* For example, changing the notation of the previous section, the standard basis $e_1, \ldots, e_7$ (rather than $e_0, \ldots, e_6$) of ^{7} is a G_2-basis.

The elements of $\boldsymbol{SO}(7)$ which fix e_1 and commute with J_{e_1} on $(e_1)^\perp$ form a subgroup $\boldsymbol{SU}(3) \subset G_2 \subset \boldsymbol{SO}(7)$. Care is needed here since

$$J_{e_1}(e_2 - ie_3) = i(e_2 - ie_3), \ J_{e_1}(e_4 - ie_5) = i(e_4 - ie_5)$$

but

$$J_{e_1}(e_6 - ie_7) = -i(e_6 - ie_7).$$

so that this subgroup $\boldsymbol{SU}(3)$ is a conjugate of the standard one in $\boldsymbol{SO}(7)$.

The maximal torus T^2 of $\boldsymbol{SU}(3)$ and of G_2 consists of elements of the form

$$\mathrm{diag}(1, R(\theta_1), R(\theta_2), R(\theta_3)), \theta_1 + \theta_2 = \theta_3.$$

Moreover, $G_2/T^2 \subset \boldsymbol{SO}(7)/T^3$, and, in the notation of section 5, consists of those elements $X = (X_0, X_1, X_2, X_3) \in \boldsymbol{SO}(7)/T^3$ for which $X_1, X_2,$ (resp. X_3) are $+i$(resp. $-i$)-eigenspaces for $J_{\rho(x)}$. In fact the condition on X_3 follows from those on X_1 and X_2. One may also show that, equivalently, if $u_0 = \rho(X)$ then there is a basis $v_1, v_2, \overline{v}_3$ of X_1, X_2, X_3 respectively such that $|v_k| = \frac{1}{\sqrt{2}}$ and

$$\left.\begin{aligned} &u_0 \times v_k = iv_k \\ &v_k \times v_\ell = \epsilon_{k\ell m}\overline{v}_m \\ &v_k \times \overline{v}_\ell = -\frac{i}{2}\delta_{k\ell}u_0 \end{aligned}\right\} k, \ell, m \in \{1, 2, 3\},$$

where $\times$ denotes the extension of the vector cross-product to ^{7}. (c.f. [BVW].)

An *almost complex curve* in S^6 is a non-constant smooth map $f: S \to S^6$ whose differential is complex linear. In terms of a local complex coordinate $z = x + iy$ the condition for f to be almost complex may be written

$$f \times f_z = if_z$$

or equivalently, in terms of the notation of section 4, (4.1),

$$f \times f_1 = if_1. \tag{6.1}$$

It is easy to check (c.f. [BVW]) that an almost complex curve which is not totally isotropic is a superconformal harmonic map.

Proposition 6.1. *Let $f: S \to S^6$ be superconformal with lift $\widetilde{f}: S \to \mathbf{SO}(7)/T^3$. Then f is almost complex if and only if $\widetilde{f}(S) \subseteq G_2/T^2$.*

Proof. If $\widetilde{f}(S) \subseteq G_2/T^2$ then writing $\widetilde{f} = (X_0, X_1, X_2, X_3)$ we see that, since $f = \rho \circ \widetilde{f}$ and $f_1 \in X_1$, it follows that (6.1) holds.

Conversely suppose that $f: S \to S^6$ is an almost complex curve. Then (6.1) holds, and differentiating (6.1) gives

$$f \times f_2 = if_2.$$

Since $f_k \in X_k$, $(k = 1, 2)$, the result follows. □

Now we consider the roots of G_2 in order to consider τ-primitive maps. Recall from section 5 that we may take $\sigma_1, \sigma_2 - \sigma_1, \sigma_3 - \sigma_2$ as a set of positive simple roots of $\boldsymbol{so}(7)$ and that the highest root is $\sigma_3 + \sigma_2$. From above, we see that the Cartan subalgebra $\mathcal{T}^2$ of $\mathcal{G}_2$ is the subspace of the Cartan subalgebra $\mathcal{T}^3$ of $\boldsymbol{so}(7)$ defined by $\sigma_1 + \sigma_2 = \sigma_3$. As a set of positive simple roots of $\mathcal{G}_2$ we may take $\sigma_1, \sigma_2 - \sigma_1$ in which case the highest root is $\sigma_3 + \sigma_2 = \sigma_1 + 2\sigma_2 = 3\sigma_1 + 2(\sigma_2 - \sigma_1)$.

The root spaces of $\boldsymbol{so}(7)$ corresponding to $\sigma_2 - \sigma_1$ and $-(\sigma_3 + \sigma_2)$ are both contained in $\mathcal{G}_2$, while the $\mathcal{G}_2$-root space corresponding to $\sigma_1 (= \sigma_3 - \sigma_2)$ is the intersection with $\mathcal{G}_2$ of the direct sum of the $\boldsymbol{so}(7)$ root spaces corresponding to σ_1 and $\sigma_3 - \sigma_2$.

It is now clear that

$$\mathcal{M}_1(\mathcal{G}_2) = \mathcal{M}_1(\mathcal{SO}(7)) \cap \mathcal{G}_2. \tag{6.2}$$

Alternatively, one may deduce this from the fact that G_2/T^2 is a 6-symmetric submanifold of the 6-symmetric space $\boldsymbol{SO}(7)/T^3$, where in both cases the symmetry of order 6 is given by $\tau = Ad\exp(2\pi i Z)$ where $Z = \mathrm{diag}(1, R(\frac{2\pi}{6}), R(\frac{4\pi}{6}), R(\frac{6\pi}{6}))$, as indicated by the general theory of section 2. For in both cases $\mathcal{M}_1$ is the $e^{2\pi i/6}$-eigenspace of the automorphism of the complexified Lie algebra given by τ.

The following proposition is now clear.

Proposition 6.2. *Let $\psi: S \to G_2/T^2$ and let $i: G_2/T^2 \to \boldsymbol{SO}(7)/T^3$ be the natural inclusion. Then ψ is τ-primitive if an only if $i \circ \psi$ is τ-primitive.*

Propositions 6.1 and 6.2 now immediately give the following theorem.

Theorem 6.3. *Let $f: S \to S^6$ be an almost complex curve which is not totally isotropic. Then the lift $\widetilde{f}: S \to G_2/T^2$ given by (4.1) is τ-primitive. Conversely, if $\psi: S \to G_2/T^2$ is τ-primitive then $f = \rho \circ \psi: S \to S^6$ is almost complex with lift ψ.*

As in the previous sections this has the following important consequence.

Corollary 6.4. *Every non-superminimal almost complex 2-torus in S^6 is of finite type.*

Together with the results on superminimal almost complex curves in [BVW] this accounts for all almost complex 2-tori in S^6.

Finally we consider Toda framings for almost complex curves. As in section 5, we first choose a local complex coordinate z in an open subset U of S such that $(f_3, f_3) = 1$. Now define a real positively oriented frame $E_1, \ldots, E_7$ on U and a real valued function η on U as follows:

$$E_1 = f_0, \quad \frac{f_k}{|f_k|} = \frac{1}{\sqrt{2}}\ (E_{2k} - iE_{2k+1}), \quad k = 1, 2,$$

$$f_3 = -\cosh\eta\ E_7 - i\sinh\eta\ E_6.$$

This is then a G_2-framing as shown in [BVW]. Note that the seemingly natural way of writing f_3 in terms of E_6, E_7 and η (as in section 5) does not give a G_2-framing. Moreover (see [BVW]), if $|f_k| = e^{w_k}$, $k = 1, 2$, then $w_1 + w_2 = \eta$, and writing $E = (E_1, \ldots, E_7)$ we have

$$E^{-1}\frac{\partial E}{\partial z} = \frac{\partial \Omega}{\partial z} + Ad(\exp\Omega)(B),$$

where

$$\Omega = i\begin{pmatrix} 0 & & & & & & \\ & & w_1 & & & & \\ & -w_1 & & & & & \\ & & & & w_2 & & \\ & & & -w_2 & & & \\ & & & & & & \eta \\ & & & & & -\eta & \end{pmatrix} \in i\mathcal{T}^2$$

and

$$B = \begin{pmatrix} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & & \\ -\frac{i}{\sqrt{2}} & & & \frac{1}{2} & -\frac{i}{2} & & \\ -\frac{1}{\sqrt{2}} & & & \frac{i}{2} & \frac{1}{2} & & \\ & -\frac{1}{2} & -\frac{i}{2} & & & \frac{1}{2} & \frac{i}{2} \\ & \frac{i}{2} & -\frac{1}{2} & & & \frac{i}{2} & -\frac{1}{2} \\ & & & -\frac{1}{2} & -\frac{i}{2} & & \\ & & & -\frac{i}{2} & \frac{1}{2} & & \end{pmatrix} \in \mathcal{M}_1(G_2)\,.$$

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