Harmonic maps via Adler–Kostant–Symes theory

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Introduction

Over the past few years significant progress has been made in the understanding of various completely integrable nonlinear PDE (soliton equations) and their relationship to classical problems in differential geometry. It has been shown in a series of recent papers [41, 31, 20, 24, 14, 7, 11] that constant mean and Gauss curvature surfaces, Willmore surfaces, minimal surfaces in spheres and projective spaces and generally harmonic maps from a Riemann surface M into various homogeneous spaces may be described as solutions to various soliton equations. Moreover, these solutions are *algebraic* in the sense that they are obtained by integrating *ordinary* differential equations of Lax type which linearise on the Jacobian of an appropriate algebraic curve.

Links between harmonic maps and integrable systems have been known to exist for some time: for instance, Uhlenbeck [54] showed that S^1-equivariant harmonic maps $\mathbb{R} \times S^1 \to S^n$ amount to solutions of the Neumann system describing motion on S^n in a quadratic potential—a classical completely integrable system (for related results on S^1-equivariant harmonic maps, see [32, 23]). However, a significant interaction between differential geometry and soliton theory did not emerge until after Wente's resolution [58] of the Hopf conjecture on the existence of a constant mean curvature torus in $\mathbb{R}^3$.

Let us consider this problem in more detail as it contains the seeds of all subsequent developments in this area. Recall that a surface M in $\mathbb{R}^3$ has constant mean curvature if and only if its Gauss map $\phi : M \to S^2$ is harmonic. We are therefore led to pose the following problem: given a compact Riemann surface M of genus g, find all harmonic maps ϕ from M into the 2-sphere. The harmonicity of ϕ implies that the $(2,0)$-part of the quadratic differential $(d\phi, d\phi)$ is holomorphic. If M has genus 0 then this differential vanishes identically, i.e. ϕ is conformal and thus $\pm$holomorphic. In this sense the harmonic map equation reduces to the linear Cauchy–Riemann equation. A similar phenomenon occurs when the target space is replaced by a compact symmetric space where the reduction to the Cauchy–Riemann equation is achieved by various twistorial constructions (see [34, 59] in this volume for further details and references).

The situation changes drastically when M has higher genus. There is no systematic theory when M has genus larger then 1 so we shall concentrate on the case where M is a 2-torus. Here, the differential $(d\phi, d\phi)^{(2,0)}$ either vanishes identically (in which case we again have $\pm$holomorphic maps) or is nowhere zero. In the latter case one can choose a global complex coordinate z so that $(d\phi, d\phi)^{(2,0)} = dz^2$ and $(d\phi, d\phi)^{(1,1)} = \cosh(\omega)\,|dz|^2$. Then the harmonicity of ϕ amounts to the sinh-Gordon equation for ω, a well known soliton equation. Starting from this observation, Pinkall–Sterling [41] show that all doubly-periodic solutions to the sinh-Gordon equation (and thus all constant mean curvature tori) are obtained by integration of a family of completely integrable finite dimensional systems of ODE. These ODE linearise on the Jacobian of an algebraic curve the study of which enabled Bobenko [6] to show that the solutions can be expressed rather explicitly in terms of theta functions. That this behaviour extends to more general target spaces was indicated in [31] for S^3, [24] for S^4 and culminated in [14] where a rather comprehensive theory of harmonic 2-tori in compact Riemannian symmetric spaces is developed.

One may summarise this theory as follows: the starting point is the basic fact [42, 55, 64, 65] that the harmonic map equation can be reformulated as a zero-curvature equation involving an auxiliary parameter or, in other words, as the Maurer–Cartan equation for a certain loop algebra valued 1-form. One can obtain solutions to these equations by integrating a pair of commuting Hamiltonian vector fields on certain finite-dimensional subspaces of loop algebras. In this way, we get harmonic maps of $\mathbb{R}^2$ which we call *harmonic maps of finite type*. Finally, under suitable nondegeneracy assumptions, one shows that these procedures account for all doubly periodic harmonic maps, that is, for all harmonic 2-tori.

It is the purpose of this article to provide a unified account of these ideas and, in particular, the results of [7, 11, 14, 24, 41]. A framework for this is given by the theory of Adler–Kostant–Symes [1, 36, 53]. This theory provides a scheme for producing and integrating non-trivial commuting Hamiltonian flows on Lie algebras. The basic setting for the scheme is this: one has a Lie algebra $\mathfrak{g}$ which admits a (vector space) direct sum decomposition into subalgebras

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}. \tag{0.1}$$

In the presence of a suitable inner product, $\mathfrak{k}$ inherits a Poisson structure from the Lie–Poisson structure on $\mathfrak{b}^*$. With this Poisson structure, $\mathfrak{k}$ has a large supply of Poisson commuting functions provided by the restriction to $\mathfrak{k}$ of the invariant functions on $\mathfrak{g}$. To integrate the Hamiltonian flows so obtained, we use the method of Symes: let G be a Lie group with Lie algebra $\mathfrak{g}$ and let K, B be the subgroups corresponding to $\mathfrak{k}$, $\mathfrak{b}$. Suppose that there is a decomposition

$$G = K \times B \tag{0.2}$$

corresponding to (0.1). Then the Hamiltonian flow through an initial condition $\xi_o \in \mathfrak{k}$ is given by

$$Ad\, k^{-1}\, \xi_o,$$

where k is the projection onto K via (0.2) of a suitable geodesic in G.

We apply this theory to a family of twisted loop algebras which admit a decomposition (0.1) of Iwasawa type. A key point is that the subspaces of Laurent polynomial loops of fixed degree are Poisson submanifolds so that we obtain commuting ODE on *finite-dimensional* Poisson manifolds. From the Hamiltonian flows on these subspaces we get solutions to zero-curvature equations which give rise to (framings of) harmonic maps of $\mathbb{R}^2$ into various homogeneous spaces.

In fact, we get in this way both harmonic maps into Riemannian symmetric spaces and primitive maps into k-symmetric spaces. These last are rather special harmonic maps into a class of reductive homogeneous spaces that generalise Riemannian symmetric spaces (the involutions of a Riemannian symmetric space are replaced by automorphisms of order k) and include all flag manifolds. A map into such a space is primitive if it satisfies a first order condition not unlike a Cauchy–Riemann equation defined by the geometry of the k-symmetric space. The motivation for studying primitive maps comes from the fact that they include twistor lifts of minimal maps in spheres and complex projective spaces and also, in case the target is a flag manifold, are in bijective correspondence with (periodic) Toda fields.

The method of Symes also works in this infinite-dimensional loop algebra setting. There are Iwasawa decompositions of the loop groups that correspond to our loop algebras and projection of (complex) geodesics provides maps into loop groups which are essentially the *extended solutions* in the sense of Uhlenbeck [55] for the harmonic maps we have produced.

Finally, we obtain sufficient conditions on a harmonic 2-torus to arise from our constructions. The basic result here is due to Burstall–Ferus–Pedit–Pinkall [14] and the method of Symes is applied to translate that result to our present setting.

1 The Adler–Kostant–Symes scheme

One of the main themes of this volume is the construction of harmonic maps from commuting Hamiltonian flows on loop algebras. A setting for such results is provided by the celebrated analysis of the (open) Toda lattice by Kostant and Symes [36, 53], where a general scheme is described for producing and integrating commuting Hamiltonian flows on Lie algebras (see also [1]). Since this beautiful circle of ideas may be unfamiliar to Riemannian geometers, we begin by rehearsing the main points of the theory in a form suitable for our applications.

1.1 Poisson structures and Lie algebra decompositions

Let M be a manifold. A *Poisson structure* on M is a Lie algebra structure on $C^\infty(M)$, with bracket denoted $(f,g) \mapsto \{f,g\}$, for which $ad\, f$ is a derivation over multiplication:

$$\{f, gh\} = \{f,g\}h + g\{f,h\},$$

for all f, g, $h \in C^\infty(M)$. As a consequence, each $f \in C^\infty(M)$ gives rise to a *Hamiltonian vector field* X_f by

$$X_f g = \{f,g\}.$$

It follows from the Jacobi identity for $\{\,,\,\}$ that $f \mapsto X_f$ is a Lie algebra homomorphism $C^\infty(M) \to C^\infty(TM)$.

For our basic example of a Poisson structure, let $\mathfrak{b}$ be a (real) Lie algebra with dual $\mathfrak{b}^*$. When $f \in C^\infty(\mathfrak{b}^*)$ and $x \in \mathfrak{b}^*$, we have $df_x \in \mathfrak{b}^{**} \cong \mathfrak{b}$ and, using this identification, we define a Poisson structure on $\mathfrak{b}^*$ by

$$\{f,g\}(x) = \langle x, [df_x, dg_x]\rangle.$$

The Ad^*-invariant functions on $\mathfrak{b}^*$ commute with respect to this Poisson structure but, unfortunately, they also commute with all other functions on $\mathfrak{b}^*$ and so have trivial Hamiltonian flows. To get non-trivial commuting flows, we must introduce some extra structure.

Suppose then that we have a Lie algebra $\mathfrak{g}$ which admits a (vector space) direct sum decomposition into subalgebras:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}.$$

Suppose further that there is a (real) non-degenerate symmetric bilinear form on $\mathfrak{g}$, denoted $(\,,\,)$, which is invariant:

$$([\xi,\eta],\zeta) = -(\eta, [\xi,\zeta]).$$

The bilinear form induces a musical isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ and so, by restriction, an isomorphism $\mathfrak{k}^\circ \cong \mathfrak{b}^*$. Here $\mathfrak{k}^\circ$ is the polar of $\mathfrak{k}$ with respect to $(\,,\,)$:

$$\mathfrak{k}^\circ = \{\eta \in \mathfrak{g} : (\eta,\xi) = 0 \text{ for all } \xi \in \mathfrak{k}\}.$$

Thus $\mathfrak{k}^\circ$ acquires a Poisson structure from that of $\mathfrak{b}^*$ and this is the Poisson structure that will be important for us.

To describe the Poisson structure on $\mathfrak{k}^\circ$ explicitly, we introduce some notation. Let $\pi_{\mathfrak{k}}$, $\pi_{\mathfrak{b}}$ be the projections onto $\mathfrak{k}$, $\mathfrak{b}$ along $\mathfrak{b}$, $\mathfrak{k}$ respectively. Further, for $f \in C^\infty(\mathfrak{k}^\circ)$, let $\tilde f$ be some extension of f to $\mathfrak{g}$ and let $\nabla \tilde f$ denote its gradient with respect to $(\,,\,)$. It is easy to check that the projected gradient

$$\pi_{\mathfrak{b}} \nabla \tilde f$$

depends only on f. A straightforward calculation now shows that the Poisson bracket on $\mathfrak{k}^\circ$ is given by

$$\{f,g\}(\xi) = (\xi, [\pi_{\mathfrak{b}}\nabla \tilde{f}_\xi, \pi_{\mathfrak{b}}\nabla \tilde{g}_\xi]). \tag{1.1}$$

To see the Hamiltonian vector field of f, let $\pi_{\mathfrak{k}^\circ}$ denote projection onto $\mathfrak{k}^\circ$ along $\mathfrak{b}^\circ$. Then, using the invariance of $(\, , \,)$, we have

$$\begin{aligned}
\{f,g\}(\xi) &= ([\xi, \pi_{\mathfrak{b}}\nabla \tilde{f}_\xi], \pi_{\mathfrak{b}}\nabla \tilde{g}_\xi) \\
&= (\pi_{\mathfrak{k}^\circ}[\xi, \pi_{\mathfrak{b}}\nabla \tilde{f}_\xi], \nabla \tilde{g}_\xi) \\
&= d\tilde{g}(\pi_{\mathfrak{k}^\circ}[\xi, \pi_{\mathfrak{b}}\nabla \tilde{f}_\xi]) = dg(\pi_{\mathfrak{k}^\circ}[\xi, \pi_{\mathfrak{b}}\nabla \tilde{f}_\xi]).
\end{aligned}$$

Thus, for $\xi \in \mathfrak{k}^\circ$,

$$X_f(\xi) = \pi_{\mathfrak{k}^\circ}[\xi, \pi_{\mathfrak{b}}\nabla \tilde{f}_\xi]. \tag{1.2}$$

1.2 Poisson submanifolds

Let M be a Poisson manifold with bracket $\{ \, , \, \}_M$. A submanifold V of M is a *Poisson submanifold* if

1. V has a Poisson structure $\{ \, , \, \}_V$;
2. the inclusion $i : V \hookrightarrow M$ is a *Poisson morphism*:

$$\{f \circ i, g \circ i\}_V = \{f, g\}_M \circ i, \tag{1.3}$$

   for f, $g \in C^\infty(M)$.

Since $i^* : C^\infty(M) \to C^\infty(V)$ is surjective, it is clear that (1.3) completely determines the Poisson structure on V. Thus a submanifold V is Poisson as soon as the prescription (1.3) gives a well-defined bracket $\{ \, , \, \}_V$ on $C^\infty(V)$.

There is a simple condition for this to be the case: a submanifold V is Poisson if and only if the restriction of any Hamiltonian vector field to V is tangent to V (see [57, Lemma 1.1]). In the setting of Section 1.1, we use this and the form of the Hamiltonian vector fields (1.2) to obtain the following

Proposition 1.1 *Let $V \subset \mathfrak{k}^\circ$ be a linear subspace satisfying*

$$\pi_{\mathfrak{k}^\circ}[V, \mathfrak{b}] \subset V. \tag{1.4}$$

Then V is a Poisson submanifold of $\mathfrak{k}^\circ$.

Such considerations will be important when we come to discuss finite-dimensional subspaces of infinite-dimensional loop algebras.

Remark The above discussion is merely the translation into our setting of the fact that $ad^*\,\mathfrak{b}$-invariant subspaces of $\mathfrak{b}^*$ are Poisson submanifolds of $\mathfrak{b}^*$ which in turn follows from the fact that the symplectic leaves of $\mathfrak{b}^*$ are precisely the co-adjoint orbits.

1.3 Commuting Hamiltonians

We now come to the main point of our constructions: ad-invariant functions on $\mathfrak{g}$ restrict to Poisson commuting functions on $\mathfrak{k}^\circ$. Recall that $f \in C^\infty(\mathfrak{g})$ is ad-invariant if

$$df_\xi([\xi, \mathfrak{g}]) = 0,$$

or, equivalently,

$$[\nabla f_\xi, \xi] = 0, \tag{1.5}$$

for all $\xi \in \mathfrak{g}$. If f, $g \in C^\infty(\mathfrak{g})$ are two ad-invariant functions, then their restrictions to $\mathfrak{k}^\circ$ have Poisson bracket

$$\begin{aligned} \{f, g\}(\xi) &= (\xi, [\pi_{\mathfrak{b}}\nabla f_\xi, \pi_{\mathfrak{b}}\nabla g_\xi]) \\ &= ([\xi, \pi_{\mathfrak{b}}\nabla f_\xi], \pi_{\mathfrak{b}}\nabla g_\xi). \end{aligned}$$

We now use the invariance of f which implies $[\xi, \pi_{\mathfrak{k}}\nabla f_\xi] = -[\xi, \pi_{\mathfrak{b}}\nabla f_\xi]$ to get

$$\begin{aligned} \{f, g\}(\xi) &= -([\xi, \pi_{\mathfrak{k}}\nabla f_\xi], \pi_{\mathfrak{b}}\nabla g_\xi) \\ &= -(\xi, [\pi_{\mathfrak{k}}\nabla f_\xi, \pi_{\mathfrak{b}}\nabla g_\xi]) \\ &= ([\xi, \pi_{\mathfrak{b}}\nabla g_\xi], \pi_{\mathfrak{k}}\nabla f_\xi) \\ &= -([\xi, \pi_{\mathfrak{k}}\nabla g_\xi], \pi_{\mathfrak{k}}\nabla f_\xi) = (\xi, [\pi_{\mathfrak{k}}\nabla f_\xi, \pi_{\mathfrak{k}}\nabla g_\xi]) \end{aligned}$$

which last vanishes since $[\pi_{\mathfrak{k}}\nabla f_\xi, \pi_{\mathfrak{k}}\nabla g_\xi] \in \mathfrak{k}$ and $\xi \in \mathfrak{k}^\circ$. As for the corresponding Hamiltonian vector fields, from (1.2) and (1.5), we see that the restriction to $\mathfrak{k}^\circ$ of an ad-invariant f has Hamiltonian vector field

$$X_f(\xi) = \pi_{\mathfrak{k}^\circ}[\pi_{\mathfrak{k}}\nabla f_\xi, \xi] = [\pi_{\mathfrak{k}}\nabla f_\xi, \xi], \tag{1.6}$$

where the last equality comes from $[\mathfrak{k}, \mathfrak{k}^\circ] \subset \mathfrak{k}^\circ$.

1.4 Group decompositions and the method of Symes

We have therefore found a family of commuting flows on $\mathfrak{k}^\circ$ and our next task is to integrate them. For this, we globalise the situation following Symes [53]: let G be a Lie group with Lie algebra $\mathfrak{g}$, let K, B be subgroups with Lie algebras $\mathfrak{k}$, $\mathfrak{b}$ and suppose that multiplication $K \times B \to G$ is a diffeomorphism onto. Thus any $g \in G$ may be uniquely written as a product

$$g = kb,$$

with $k \in K$, $b \in B$. We further demand that the adjoint action of G on $\mathfrak{g}$ preserve $(\, , \,)$ (which follows from the invariance of $(\, , \,)$ if G is connected) and we consider Ad-invariant functions on $\mathfrak{g}$: that is, functions $f \in C^\infty(\mathfrak{g})$ satisfying

$$f(Ad\, g\, \xi) = f(\xi), \tag{1.7}$$

for all $\xi \in \mathfrak{g}$, $g \in G$ (again, when G is connected, this is equivalent to (1.5)). Differentiating equation (1.7) with respect to ξ, we deduce that ∇f is equivariant in the sense that

$$\nabla f_{Ad\, g\, \xi} = Ad\, g\, \nabla f_\xi, \tag{1.8}$$

for all $\xi \in \mathfrak{g}$, $g \in G$.

So let $f \in C^\infty(\mathfrak{g})$ be Ad-invariant and fix an initial condition $\xi_o \in \mathfrak{k}^\circ$. Consider the geodesic $g : \mathbb{R} \to G$ given by $g(t) = \exp -t\nabla f_{\xi_o}$ and factorise to get curves $k : \mathbb{R} \to K$, $b : \mathbb{R} \to B$ with

$$g(t) = k(t)b(t),$$

for all $t \in \mathbb{R}$. Define $\xi : \mathbb{R} \to \mathfrak{k}^\circ$ by $\xi = Ad\, k^{-1}\xi_o$. Then ξ is the integral curve of X_f with $\xi(0) = \xi_o$. Indeed,

$$\dot{\xi} = -[k^{-1}\dot{k}, \xi]$$

while

$$-\nabla f_{\xi_o} = \dot{g}g^{-1} = \dot{k}k^{-1} + Ad\, k\, \dot{b}b^{-1}$$

so that taking $Ad\, k^{-1}$ of both sides and projecting onto $\mathfrak{k}$ gives

$$k^{-1}\dot{k} = -\pi_{\mathfrak{k}} Ad\, k^{-1}\nabla f_{\xi_o} = -\pi_{\mathfrak{k}}\nabla f_\xi,$$

where the last inequality follows from the equivariance (1.8) of ∇f. Thus

$$\dot{\xi} = [\pi_{\mathfrak{k}}\nabla f_\xi, \xi].$$

To summarise: we have seen how to equip $\mathfrak{k}^\circ$ with a Poisson structure for which the restriction of Ad-invariant functions on $\mathfrak{g}$ commute. Moreover, the corresponding Hamiltonian vector fields are of Lax form (1.6) and are integrated using the projection onto K of geodesics in G.

Remark In fact, the invariant inner product $(\ ,\ )$, although present in all our applications, is not strictly necessary in the above development. If one replaces $\mathfrak{k}^\circ$ with $\mathfrak{k}^\perp \subset \mathfrak{g}^*$, the annihilator of $\mathfrak{k}$, then, as above, $\mathfrak{k}^\perp \cong \mathfrak{b}^*$ and so acquires a Poisson structure with respect to which the restrictions to $\mathfrak{k}^\perp$ of Ad^*-invariant functions on $\mathfrak{g}^*$ commute. Moreover, the Symes method of integration also works in this setting *mutatis mutandis*.

1.5 The Toda lattice

We conclude this discussion with a brief exposition of the original application of the theory to the open Toda lattice. Besides completing the circle of ideas, this will also give us an opportunity to develop concepts we shall need later.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{k}$ a compact real form. Fix a maximal toral subalgebra $\mathfrak{t} \subset \mathfrak{k}$ and set $\mathfrak{a} = i\mathfrak{t}$ where $i = \sqrt{-1}$. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and fix a choice of positive roots $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{h})$ with corresponding simple roots $\alpha_1, \dots, \alpha_l$.

The Toda lattice is a Hamiltonian system on $T^*\mathfrak{a} = \mathfrak{a} \times \mathfrak{a}^*$, equipped with its canonical symplectic structure. The Hamiltonian is given by

$$H(q,p) = |p|^2/2 + \sum_{i=1}^{l} e^{2\alpha_i(q)}$$

where the metric is given by the real part of the Killing form κ of $\mathfrak{g}$. One may think of this system as describing l particles moving on a line with exponential interactions governed by the Dynkin diagram of $\mathfrak{g}$. (Note that the roots are *real* linear functionals on $\mathfrak{a}$.)

To put this Hamiltonian system into our frame-work, we follow Bloch–Flaschka–Ratiu [5] and use the Iwasawa decomposition of $\mathfrak{g}$: for each root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, let $\mathfrak{g}^\alpha$ be the corresponding root space and define a nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$ by

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha.$$

We then have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

the *Iwasawa decomposition* of $\mathfrak{g}$ (cf. [29, page 275]). Moreover, there is a global analogue of this decomposition: let G be a Lie group with Lie algebra $\mathfrak{g}$, K the maximal compact subgroup with Lie algebra $\mathfrak{k}$ and A, N the analytic subgroups with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$. Then multiplication $K \times A \times N \to G$ is a diffeomorphism onto.

Now $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$ so that $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$ and we may apply our preceding discussion to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$. As our invariant symmetric form, we take the imaginary part of the Killing form: $(\, , \,) = Im\, \kappa$. The Killing form is positive definite on $\mathfrak{a}$, negative definite on $\mathfrak{k}$ and vanishes identically on $\mathfrak{n}$ so that both $\mathfrak{k}$ and $\mathfrak{b}$ are isotropic for $(\, , \,)$ giving

$$\mathfrak{k}^\circ = \mathfrak{k}, \qquad \mathfrak{b}^\circ = \mathfrak{b}.$$

Thus we have a Poisson structure on $\mathfrak{k} \cong \mathfrak{b}^*$.

For $1 \le i \le l$, choose non-zero vectors $X_i \in \mathfrak{g}^{\alpha_i}$ and consider

$$\mathcal{O} = \{H + \sum_{i=1}^{l} a_i(X_i + \overline{X}_i) : H \in \mathfrak{t},\, a_i > 0\}.$$

It is not too difficult to see that $\mathcal{O}$ is a symplectic leaf of the Poisson structure on $\mathfrak{k}$. Indeed, the co-adjoint action of $B = AN$ on $\mathfrak{b}^*$ induces an action of B on $\mathfrak{k}$, the orbits of which are the symplectic leaves on $\mathfrak{k}$ and one can show that $\mathcal{O}$ is the B-orbit of $\sum_i (X_i + \overline{X}_i)$.

In fact, $\mathcal{O}$ is symplectomorphic with the phase space $T^*\mathfrak{a}$ of the Toda lattice: use $(\ ,\ )$ to identify $\mathfrak{a}^*$ with $\mathfrak{t}$ and define $\phi : T^*\mathfrak{a} \to \mathcal{O}$ by

$$\phi(q,p) = p + \sum_{i=1}^{l} c_i e^{\alpha_i(q)}(X_i + \overline{X}_i),$$

where the constants c_i are positive constants to be chosen later. By checking the Poisson brackets of suitable co-ordinate functions, it is straight-forward to prove that ϕ is a Poisson morphism and hence a symplectomorphism for any choice of the constants c_i.

Now take as an Ad-invariant function $f : \mathfrak{g} \to \mathbb{R}$ half the negative of the real part of the Killing form:

$$f(\eta) = -\,Re\,\kappa(\eta,\eta)/2.$$

On $\mathcal{O}$, we have

$$f(H + \sum_i a_i(X_i + \overline{X})) = -\kappa(H,H)/2 - \sum_i a_i^2 \kappa(X_i, \overline{X}_i)$$

so that

$$\begin{aligned} f \circ \phi(q,p) &= |p|^2/2 - \sum_i c_i^2 \kappa(X_i, \overline{X}_i) e^{2\alpha_i(q)} \\ &= |p|^2/2 + \sum_i c_i^2 |X_i|^2 e^{2\alpha_i(q)}. \end{aligned}$$

Thus choosing $c_i = 1/|X_i|$, we see that $f \circ \phi$ is the Toda Hamiltonian.

The preceding theory may now be applied in a number of ways: the invariant polynomials on $\mathfrak{g}$ restrict to give l functionally independent Poisson commuting conserved quantities on $\mathcal{O}$ so that the Toda lattice is completely integrable in the sense of Liouville. Moreover, the method of Symes can be used to integrate the Toda flow leading eventually to explicit formulae for the flows in terms of matrix coefficients of the fundamental representations of G. For this and for further details on the ideas we have been discussing, the Reader is referred to the original papers and the books [28, 40].

2 Twisted loop algebras and zero-curvature equations

2.1 What the Maurer–Cartan equations are for

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let θ be the (left) Maurer–Cartan form of G. Thus θ is the $\mathfrak{g}$-valued 1-form on G given by

$$\theta_g(X) = (g^{-1})_* X \in T_e G = \mathfrak{g},$$

for $X \in T_g G$. A simple calculation using the left-invariance of θ shows that

$$d\theta + \tfrac{1}{2}[\theta \wedge \theta] = 0. \tag{2.1}$$

These equations are the *Maurer–Cartan equations*.

Now let $\phi : M \to G$ be a map of a manifold M and set $\alpha = \phi^*\theta$. Then ϕ pulls back (2.1) to give

$$d\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0 \tag{2.2}$$

so that α also satisfies the Maurer–Cartan equations.

Fundamental to what follows is that a partial converse of this is true. We have the following classical theorem (which is proved, for example, in [52]):

Theorem 2.1 *Let α be a $\mathfrak{g}$-valued 1-form on a* simply-connected *manifold M. Then $\alpha = \phi^*\theta$ for some map $\phi : M \to G$ if and only if α satisfies the Maurer–Cartan equations (2.2).*

In this case, ϕ is unique up to left translation by a constant element of G.

Otherwise said, (2.2) amounts to the assertion that the connection $d + \alpha$ on the trivial principal G-bundle $M \times G \to M$ has vanishing curvature (i.e., is *flat*) while Theorem 2.1 says that there is a gauge transformation ϕ which gauges this flat connection to the trivial connection. It is for this reason that the Maurer–Cartan equations are often called the *zero-curvature equations*.

Notation When G is a matrix group, $\phi^*\theta = \phi^{-1}d\phi$. In previous sections, we have used this latter notation even when G is not a matrix group and we shall continue to do so.

The application of the above constructions to the theory of harmonic maps begins with two observations. Firstly, it is well known [42, 55, 64, 65] that, for maps of a Riemann surface into suitable homogeneous spaces, the harmonic map equations are equivalent to the flatness of a certain loop of connections, that is, to certain solutions of the Maurer–Cartan equations (2.2) where α is a 1-form with values in an appropriate loop algebra.

On the other hand, solutions of (2.2) arise in the setting of the previous sections from the simultaneous integration of several commuting Hamiltonian flows on $\mathfrak{k}^\circ$. Indeed, let $f^1, \dots, f^m$ be Ad-invariant functions on $\mathfrak{g}$ with corresponding Hamiltonian vector fields X_i on $\mathfrak{k}^\circ$. Fixing an initial condition $\xi_o \in \mathfrak{k}^\circ$, we may integrate these vector fields (when they are complete) to get $\xi : \mathbb{R}^m \to \mathfrak{k}^\circ$ satisfying

$$\begin{aligned} d\xi &= X_i(\xi)\, dt^i; \\ \xi(0) &= \xi_o. \end{aligned}$$

Moreover, the Symes method shows that $\xi = Ad\, k^{-1}\xi_o$ where $k : \mathbb{R}^m \to K$ satisfies

$$k^{-1}dk = -\pi_{\mathfrak{k}} Ad\, k^{-1} \nabla f^i_{\xi_o}\, dt^i = -\pi_{\mathfrak{k}} \nabla f^i_{\xi}\, dt^i.$$

In particular, $\alpha = -\pi_{\mathfrak{k}} \nabla f^i_{\xi}\, dt^i$ is the pull-back by k of the Maurer–Cartan form on K and so satisfies the Maurer–Cartan equations.

The point now is that when $\mathfrak{k}$ is an appropriate loop algebra and the f^i are suitably chosen, the 1-form α is of precisely the right form to produce a harmonic map. Moreover, the map k will be an "extended solution" as described in the contribution of Guest–Ohnita to this volume [27].

2.2 Passage to infinite dimension

Our intention is to carry out this programme for a family of twisted loop algebras. We therefore begin by considering the adjustments that must be made when carrying the Adler–Kostant–Symes scheme over to an infinite-dimensional setting. So let $\mathcal{G}$ be a reflexive Banach Lie algebra and suppose we have

(*i*) a decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{B}$ into *closed* subalgebras;

(*ii*) a continuous *ad*-invariant symmetric bilinear form $(\ ,\ )$ which is non-degenerate in the *weak* sense that the induced map $i : \mathcal{G} \to \mathcal{G}^*$ is injective;

(*iii*) $\mathcal{K}$, $\mathcal{B}$ are both isotropic for $(\ ,\ )$.

It follows that $\mathcal{B}$ is reflexive so that we may canonically identify $\mathcal{B}$ with $\mathcal{B}^{**}$ and so obtain a Poisson structure on $\mathcal{B}^*$ as before. By virtue of the isotropy of $\mathcal{K}$, the injection $\mathcal{G} \to \mathcal{G}^*$ restricts to give an injection $\mathcal{K} \to \mathcal{B}^*$ but this need no longer be an isomorphism and so we cannot transfer the Poisson structure of $\mathcal{B}^*$ to $\mathcal{K}$.

However, suppose that $V \subset \mathcal{K}$ is a linear subspace satisfying

1. $\pi_{\mathcal{K}}[V, \mathcal{B}] \subset V$;
2. $i(V)$ is closed and has a closed complement in $\mathcal{B}^*$ (trivially true if V is finite-dimensional).

In this case, V has a Poisson structure for which $i : V \to \mathcal{B}^*$ is a Poisson morphism. Indeed, by virtue of the second hypothesis, $i^* : C^\infty(\mathcal{B}^*) \to C^\infty(V)$ is a surjection so it suffices to show that the prescription

$$\{f \circ i, g \circ i\}_V = \{f, g\}_{\mathcal{B}^*} \circ i$$

is well-defined. This, in turn, follows from the first hypothesis as in Proposition 1.1.

We can now proceed as before with the *caveat* that not all functions on $\mathcal{G}$ have gradients with respect to $(\ ,\ )$. This prompts the following terminology:

Definition $F \in C^\infty(\mathcal{G})$ is *admissible* if it has a gradient with respect to $(\ ,\ )$.

For admissible functions F, $G \in C^\infty(\mathcal{G})$, we can compute the Poisson bracket of their restrictions to V as before:

$$\{F, G\}(\xi) = (\xi, [\pi_{\mathcal{B}} \nabla F_\xi, \pi_{\mathcal{B}} \nabla G_\xi]),$$

for $\xi \in V$.

Finally, an admissible function $F \in C^\infty(\mathcal{G})$ is *invariant* if

$$[\xi, \nabla F_\xi] = 0,$$

for all $\xi \in \mathcal{G}$. For such functions, the analysis of the previous section goes through as before and is summarised in the following theorem.

Theorem 2.2 *Let F and G be invariant, admissible functions on $\mathcal{G}$. Then their restrictions to V Poisson commute.*

Moreover, the Hamiltonian vector field corresponding to F is given by

$$X_F(\xi) = [\pi_{\mathcal{K}} \nabla F_\xi, \xi],$$

for $\xi \in V$.

2.3 Twisted loop algebras

To introduce the loop algebras to which these ideas will be applied, we begin by fixing the following ingredients:

1. A compact semisimple Lie algebra $\mathfrak{g}$.
2. An automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ of finite order k with fixed set $\mathfrak{k}$.
3. The primitive k-th root of unity $\omega = e^{2\pi i/k}$.

In addition, we fix an Iwasawa decomposition of $\mathfrak{k}^{\mathbb{C}}$:

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{b},$$

where $\mathfrak{b}$ is a solvable subalgebra of $\mathfrak{k}^{\mathbb{C}}$. Such a decomposition exists since $\mathfrak{k}$ is compact so that $\mathfrak{k}^{\mathbb{C}}$ is reductive.

Define a loop algebra by

$$\Lambda\mathfrak{g}^{\mathbb{C}}_\tau = \{\xi : S^1 \to \mathfrak{g}^{\mathbb{C}} : \xi(\omega\lambda) = \tau\,\xi(\lambda) \text{ for all } \lambda \in S^1\}$$

and equip it with the Sobolev H^r-topology for some $r > \frac{1}{2}$. Thus $\Lambda\mathfrak{g}^{\mathbb{C}}_\tau$ is a Banach (indeed, Hilbert) Lie algebra under point-wise bracket. There is an analogue of the Iwasawa decomposition for $\Lambda\mathfrak{g}^{\mathbb{C}}_\tau$: let $\Lambda\mathfrak{g}_\tau$ be the real form

$$\Lambda\mathfrak{g}_\tau = \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau : \xi : S^1 \to \mathfrak{g}\}$$

and define a complementary subalgebra by

$$\Lambda_+\mathfrak{g}^{\mathbb{C}}_\tau = \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau : \xi \text{ extends holomorphically to } \xi : D \to \mathfrak{g}^{\mathbb{C}} \text{ and } \xi(0) \in \mathfrak{b}\},$$

where D is the disc $\{|\lambda| < 1\}$.

A loop $\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau$ has a Fourier decomposition:

$$\xi = \sum_{n\in\mathbb{Z}} \xi_{-n}\lambda^n$$

with each $\xi_n \in \mathfrak{g}^{\mathbb{C}}$ satisfying $\tau\xi_n = \omega^{-n}\xi_n$. (Our convention of labelling the coefficient of λ^n by $-n$ is to prevent a welter of negative indices in subsequent formulae.) In terms of this decomposition, the conjugation on $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ across the real form $\Lambda\mathfrak{g}_\tau$ is given by

$$\overline{\xi} = \sum_{n\in\mathbb{Z}} \overline{\xi_{-n}}\lambda^{-n}$$

where the conjugation in $\mathfrak{g}^{\mathbb{C}}$ is with respect to the real form $\mathfrak{g}$. Moreover, our subalgebras are given by

$$\begin{aligned} \Lambda\mathfrak{g}_\tau &= \{\xi \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}} : \overline{\xi_n} = \xi_{-n}\}; \\ \Lambda_+\mathfrak{g}_\tau^{\mathbb{C}} &= \{\xi \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}} : \xi_n = 0 \text{ for } n > 0;\ \xi_0 \in \mathfrak{b}\}. \end{aligned}$$

From this it is clear that

$$\Lambda\mathfrak{g}_\tau^{\mathbb{C}} = \Lambda\mathfrak{g}_\tau \oplus \Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$$

is a vector space decomposition into closed subalgebras.

Now let κ be the Killing form of $\mathfrak{g}^{\mathbb{C}}$ and introduce a (weakly) non-degenerate invariant symmetric bilinear form $(\ ,\ )$ on $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ by taking the imaginary part of the L^2 inner product:

$$(\xi, \eta) = Im \int_{S^1} \kappa(\xi, \eta).$$

Observe that both $\Lambda\mathfrak{g}_\tau$ and $\Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$ are isotropic for this form so that we are in the situation of the previous discussion.

Let $\pi_{\mathcal{K}}$, $\pi_{\mathcal{B}}$ be the projections onto $\Lambda\mathfrak{g}_\tau$, $\Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$ along $\Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$, $\Lambda\mathfrak{g}_\tau$ respectively. To calculate these, write $\xi \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ as

$$\xi = \xi^+ + \xi^0 + \xi^-,$$

where

$$\xi^+ = \sum_{n>0} \xi_{-n}\lambda^n; \qquad \xi^0 = \xi_0 \in \mathfrak{k}^{\mathbb{C}}; \qquad \xi^- = \sum_{n<0} \xi_{-n}\lambda^n.$$

Further, write $\xi^0 = \xi^0_{\mathfrak{k}} + \xi^0_{\mathfrak{b}}$ for the Iwasawa decomposition of ξ^0 into $\mathfrak{k}$ and $\mathfrak{b}$ parts. Then

$$\xi = (\xi^- + \xi^0_{\mathfrak{k}} + \overline{\xi^-}) + (\xi^+ - \overline{\xi^-} + \xi^0_{\mathfrak{b}})$$

and the first summand is in $\Lambda\mathfrak{g}_\tau$ while the second is in $\Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$. In particular, we have

$$\pi_{\mathcal{K}}\xi = \xi^- + \xi^0_{\mathfrak{k}} + \overline{\xi^-}. \tag{2.3}$$

Recall that finite-dimensional subspaces $V \subset \Lambda\mathfrak{g}_\tau$ satisfying

$$\pi_{\mathcal{K}}[V, \Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}] \subset V$$

inherit a Poisson structure from $(\Lambda_+\mathfrak{g}_\tau^{\mathbb{C}})^*$. Our main examples of such subspaces are the spaces of polynomial loops given by

$$\Lambda_d = \{\xi \in \Lambda\mathfrak{g}_\tau : \xi_n = 0 \text{ for all } |n| > d\},$$

where $d \in \mathbb{N}$. Indeed, if $\xi \in \Lambda_d$ and $\eta \in \Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$ then $\lambda^d\xi$ extends holomorphically to D so that $[\lambda^d\xi, \eta]$ does also. This means that $[\xi, \eta] = \lambda^{-d}[\lambda^d\xi, \eta]$ has a pole of order at most d at $0 \in D$ so that

$$\pi_{\mathcal{K}}[\xi, \eta] = [\xi, \eta]^- + [\xi, \eta]^0_{\mathfrak{k}} + \overline{[\xi, \eta]^-} \in \Lambda_d,$$

as required. We have therefore proved:

Proposition 2.3 *For $d \in \mathbb{N}$, Λ_d is a Poisson manifold on which the restrictions of invariant admissible functions on $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ Poisson commute.*

2.4 Zero-curvature equations

We now come to our main construction: for $d \in \mathbb{N}$, we will integrate a pair of commuting Hamiltonian flows on the finite-dimensional Poisson manifold Λ_d and, from that data, construct a $\Lambda\mathfrak{g}_\tau$-valued 1-form which solves the Maurer–Cartan equations. In Section 3, we shall see how such 1-forms are related to harmonic maps.

Fix $d \in \mathbb{N}$ with $d \equiv 1 \bmod k$. Define f^1, $f^2 \in C^\infty(\Lambda\mathfrak{g}_\tau^{\mathbb{C}})$ by

$$f^1(\xi) = -\tfrac{1}{2}\, Im \int_{S^1} \lambda^{d-1}\kappa(\xi, \xi), \qquad f^2(\xi) = -\tfrac{1}{2}\, Re \int_{S^1} \lambda^{d-1}\kappa(\xi, \xi).$$

It is easy to see that the f^i are invariant admissible functions with gradients given by

$$\nabla f^1_\xi = -\lambda^{d-1}\xi, \qquad \nabla f^2_\xi = -i\lambda^{d-1}\xi,$$

(here we have used $d - 1 \equiv 0 \bmod k$ to ensure that the ∇f^i so defined take values in $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$).

We restrict f^1, f^2 to Λ_d and let X_1, X_2 denote the corresponding Hamiltonian vector fields. Thus

$$X_1(\xi) = [\xi, \pi_{\mathcal{K}}\lambda^{d-1}\xi]; \qquad X_2(\xi) = [\xi, \pi_{\mathcal{K}} i\lambda^{d-1}\xi],$$

for $\xi \in \Lambda_d$. That these vector fields are complete is a consequence of two observations: firstly, the L^2 inner product

$$\langle \xi, \eta \rangle = \int_{S^1} \kappa(\xi, \eta).$$

is negative definite on $\Lambda\mathfrak{g}_\tau$ (and thus on Λ_d). Secondly, the X_i are of Lax form, i.e., of the form

$$X(\xi) = [\xi, A(\xi)].$$

For any integral curve ξ of such an X we have

$$\begin{aligned} d/dt|\xi|^2 &= 2\langle\dot{\xi}, \xi\rangle \\ &= 2\langle[\xi, A(\xi)], \xi\rangle = -2\langle A(\xi), [\xi, \xi]\rangle = 0 \end{aligned}$$

so that ξ takes values in a sphere in Λ_d whence X is complete.

Let X_1^s, X_2^t be the flows corresponding to the X_i. Since f^1 and f^2 Poisson commute, the commutator $[X_1, X_2]$ vanishes so that

$$X_1^s \circ X_2^t = X_2^t \circ X_1^s,$$

for all s, $t \in \mathbb{R}$. Now fix an initial condition $\xi_o \in \Lambda_d$ and define $\xi : \mathbb{R}^2 \to \Lambda_d$ by

$$\xi(s,t) = X_1^s \circ X_2^t(\xi_o).$$

Then ξ satisfies

$$d\xi = X_1(\xi)\, ds + X_2(\xi)\, dt; \qquad \xi(0) = \xi_o. \tag{2.4}$$

Our interest in such maps comes from the possibility of constructing loops of flat connections from them:

Theorem 2.4 *Let $\xi : \mathbb{R}^2 \to \Lambda_d$ be the solution of (2.4) and define a $\Lambda\mathfrak{g}_\tau$-valued 1-form α on $\mathbb{R}^2$ by*

$$\alpha = -\pi_{\mathcal{K}}(\nabla f_\xi^1\, ds + \nabla f_\xi^2\, dt).$$

Then α solves the Maurer–Cartan equations:

$$d\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0.$$

This theorem can be proved by direct calculation or by recourse to the method of Symes. The first approach is an exercise for the Reader while the second will be carried out in Section 4.

We have thus obtained solutions to the Maurer–Cartan equations by integrating Hamiltonian ordinary differential equations on some Λ_d. Our application to harmonic maps will depend on the precise algebraic form of the solutions so obtained and it is to this that we now turn. We have

$$\alpha = -\pi_{\mathcal{K}}(\nabla f_\xi^1\, ds + \nabla f_\xi^2\, dt) = \pi_{\mathcal{K}}(\lambda^{d-1}\xi\, ds + i\lambda^{d-1}\xi\, dt).$$

This simplifies if we introduce the complex co-ordinate $z = s + it$ on $\mathbb{R}^2$:

$$\alpha = \pi_{\mathcal{K}}(\lambda^{d-1}\xi\, dz).$$

We now use (2.3), together with the fact that ξ takes values in Λ_d, to get

$$\begin{aligned} \alpha &= (\lambda^{d-1}\xi)^-\, dz + (\lambda^{d-1}\xi\, dz)_{\mathfrak{k}}^0 + \overline{(\lambda^{d-1}\xi)^-}\, d\bar{z} \\ &= \lambda^{-1}\xi_d\, dz + (\xi_{d-1}\, dz)_{\mathfrak{k}} + \lambda\xi_{-d}\, d\bar{z}. \end{aligned}$$

To compute the λ-independent term, we must understand the Iwasawa decomposition of $\mathfrak{k}^{\mathbb{C}}$: let $\mathfrak{t}$ be the given maximal torus in $\mathfrak{k}$, $\mathfrak{n}$ the nilpotent subalgebra given by the positive root spaces and set $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$. Then we have

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{n} \oplus \mathfrak{h} \oplus \overline{\mathfrak{n}}, \qquad \mathfrak{b} = (i\mathfrak{t}) \oplus \mathfrak{n}.$$

Corresponding to this decomposition of $\mathfrak{k}^{\mathbb{C}}$, write $\eta \in \mathfrak{k}^{\mathbb{C}}$ as

$$\eta = \eta_{\mathfrak{n}} + \eta_{\mathfrak{h}} + \eta_{\overline{\mathfrak{n}}}$$

It is then easy to check that

$$(\eta\, dz)_{\mathfrak{k}} = (\eta_{\mathfrak{n}} + \tfrac{1}{2}\eta_{\mathfrak{h}})\, dz + \overline{(\eta_{\mathfrak{n}} + \tfrac{1}{2}\eta_{\mathfrak{h}})}\, d\overline{z}. \tag{2.5}$$

This suggests that we define $r : \mathfrak{k}^{\mathbb{C}} \to \mathfrak{k}^{\mathbb{C}}$ by

$$r(\eta) = \eta_{\mathfrak{n}} + \tfrac{1}{2}\eta_{\mathfrak{h}}$$

Then, denoting the $(1,0)$-part of α by α', we conclude that

$$\alpha' = (\lambda^{-1}\xi_d + r(\xi_{d-1}))\, dz. \tag{2.6}$$

We can now summarise our discussion in the following theorem:

Theorem 2.5 *For each $d \equiv 1 \bmod k$ and $\xi_o \in \Lambda_d$, there is a unique solution $\xi : \mathbb{R}^2 \to \Lambda_d$ to*

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda^{-1}\xi_d + r(\xi_{d-1})]; \qquad \xi(0) = \xi_o \tag{2.7}$$

and then the $\Lambda\mathfrak{g}_\tau$-valued 1-form α given by

$$\alpha = (\lambda^{-1}\xi_d + r(\xi_{d-1}))\, dz + (\lambda\xi_{-d} + \overline{r(\xi_{d-1})})\, d\overline{z} \tag{2.8}$$

satisfies the Maurer–Cartan equations.

Remark When $\mathfrak{g}$ has rank greater then one, we may generalise these methods to produce Maurer–Cartan solutions on $\mathbb{C}^n$ by simultaneous integration of $2n$ commuting flows. Briefly, the idea is to consider *ad*-invariant polynomials $P : \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ and define functions f^1, f^2 on $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ by

$$f^1(\xi) = Im \int_{S^1} \lambda^m P(\xi); \qquad f^2(\xi) = Re \int_{S^1} \lambda^m P(\xi)$$

which are easily seen to be admissible and invariant. Any collection of such functions, with P ranging over *ad*-invariant polynomials, will Poisson commute and so give rise to a solution to the Maurer–Cartan equations. By choosing the exponents m appropriately, one obtains solutions α with the same kind of λ-dependence as in (2.6). From such α one can construct *pluriharmonic* maps of $\mathbb{C}^n$.

We shall not pursue this topic further in this chapter but, instead, refer the Reader to the discussion in [14].

2.5 The r-matrix approach

Elsewhere in this volume (and in the papers [24, 14]), a different approach to the construction of commuting flows on loop algebras is employed, to wit: the r-matrix formalism of Reyman–Semenov-Tian-Shansky [48]. In this section, we take a break from our main development to briefly review this method and show how it gives the same Poisson structures on $\Lambda\mathfrak{g}_\tau$ as the Adler–Kostant–Symes scheme. For brevity of exposition, we shall ignore questions of completeness of metrics, admissibility of functions and so on.

We begin with a (possibly infinite-dimensional) Lie algebra $\mathcal{G}$. A linear map $R : \mathcal{G} \to \mathcal{G}$ is called an *r-matrix* if the bracket defined by

$$[\xi, \eta]_R = [R\xi, \eta] + [\xi, R\eta]$$

satisfies the Jacobi identity. This is easily seen to be the case if R solves the (modified) classical Yang–Baxter equations:

$$R[\xi, \eta]_R - [R\xi, R\eta] = a[\xi, \eta], \tag{2.9}$$

for some fixed $a \in \mathbb{C}$. Given an r-matrix, $\mathcal{G}$ acquires a second Lie algebra structure so that the construction which opens Section 1.1 equips $\mathcal{G}^*$ with a second Poisson structure.

Suppose also that $\mathcal{G}$ is equipped with an ad-invariant inner product $\langle\, , \rangle$. We use this to transfer the new Poisson structure from $\mathcal{G}^*$ to $\mathcal{G}$ and thus arrive at a Poisson structure on $\mathcal{G}$ with bracket:

$$\{f, g\}_R(\xi) = \langle \xi, [grad\, f_\xi, grad\, g_\xi]_R \rangle, \tag{2.10}$$

where $grad$ denotes the gradient with respect to $\langle\, , \rangle$.

Such Poisson structures enjoy many of the properties of those constructed by the Adler–Kostant–Symes scheme. In particular, ad-invariant functions on $\mathcal{G}$ Poisson commute and give rise to solutions of the Maurer–Cartan equations.

Let us show that both these constructions coincide in our setting: define a linear map $R : \Lambda\mathfrak{g}_\tau \to \Lambda\mathfrak{g}_\tau$ by

$$R\xi = i\xi^+ + R_0\xi^0 - i\xi^-.$$

Here $R_0 : \mathfrak{k} \to \mathfrak{k}$ is given by

$$R_0\eta = i\eta_{\mathfrak{n}} - i\eta_{\overline{\mathfrak{n}}}.$$

One observes that the $\pm i$-eigenspaces of R are subalgebras of $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ while the 0-eigenspace is the abelian subalgebra $\mathfrak{h}$. From this, it is easy to check that R solves the modified classical Yang–Baxter equations (2.9) with $a = -1$ and so is an r-matrix.

Recall that $\Lambda\mathfrak{g}_\tau$ carries the ad-invariant, negative definite inner product,

$$\langle \xi, \eta \rangle = \int_{S^1} \kappa(\xi, \eta)$$

so that R induces a Poisson structure on $\Lambda\mathfrak{g}_\tau$.

We claim that this Poisson structure is the same as the one described in Section 2.3. To see this, it suffices to compare Hamiltonian vector fields. So let f be some function on $\Lambda\mathfrak{g}_\tau$ and recall that its Hamiltonian vector field with respect to the Adler–Kostant–Symes Poisson structure is given by

$$\begin{aligned}\pi_{\mathcal{K}}[\xi, \pi_{\mathcal{B}}\nabla f_\xi] &= [\xi, \pi_{\mathcal{B}}\nabla f_\xi] - \pi_{\mathcal{B}}[\xi, \pi_{\mathcal{B}}\nabla f_\xi] \\ &= [\xi, \pi_{\mathcal{B}}\nabla f_\xi] - \pi_{\mathcal{B}}[\xi, \nabla f_\xi],\end{aligned}$$

for $\xi \in \Lambda\mathfrak{g}_\tau$. Here the last equality is due to the fact that $\pi_{\mathcal{B}}[\xi, \pi_{\mathcal{K}}\nabla f_\xi]$ vanishes since $\Lambda\mathfrak{g}_\tau$ is a subalgebra.

On the other hand, we note that R is skew-symmetric for $\langle\,,\rangle$ and use this, together with (2.10), to see that the Hamiltonian vector field of f with respect to the R-Poisson structure is given by

$$[\xi, R\, grad\, f_\xi] - R[\xi, grad\, f_\xi] = [\xi, (R+i)\, grad\, f_\xi] - (R+i)[\xi, grad\, f_\xi].$$

That these vector fields are the same is an immediate consequence of two easily verified facts. Firstly, the relation between the two gradients is given by

$$grad\, f_\xi = \tfrac{1}{2i}(\nabla f_\xi - \overline{\nabla f_\xi}),$$

for $\xi \in \Lambda\mathfrak{g}_\tau$. Secondly, one deduces from (2.3) and (2.5) that

$$\pi_{\mathcal{B}}\xi = \tfrac{1}{2i}(R+i)(\xi - \overline{\xi}),$$

for $\xi \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$. From this it is clear that, for $\xi \in \Lambda\mathfrak{g}_\tau$,

$$[\xi, \pi_{\mathcal{B}}\nabla f_\xi] - \pi_{\mathcal{B}}[\xi, \nabla f_\xi] = [\xi, (R+i)\, grad\, f_\xi] - (R+i)[\xi, grad\, f_\xi]$$

so that the Poisson structures coincide.

Example Let $\mathfrak{g} = \mathfrak{so}(5)$ and let $\tau : \mathfrak{g} \to \mathfrak{g}$ be conjugation by

$$\begin{pmatrix} 1 & & & & \\ & & 1 & & \\ & -1 & & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}.$$

Then τ is an automorphism of order 4 with fixed set a maximal torus $\mathfrak{t}$ of $\mathfrak{so}(5)$. The Iwasawa decomposition of $\mathfrak{t}^{\mathbb{C}}$ is $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}$ with no $\mathfrak{n}$-part so that the r-matrix in this case is given by

$$R\xi = i\xi^+ - i\xi^-.$$

This is (up to a factor of $\frac{1}{2}$) the r-matrix used in [24] to construct commuting flows from which all minimal non-superminimal 2-tori are obtained.

Remark In fact the r-matrix formalism constitutes a strict generalisation of the Adler–Kostant–Symes scheme. In the notation of Section 1.1, one may define an r-matrix on $\mathfrak{g}$ by setting $R = \frac{1}{2}(\pi_{\mathfrak{b}} - \pi_{\mathfrak{k}})$ and then check that $\mathfrak{k}^{\circ}$ is a Poisson submanifold of $\mathfrak{g}$ with respect to the Poisson structure obtained from R. Moreover, the induced Poisson structure on $\mathfrak{k}^{\circ}$ coincides with that provided by the Adler–Kostant–Symes scheme (c.f., [40]).

However, when it can be applied, the Adler–Kostant–Symes scheme has advantages. In particular, the symplectic leaves are known to be the B-orbits on $\mathfrak{k}^{\circ}$ and so Poisson submanifolds are more readily identified.

3 Harmonic and primitive maps

We now turn from symplectic geometry to Riemannian geometry and apply the theory of the preceding sections to the construction of harmonic maps from $\mathbb{R}^2$ into certain homogeneous spaces. We begin by describing the class of such spaces with which we shall be concerned.

3.1 Symmetric and k-symmetric spaces

Recall the data of Section 2.3: a compact semisimple Lie algebra $\mathfrak{g}$; an automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ of order k with fixed set $\mathfrak{k}$; the primitive k-th root of unity $\omega = e^{2\pi i/k}$. We have an eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i \in \mathbb{Z}_k} \mathfrak{g}_i$$

where $\mathfrak{g}_i$ is the ω^i-eigenspace of τ. Clearly, $\mathfrak{g}_0 = \mathfrak{k}^{\mathbb{C}}$, $\overline{\mathfrak{g}_i} = \mathfrak{g}_{-i}$ and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

for all $i, j \in \mathbb{Z}_k$ (here, of course, all arithmetic is modulo k). In particular, define $\mathfrak{m} \subset \mathfrak{g}$ by

$$\mathfrak{m}^{\mathbb{C}} = \sum_{i \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_i .$$

Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{3.1}$$

and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ so that (3.1) is a reductive decomposition. Moreover, when $k = 2$, $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}_1$ so that

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$$

and, in this case, (3.1) is a symmetric decomposition of $\mathfrak{g}$.

Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$ and suppose that τ exponentiates to give an order k automorphism, also called τ, of G. Further let $(G^\tau)_0 \subset K \subset G^\tau$ so that K has Lie algebra $\mathfrak{k}$.

Consider the coset space $N = G/K$ with base-point $o = eK$. Define $\hat{\tau} : N \to N$ by

$$\hat{\tau}(g \cdot o) = \tau(g) \cdot o,$$

for $g \in G$. Similarly, for $x = g \cdot o \in N$, define $\hat{\tau}_x : N \to N$ by

$$\hat{\tau}_x = g \circ \hat{\tau} \circ g^{-1}.$$

Then each $\hat{\tau}_x$ is a diffeomorphism or order k having x as an isolated fixed point. Moreover, we may use the Killing form of $\mathfrak{g}$ to equip N with a metric for which each of the $\hat{\tau}_x$ is an isometry so that N has the structure of a (regular) k-symmetric space in the sense of [37]. Of course, the 2-symmetric spaces are just the familiar Riemannian symmetric spaces of compact type.

The algebraic situation in $\mathfrak{g}$ transfers to the reductive homogeneous space N as follows: for $x = g \cdot o \in N$, the map $\mathfrak{g} \to T_x N$ given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot x$$

restricts to an isomorphism $Ad\, g\, \mathfrak{m} \to T_x N$. The inverse map $\beta_x : T_x N \to Ad\, g\, \mathfrak{m} \subset \mathfrak{g}$ may be viewed as a $\mathfrak{g}$-valued 1-form β on N which, after [15], we call the *Maurer–Cartan form* of N.

This prompts in part the following notation: if $\mathfrak{l} \subset \mathfrak{g}^{\mathbb{C}}$ is an $Ad\, K$-invariant subspace, let $[\mathfrak{l}]$ denote the subbundle of the trivial bundle $\underline{\mathfrak{g}}^{\mathbb{C}} = N \times \mathfrak{g}^{\mathbb{C}}$ defined by

$$[\mathfrak{l}]_{g \cdot o} = Ad\, g\, \mathfrak{l}.$$

In particular, $\beta : TN \to [\mathfrak{m}]$ is a bundle isomorphism while $[\mathfrak{k}]_x$ is the Lie algebra of the stabiliser of $x \in N$. Moreover, the ω^i-eigenspace of $d\hat{\tau}_x$ at $x \in N$ corresponds to $[\mathfrak{g}_i]_x$ under the isomorphism β_x.

3.2 Harmonic maps

A map $\phi : M \to N$ of Riemannian manifolds is *harmonic* if it extremizes the energy functional

$$E(\phi) = \int_D |d\phi|^2 \, dvol \tag{3.2}$$

on all compact sub-domains $D \subset M$ (for an introduction to harmonic maps in this volume, see [59]). In case that the target is a reductive homogeneous space $N = G/K$, the corresponding Euler–Lagrange equations have a particularly simple form. Indeed, if such an N has metric induced from an invariant metric on $\mathfrak{g}$, we may use these metrics to identify TN with T^*N and $\mathfrak{g}^*$ with $\mathfrak{g}$. In this way, TN becomes a symplectic manifold on which G acts symplectically and the moment map for this action is precisely the Maurer–Cartan form $\beta : TN \to \mathfrak{g} \cong \mathfrak{g}^*$. A Noether analysis of the energy functional (3.2) now gives [44] (c.f. also, [59]):

Lemma 3.1 *ϕ is harmonic if and only if the pull-back of the Maurer–Cartan form is co-closed:*

$$d^*\phi^*\beta = 0.$$

We can gain a different perspective on the harmonic map equations by lifting everything to the frame bundle. Let $\pi : G \to N = G/K$ be the coset projection and $\phi : M \to N$ be a map. A *framing of* ϕ is a map $\Phi : M \to G$ such that $\pi \circ \Phi = \phi$ so that the following diagram commutes:

$$\begin{tikzcd} & G \arrow[d, "\pi"] \\ M \arrow[ur, "\Phi"] \arrow[r, "\phi"'] & N \end{tikzcd}$$

Such lifts always exist locally.

So let $\phi : M \to N$ be a map with framing Φ and set $\alpha = \Phi^{-1} d\Phi$. Corresponding to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a decomposition of α,

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$$

and one may easily verify that

$$\phi^*\beta = Ad\,\Phi\,\alpha_{\mathfrak{m}}$$

We can now express the harmonic map equations in terms of α:

$$\begin{aligned} d * \phi^*\beta &= d(Ad\,\Phi * \alpha_{\mathfrak{m}}) \\ &= Ad\,\Phi\{d * \alpha_{\mathfrak{m}} + [\alpha \wedge *\alpha_{\mathfrak{m}}]_{\mathfrak{m}}\}, \end{aligned}$$

where we have used $d(Ad\,\Phi) = Ad\,\Phi \circ ad\,\alpha$. Thus ϕ if harmonic if and only if

$$d * \alpha_{\mathfrak{m}} + [\alpha \wedge *\alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0. \tag{3.3}$$

Henceforth, we will restrict attention to the case where M is two-dimensional. Then the energy is conformally invariant so that we may take M to be a Riemann surface. We now have a type decomposition $\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$ where $\alpha'_{\mathfrak{m}}$ is an $\mathfrak{m}^{\mathbb{C}}$-valued $(1,0)$-form with complex conjugate $\alpha''_{\mathfrak{m}}$. Then

$$*\alpha_{\mathfrak{m}} = -i\alpha'_{\mathfrak{m}} + i\alpha''_{\mathfrak{m}}$$

so that (3.3) becomes

$$\begin{aligned} 0 &= -(d\alpha'_{\mathfrak{m}} + [\alpha \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}}) + (d\alpha''_{\mathfrak{m}} + [\alpha \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}) \\ &= -(d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}} + [\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}}) + (d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}) \\ &= -(d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]_{\mathfrak{m}}) + (d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}). \end{aligned} \tag{3.4}$$

On the other hand, we have the Maurer–Cartan equations for α whose $\mathfrak{m}$- and $\mathfrak{k}$-parts read

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] + d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \tag{3.5}$$

$$d\alpha_{\mathfrak{k}} + \tfrac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{k}} = 0. \tag{3.6}$$

In particular, (3.4) and (3.5) are equivalent to

$$d\alpha'_{\mathfrak{m}} + [\alpha_k \wedge \alpha'_{\mathfrak{m}}] = -\tfrac{1}{2}[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}$$

Suppose now that $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}$ vanishes (certainly true when N is a Riemannian symmetric space since then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$). The harmonic map equations combine with the Maurer–Cartan equations to give

$$d\alpha'_{\mathfrak{m}} + [\alpha_k \wedge \alpha'_{\mathfrak{m}}] = d\alpha''_{\mathfrak{m}} + [\alpha_k \wedge \alpha''_{\mathfrak{m}}] = 0 \tag{3.7}$$

$$d\alpha_{\mathfrak{k}} + \tfrac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0. \tag{3.8}$$

Remark These last equations have a gauge-theoretic formulation: let A denote the connection $d + \alpha_{\mathfrak{k}}$ with curvature F_A and (temporarily) denote $\alpha'_{\mathfrak{m}}$ by Ψ. Then our equations read

$$\begin{aligned} d^A\Psi &= 0 \\ F_A &= -[\Psi \wedge \overline{\Psi}] \end{aligned}$$

which are the Yang–Mills–Higgs equations for the connection A and the Higgs field Ψ. (See Hitchin [30] for a far-reaching study of these equations when N is the *non-compact* symmetric space $G^{\mathbb{C}}/G$.)

The equations (3.7) and (3.8) are invariant under an S^1-action: for $\lambda \in S^1$ and α a $\mathfrak{g}$-valued 1-form, set

$$\lambda \cdot \alpha = \alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}. \tag{3.9}$$

This is clearly an action of S^1 on $\mathfrak{g}$-valued 1-forms which preserves the solution set of (3.7) and (3.8). In fact, more is true: compare coefficients of λ to see that (3.7) and (3.8) hold for α precisely when

$$d\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in S^1$. Otherwise said, we have written the harmonic map equations as a loop of zero-curvature equations (c.f. [42, 55, 64, 65]).

Conversely, suppose that M is simply connected and let α_λ be a loop of 1-forms of the form (3.9) such that

1. $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0$;
2. $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$, for all $\lambda \in S^1$.

Then, by Theorem 2.1, for each $\lambda \in S^1$ we can find a map $\Phi_\lambda : M \to G$ such that

$$\Phi_\lambda^{-1} d\Phi_\lambda = \alpha_\lambda$$

and, since α_λ satisfies (3.7) and (3.8), we conclude that each $\phi_\lambda = \pi \circ \Phi_\lambda : M \to N$ is harmonic. In particular, harmonic maps of this kind come in S^1-families.

This analysis applies to harmonic maps into any reductive homogeneous space N (with metric induced by one on $\mathfrak{g}$) which satisfy the auxiliary condition

$$[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0$$

(which is the same as demanding that $[\phi^*\beta' \wedge \phi^*\beta'']$ be $[\mathfrak{k}]$-valued). This last condition is trivially satisfied when N is a 2-symmetric space and then our discussion is valid for all harmonic maps.

So let us consider the case when N is a Riemannian symmetric space and show how the results of Section 2 provide harmonic maps of $\mathbb{R}^2 \to N$. For this, let τ be the involution determining N so that

$$\mathfrak{g}_0 = \mathfrak{k}^{\mathbb{C}}; \qquad \mathfrak{g}_1 = \mathfrak{m}^{\mathbb{C}}$$

and recall the loop algebra $\Lambda\mathfrak{g}_\tau$. Consider the algebraic structure of α_λ in (3.9): the coefficient of λ^{-1} is a $\mathfrak{g}_1$-valued $(1,0)$-form, the constant (in λ) term is $\mathfrak{k}$-valued while the λ-coefficient is a $\mathfrak{g}_1$-valued $(0,1)$-form. In particular, α_λ may be viewed as a $\Lambda\mathfrak{g}_\tau$-valued 1-form. Moreover, we see from (2.8) that this is precisely the type of $\Lambda\mathfrak{g}_\tau$-valued 1-form that arises as the solution to the zero-curvature equations in Theorem 2.5 so that combining that theorem with our present discussion gives

Theorem 3.2 *For each $d \in 2\mathbb{N}+1$ and $\xi_o \in \Lambda_d$, there is a unique solution to*

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda^{-1}\xi_d + r(\xi_{d-1})]; \qquad \xi(0) = \xi_o$$

and then there is a harmonic map $\phi : \mathbb{R}^2 \to N$ with framing $\Phi : \mathbb{R}^2 \to G$ satisfying $\Phi^{-1}\partial\Phi/\partial z = \xi_d + r(\xi_{d-1})$.

We have therefore succeeded in constructing harmonic maps of $\mathbb{R}^2$ into Riemannian symmetric spaces N from our commuting Hamiltonian flows on the Poisson manifolds Λ_d. We call the maps so obtained *harmonic maps of finite type*.

When is a harmonic map $\mathbb{R}^2 \to N$ of finite type? The framing Φ of such a map has $\alpha'_{\mathfrak{m}} = \xi_d\, dz$ and this leads to necessary conditions. Indeed, comparing coefficients in the differential equation for ξ gives

$$d\xi_d = [\xi_d, (r(\xi_{d-1}) - \xi_{d-1})\, dz + \overline{r(\xi_{d-1})}\, d\overline{z}]$$

which takes values in $[\xi_d, \mathfrak{k}^{\mathbb{C}}]$. Otherwise said, $d\xi$ takes values in the tangent space at ξ_d to the $Ad\, K^{\mathbb{C}}$-orbit through ξ_d. We therefore deduce from the uniqueness of solutions to ODE:

Lemma 3.3 *$\xi_d : \mathbb{R}^2 \to \mathfrak{m}^{\mathbb{C}}$ (and hence $\alpha'_{\mathfrak{m}}(\partial/\partial z)$) takes values in a single $Ad\, K^{\mathbb{C}}$-orbit in $\mathfrak{m}^{\mathbb{C}}$.*

This has the effect of ensuring that certain polynomial invariants of a finite type harmonic map are constant: if $P : \mathfrak{m}^{\mathbb{C}} \to \mathbb{C}$ is an $Ad\,K^{\mathbb{C}}$-invariant polynomial of degree r then clearly $P(\alpha'_{\mathfrak{m}}(\partial/\partial z))$ is constant. On the other hand, P gives rise to a G-invariant, and hence parallel, section $\widehat{P}$ of $S^rT^*N^{\mathbb{C}} \cong S^r[\mathfrak{m}^{\mathbb{C}}]$ by

$$\widehat{P}_{g\cdot o} = P \circ Ad\,g^{-1}$$

so that we have

$$\phi^*\widehat{P}^{(r,0)} = P(\alpha'_{\mathfrak{m}}(\partial/\partial z))\,dz^r = c\,dz^r$$

for some constant c.

One important class of harmonic maps which satisfy such constraints is that of doubly periodic harmonic maps or, equivalently, lifts to the universal cover of harmonic 2-tori. Indeed, for any harmonic map, it is known that these polynomial invariants are holomorphic differentials: one may write the harmonic map equations as

$$\nabla^{\phi}_{\partial/\partial\bar{z}}\partial\phi/\partial z = 0,$$

where ∇^{ϕ} is the pull-back of the Levi-Civita connection on N (see [59]) and then, for any parallel field of polynomials $\widehat{P}$ we have

$$\frac{\partial}{\partial\bar{z}}\widehat{P}(\partial\phi/\partial z,\ldots,\partial\phi/\partial z) = r\widehat{P}(\nabla^{\phi}_{\partial/\partial\bar{z}}\partial\phi/\partial z,\partial\phi/\partial z,\ldots,\partial\phi/\partial z) = 0.$$

In particular, when ϕ is doubly periodic, each $\phi^*\widehat{P}^{(r,0)}$ is constant by Liouville's theorem.

Each common level set of the invariant polynomials on $\mathfrak{m}^{\mathbb{C}}$ comprises but a finite number of $Ad\,K^{\mathbb{C}}$-orbits so that harmonic 2-tori come close to satisfying the necessary condition to be of finite type provided by Lemma 3.3. In fact, in this case, that necessary condition is almost sufficient: we shall see in Section 5 that the following theorem is a consequence of the results of [14].

Theorem 3.4 *Let $\phi : \mathbb{R}^2 \to N$ be a doubly periodic harmonic map into a Riemannian symmetric space. Suppose that some (and hence every) framing of ϕ has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in a single* semisimple *$Ad\,K^{\mathbb{C}}$-orbit in $\mathfrak{m}^{\mathbb{C}}$. Then ϕ is of finite type.*

Here an orbit is semisimple if it consists of elements $\xi \in \mathfrak{m}^{\mathbb{C}}$ which are semisimple in the sense that $ad\,\xi$ is diagonalisable.

Let us conclude this section with an example which gives geometric meaning to the "semisimple orbit" hypothesis. For a rank 1 symmetric space, the semisimple orbits are just the non-zero level sets of the Killing form. The corresponding quadratic form on $TN^{\mathbb{C}}$ is the complexified metric h so that the constant holomorphic differential for a doubly periodic map ϕ is just the obstruction to conformality $\phi^*h^{(2,0)}$. We therefore conclude

Theorem 3.5 *A doubly periodic non-conformal harmonic map* $\phi : \mathbb{R}^2 \to N$ *into a rank* 1 *symmetric space is of finite type.*

While this result explicitly excludes the geometrically interesting case of minimal 2-tori, it is already strong enough to be very useful. For instance, the non-conformal harmonic 2-tori in S^2 are precisely the Gauss maps of constant mean curvature tori in $\mathbb{R}^3$ and, in this case, the theorem reproduces the basic result of Pinkall–Sterling [41].

For spheres and complex projective spaces, this result will be improved in Section 3.5.

3.3 Primitive maps

We now extend our theory to treat certain harmonic maps into k-symmetric spaces N where $k > 2$. Let us begin by introducing the class of harmonic maps that our constructions will produce.

So let $N = G/K$ be a k-symmetric space, $k > 2$, with automorphism τ and associated eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{i \in \mathbb{Z}_k} \mathfrak{g}_i.$$

In particular, $\mathfrak{g}_{-1} = \overline{\mathfrak{g}_1}$ and, since $k > 2$, $\mathfrak{g}_1 \cap \mathfrak{g}_{-1} = \{0\}$.

Definition A map $\phi : M \to N$ of a Riemann surface is *primitive* if $\phi^*\beta'$ takes values in $[\mathfrak{g}_{-1}]$. Equivalently, ϕ is primitive if and only if any framing Φ has $\alpha'_{\mathfrak{m}}$ taking values in $\mathfrak{g}_{-1}$.

Example Consider the case $k = 3$: here we have

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$$

so that N acquires an invariant almost complex structure with $T^{(1,0)}N \cong [\mathfrak{g}_{-1}]$. This is the non-integrable almost complex structure discussed by Salamon [49] who viewed 3-symmetric spaces as twistor spaces for Riemannian symmetric spaces (in this regard, see also [10, 15]). In this setting, primitive maps are just (almost) holomorphic maps.

In general, primitive maps are examples of maps which are f-holomorphic with respect to a horizontal f-structure in the sense of Black [4]. Under rather general conditions, such maps enjoy a number of interesting and useful properties which, in our setting, stem ultimately from the relation

$$[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0 = \mathfrak{k}^{\mathbb{C}}. \tag{3.10}$$

To be more precise, under the additional assumption that all irreducible subrepresentations of the adjoint representation of K on $\mathfrak{m}^{\mathbb{C}}$ occur with multiplicity one, Black proves:

1. All primitive maps are *equiharmonic*, that is, harmonic with respect to any invariant metric on N;

2. If $p : N = G/K \to G/H$ is any homogeneous projection and $\phi : M \to N$ is equiharmonic then so is $p \circ \phi$.

Even without such extra assumptions, restricted versions of these results are still available. For this, consider a framing of a primitive map with associated 1-form α. We know that $\alpha'_{\mathfrak{m}}$ takes values in $\mathfrak{g}_{-1}$ so that, in view of (3.10), we have

$$[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0.$$

The projections of the Maurer–Cartan equations for α onto $\mathfrak{g}_{-1}$, $\mathfrak{g}_1$ and $\mathfrak{g}_0$ therefore read

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = 0 \tag{3.11}$$

$$d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] = 0 \tag{3.12}$$

$$d\alpha_{\mathfrak{k}} + \tfrac{1}{2}[\alpha_k \wedge \alpha_k] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0. \tag{3.13}$$

However, these are just the harmonic maps equations (3.7) and (3.8) and we conclude:

Theorem 3.6 *A primitive map $\phi : M \to N$ is harmonic with respect to the metric on N induced by that on $\mathfrak{g}$.*

Primitive maps are also well-behaved with respect to projections: let G/H be a reductive homogeneous space with $K \subset H$ and reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

which we assume to be orthogonal and stable under τ. As usual, equip G/H with the metric induced by that on $\mathfrak{g}$ and let $p : N = G/K \to G/H$ be the homogeneous projection. We have

Theorem 3.7 *If $\phi : M \to N$ is primitive then $p \circ \phi : M \to G/H$ is harmonic.*

Proof. Let $\mathfrak{l}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{h}$ so that we have a τ-stable decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l} \oplus \mathfrak{p}$$

with

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{l}; \qquad \mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}.$$

Both $\mathfrak{l}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ decompose into eigenspaces of τ and we have various relations between these eigenspaces: $[\mathfrak{l}_1, \mathfrak{p}_{-1}] \subset \mathfrak{g}_0 = \mathfrak{k}^{\mathbb{C}}$ but $[\mathfrak{l}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}}] \subset [\mathfrak{h}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}}] \subset \mathfrak{p}^{\mathbb{C}}$ so that

$$[\mathfrak{l}_1, \mathfrak{p}_{-1}] = 0. \tag{3.14}$$

We also have $[\mathfrak{p}_1, \mathfrak{p}_{-1}] \subset \mathfrak{k}^{\mathbb{C}}$ *so that*

$$[\mathfrak{p}_1, \mathfrak{p}_{-1}]_{\mathfrak{p}} = 0. \tag{3.15}$$

Any framing for ϕ *is one for* $p \circ \phi$ *also so it suffices to prove that, for such a framing,*

$$[\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} = 0; \tag{3.16}$$
$$d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{p}}] = 0.$$

However, $\alpha'_{\mathfrak{m}} = \alpha'_{\mathfrak{l}} + \alpha'_{\mathfrak{p}}$ *takes values in* $\mathfrak{g}_{-1}$ *so that* $\alpha'_{\mathfrak{p}}$ *is* $\mathfrak{p}_{-1}$*-valued and (3.16) follows from (3.15). Finally,*

$$\begin{aligned} d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{p}}] &= d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] + [\alpha''_{\mathfrak{l}} \wedge \alpha'_{\mathfrak{p}}] \\ &= d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] \end{aligned}$$

in view of (3.14) and this last is just the $\mathfrak{p}$*-part of* $d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]$ *which vanishes by (3.11).* □

In particular, primitive maps may give rise, by projection, to harmonic maps into Riemannian symmetric spaces and it is from this possibility that our principal interest in them derives.

Consider now the loop (3.9) of 1-forms

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}},$$

where $\alpha = \Phi^{-1}d\Phi$ for a framing Φ of a primitive map ϕ. Then $\alpha'_{\mathfrak{m}}$ is $\mathfrak{g}_{-1}$-valued so that we may view α_λ as a $\Lambda\mathfrak{g}_\tau$-valued 1-form.

Since ϕ is harmonic and $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}}$ vanishes, the discussion in Section 3.2 applies so that, for each $\lambda \in S^1$,

$$d\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

Thus primitive maps also give rise to zero-curvature equations.

Conversely, suppose that M is simply connected and α_λ is a $\Lambda\mathfrak{g}_\tau$-valued 1-form of the form (3.9) such that

$$d\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0.$$

Then, by Theorem 2.1, for each $\lambda \in S^1$, we can find a map $\Phi_\lambda : M \to G$ such that

$$\Phi_\lambda^{-1} d\Phi_\lambda = \alpha_\lambda$$

and then $\phi_\lambda = \pi \circ \Phi_\lambda$ will be harmonic. In fact, more is true: since

$$(\alpha_\lambda)'_{\mathfrak{m}} = \lambda^{-1}\alpha'_{\mathfrak{m}}$$

takes values in $\mathfrak{g}_{-1}$, we conclude that each ϕ_λ is primitive. In particular, primitive maps of simply connected surfaces come in S^1-families.

Finally let us show how the results of Section 2 provide primitive maps $\mathbb{R}^2 \to N$. For this, observe that $\Lambda\mathfrak{g}_\tau$-valued 1-forms of the form (3.9) are precisely the kind of 1-form that arises as the solution to the zero-curvature equations in Theorem 2.5 so that we get

Theorem 3.8 *For each $d \equiv 1 \bmod k$ and $\xi_o \in \Lambda_d$, there is a unique solution to*

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda^{-1}\xi_d + r(\xi_{d-1})]; \qquad \xi(0) = \xi_o$$

and then there is a primitive map $\phi : \mathbb{R}^2 \to N$ with framing $\Phi : \mathbb{R}^2 \to G$ satisfying $\Phi^{-1}d\Phi(\partial/\partial z) = \xi_d + r(\xi_{d-1})$.

We call the primitive maps so obtained *primitive maps of finite type*.

A similar analysis to that in Section 3.2 can be carried out for primitive maps to give sufficient conditions for a primitive map to be of finite type. Just as before, such a map must have $\xi_d = \alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in a single $Ad\, K^{\mathbb{C}}$-orbit in $\mathfrak{g}_{-1}$ and, for doubly periodic maps, this condition is almost sufficient. Indeed, as we shall see in Section 5, the results of [11, 14] imply:

Theorem 3.9 *Let $\phi : \mathbb{R}^2 \to N$ be a doubly periodic primitive map into a k-symmetric space, $k > 2$. Suppose that some (and hence every) framing of ϕ has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in a single* semisimple *$Ad\, K^{\mathbb{C}}$-orbit in $\mathfrak{g}_{-1}$. Then ϕ is of finite type.*

We conclude this section with an alternative formulation of the "semisimple orbit" condition which will be useful below. First, the holomorphic differentials argument of Section 3.2 can be extended to primitive maps so that one can conclude that a doubly periodic primitive map of $\mathbb{R}^2$ has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in a single common level set of the $Ad\, K^{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}_{-1}$. On the other hand, Vinberg [56] proves that each such level set contains a single semisimple orbit and that this is the unique closed orbit in the level set. We therefore deduce:

Theorem 3.10 *Let $\phi : \mathbb{R}^2 \to N$ be a doubly periodic primitive map into a k-symmetric space, $k > 2$. Suppose that, for some (and hence every) framing of ϕ, $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ is semisimple on a dense subset of M. Then ϕ is of finite type.*

3.4 Example: Toda fields

As an illustration of our ideas, we describe some results of Bolton–Pedit–Woodward [7] which relate certain primitive maps into flag manifolds with (periodic) Toda fields (for further details, see [8] in this volume). We begin by introducing the basic ingredients of Toda field theory.

Let $\mathfrak{g}$ be a compact, *simple* Lie algebra with Killing form κ and, as usual, fix a maximal toral subalgebra $\mathfrak{t}$ and set $\mathfrak{a} = i\mathfrak{t}$ so that $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We also fix a choice of positive roots $\Delta^+ \subset \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$ with corresponding simple roots $\alpha_1, \dots, \alpha_l$. Since $\mathfrak{g}$ is simple, we have another distinguished root: the *highest root* μ and we set

$$\Pi = \{\alpha_1, \dots, \alpha_l, -\mu\}.$$

Thus Π is the set of roots labelling the nodes of the extended Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$. From the elementary properties of root systems (see, for example, [33]), we have that if $\alpha, \beta \in \Pi$ and $X_\alpha \in \mathfrak{g}^\alpha$, $X_{-\beta} \in \mathfrak{g}^{-\beta}$ then

$$[X_\alpha, X_{-\beta}] = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \kappa(X_\alpha, X_{-\beta})H_\alpha & \text{if } \alpha = \beta \end{cases} \tag{3.17}$$

where $H_\alpha \in \mathfrak{a}$ is the Killing dual of α.

With this understood, we can describe Toda field theory. This is a nonlinear Lagrangian field theory where the fields are maps $u : \mathbb{R}^2 \to \mathfrak{a}$ and the Lagrangian is given by

$$\int_{\mathbb{R}^2} \tfrac{1}{2}\kappa(du, du) - \sum_{\alpha\in\Pi} e^{\alpha(u)}\, dvol.$$

The Euler–Lagrange equations for this functional are

$$d^*du - \sum_{\alpha\in\Pi} e^{\alpha(u)} H_\alpha = 0 \tag{3.18}$$

and solutions of this are *Toda fields*.

Remark The alert Reader will notice that our Toda field equations differ from those in [7, 8]. The difference is a matter of normalisation with that in [7, 8] chosen to ensure that zero is a solution. We have given a normalisation which emphasises the relationship with the Toda lattice of Section 1.5.

We can find a zero-curvature formulation of (3.18) via the following *ansatz*: fix, once and for all, root vectors $X_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Pi$ with

$$\kappa(X_\alpha, \overline{X_\alpha}) = -\tfrac{1}{4}$$

and set $X = \sum_{\alpha\in\Pi} X_\alpha$. Now for $u : \mathbb{R}^2 \to \mathfrak{a}$, define a loop of $\mathfrak{g}$-valued 1-forms

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{t}} + \lambda\alpha''_{\mathfrak{m}}$$

by

$$\begin{aligned} \alpha'_{\mathfrak{m}} &= Ad\exp(u/2)X\,dz = \sum_{\alpha\in\Pi} e^{\alpha(u)/2} X_\alpha\, dz \\ \alpha_{\mathfrak{t}} &= \tfrac{i}{2} * du = \tfrac{1}{2}\left(\frac{\partial u}{\partial z}\, dz - \frac{\partial u}{\partial \overline{z}}\, d\overline{z}\right) \\ \alpha''_{\mathfrak{m}} &= \overline{\alpha'_{\mathfrak{m}}} = Ad\exp(-u/2)\overline{X}\, d\overline{z}. \end{aligned}$$

In particular, $\alpha_{\mathfrak{t}}$ is $\mathfrak{t}$-valued so that $[\alpha_{\mathfrak{t}} \wedge \alpha_t]$ vanishes. For any field u, it is easy to verify that

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{t}} \wedge \alpha'_{\mathfrak{m}}] = 0 = d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{t}} \wedge \alpha''_{\mathfrak{m}}]$$

while, in view of (3.17),

$$\begin{aligned} d\alpha_{\mathfrak{t}} + \tfrac{1}{2}[\alpha_{\mathfrak{t}} \wedge \alpha_{\mathfrak{t}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] &= \tfrac{i}{2} d * du + \sum_{\alpha\in\Pi} e^{\alpha(u)}\kappa(X_\alpha, \overline{X_\alpha}) H_\alpha\, dz \wedge d\overline{z} \\ &= \tfrac{i}{2}(d * du + \sum_{\alpha\in\Pi} e^{\alpha(u)} H_\alpha * 1) \\ &= -\tfrac{i}{2} * (d^*du - \sum_{\alpha\in\Pi} e^{\alpha(u)} H_\alpha). \end{aligned}$$

Thus we conclude:

Proposition 3.11 *$u : \mathbb{R}^2 \to \mathfrak{a}$ is a Toda field if and only if*

$$d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0,$$

for all $\lambda \in S^1$.

Remark There are numerous other (gauge-equivalent) zero-curvature formulations of the Toda field equations (see, for example, [38]). We have chosen this one because it gives real-valued α_λ.

We now turn to a geometric interpretation of the Toda field equations. Let G be the adjoint group of $\mathfrak{g}$ (so that, in particular, G has trivial centre) and let T, A be the analytic subgroups with Lie algebras $\mathfrak{t}$, $\mathfrak{a}$. Consider the flag manifold G/T—we equip this with the structure of a k-symmetric space as follows. Let $\xi_j \in \mathfrak{t}$, $1 \leq j \leq l$ be defined by

$$\alpha_i(\xi_j) = \sqrt{-1}\delta_{ij}$$

and set $\xi = \sum \xi_j$. In particular, $\alpha(\xi) \in \sqrt{-1}\mathbb{Z}$, for all $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$, and we define $k \in \mathbb{N}$ by

$$\mu(\xi) = \sqrt{-1}(k-1).$$

We take τ to be conjugation by $\exp(-2\pi\xi/k)$: this is an order k automorphism of G. Then T is the identity component of the fixed set of τ so that G/T is a k-symmetric space.

Remark This is an example of a rather general construction: any generalised flag manifold $G^{\mathbb{C}}/P$, where P is a parabolic subgroup, can be given a canonical k-symmetric structure [15, p. 52].

In the case at hand, observe that in the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into eigenspaces of τ, we have

$$\mathfrak{g}_0 = \mathfrak{h}; \qquad \mathfrak{g}_{-1} = \sum_{\alpha\in\Pi} \mathfrak{g}^\alpha$$

so that we can immediately deduce the following result from Theorem 2.1 and Proposition 3.11:

Proposition 3.12 *Let $u : \mathbb{R}^2 \to \mathfrak{a}$ be a Toda field with loop of 1-forms α_λ. Then each $\alpha_\lambda = \Phi_\lambda^{-1} d\Phi_\lambda$ for a framing $\Phi_\lambda : \mathbb{R}^2 \to G$ of a primitive map $\phi_\lambda : \mathbb{R}^2 \to G/T$.*

We call such a framing of a primitive map a *Toda frame*.

When does a primitive map $\mathbb{R}^2 \to G/T$ have a Toda frame? Observe that, by construction, a Toda frame has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in the $Ad\,A$-orbit of X. In fact, the converse is true: it is easy to see that A acts simply transitively on $Ad(A)X$ while $\exp : \mathfrak{a} \to A$ is a diffeomorphism so that, if a framing has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in $Ad(A)X$ then we may take logarithms to find $u : \mathbb{R}^2 \to \mathfrak{a}$ satisfying

$$\alpha'_{\mathfrak{m}} = Ad \exp(u/2) X \, dz.$$

Moreover, one can show that $\alpha_{\mathfrak{t}}$ is determined completely by $\alpha'_{\mathfrak{m}}$ and the Maurer–Cartan equations so that

$$\alpha_{\mathfrak{t}} = \tfrac{i}{2} * du.$$

Thus

Lemma 3.13 *[7] A frame is Toda if and only if $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ takes values in $Ad(A)X$.*

To go further, we must introduce some ideas of Kostant [35]: define the set of cyclic elements $\mathcal{C} \subset \mathfrak{g}_{-1}$ by

$$\mathcal{C} = \{ \sum_{\alpha \in \Pi} c_\alpha X_\alpha : \quad c_\alpha \in \mathbb{C} \setminus \{0\} \}.$$

Concerning these, Kostant proves

1. any element of $\mathcal{C}$ is (regular) semisimple;
2. if $P_1, \dots, P_l$ are algebraically independent homogeneous generators of the invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ with $\deg P_1 \leq \cdots \leq \deg P_l$ then

   $$P_i|_{\mathfrak{g}_{-1}} \equiv 0 \quad \text{for } 1 \leq i \leq l-1$$

   and, for $\xi \in \mathfrak{g}_{-1}$, $\xi \in \mathcal{C}$ if and only if $P_l(\xi) \neq 0$.
3. For $c \neq 0$, T acts simply transitively on the set of $Ad\,A$-orbits in $P_l^{-1}\{c\}$.

From this, we deduce that if a framing Φ satisfies $P_l(\alpha'_{\mathfrak{m}}(\partial/\partial z)) \equiv P_l(X)$ then there is a *unique* gauge transformation taking Φ to a Toda frame. More generally, when $P_l(\alpha'_{\mathfrak{m}}(\partial/\partial z))$ is a non-zero constant, we can make a linear change of co-ordinate $cz = w$ so that $P_l(\alpha'_{\mathfrak{m}}(\partial/\partial w)) \equiv P_l(X)$ and so obtain a Toda frame.

To summarise:

Theorem 3.14 *[7] A primitive map $\phi : \mathbb{R}^2 \to G/T$ has a Toda frame (after a linear change of co-ordinate) if and only if, for some (and hence, every) framing we have that $P_l(\alpha'_{\mathfrak{m}}(\partial/\partial z))$ is a non-zero constant. Moreover, in this case, the Toda frame is unique.*

From the uniqueness assertion, we also get

Corollary 3.15 *[7] Let $\phi : \mathbb{R}^2 \to G/T$ be a primitive map with Toda frame Φ and corresponding Toda field u. Then ϕ is doubly periodic if and only if Φ is doubly periodic and then u is doubly periodic.*

We remark that the usual holomorphic differentials argument shows that a doubly periodic primitive map has a Toda frame if and only if $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ is cyclic at some (and hence, every) point of $\mathbb{R}^2$.

We can now apply Theorem 3.9 to obtain such doubly periodic primitive maps from commuting flows. Indeed, a primitive map with Toda frame necessarily satisfies the "semisimple orbit" hypothesis since X is semisimple so that we have

Theorem 3.16 *[7] A doubly periodic primitive map $\phi : \mathbb{R}^2 \to G/T$ with non-zero $P_l(\alpha'_{\mathfrak{m}}(\partial/\partial z))$ is of finite type.*

In particular, we obtain doubly periodic Toda fields this way from Corollary 3.15. However, more is true: while it is not the case that doubly-periodic Toda fields necessarily produce doubly periodic primitive maps (there may be holonomy), Bolton–Pedit–Woodward prove that, for doubly periodic Toda fields, the $\Lambda\mathfrak{g}_\tau$-valued 1-form α_λ *always* arises as in Theorem 2.5 so that all doubly periodic Toda fields are of finite type.

Examples Let us take $G = SO(5)$ with flag manifold $F = SO(5)/SO(2) \times SO(2)$. We have the homogeneous fibration $p : F \to S^4$ and a homogeneous diffeomorphism between F and the Grassmannian bundle $\widetilde{G}_2(TS^4)$ of oriented 2-planes in TS^4. Under this indentification, a map $\phi : M \to F$ is primitive if and only if it is the Gauss map of a minimal (i.e., conformal and harmonic) map $p \circ \phi : M \to S^4$ (a result originally due to Eells–Salamon [21]). Moreover, ϕ has a Toda frame precisely when the corresponding minimal surface is not superminimal. In this case, the above results reproduce the analysis of Ferus–Pedit–Pinkall–Sterling [24].

Again, let us take $F = SU(n)/T^{n-1}$ as our flag manifold. A map $\phi : M \to F$ may be viewed as a collection of mutually orthogonal maps $\phi_1, \dots, \phi_n : M \to \mathbb{C}P^{n-1}$ and then ϕ is primitive with Toda frame if and only if the ϕ_i comprise the Frenet frame of a *superconformal* harmonic map $M \to \mathbb{C}P^{n-1}$ in the sense of [7, 8]. It was for the study of such maps that the methods of [7] were developed.

We shall have more to say about results of this kind in the next section.

We conclude this section by briefly contemplating the geometric significance of open Toda fields as this has received a lot of recent attention in the Physics literature [18, 25, 26, 47, 50]. Open Toda fields are extremals of the modified Lagrangian

$$\int_{\mathbb{R}^2} \tfrac{1}{2}\kappa(du, du) - \sum_{i=1}^{l} e^{\alpha_i(u)}\, dvol$$

where there is no longer an exponential interaction involving $-\mu$. Again there is a zero-curvature representation but this time the primitive maps so obtained are *holomorphic* maps into G/T. In fact, they are superhorizontal holomorphic maps in the sense of [15]. As a consequence, open Toda fields are much simpler objects for which Weierstrass representation formulae are available. See [38] in this volume for a brief discussion of such fields.

3.5 Example: primitive maps and twistor lifts

We have seen how all non-conformal harmonic 2-tori in a rank-1 symmetric space are of finite type. However, this excludes the geometrically interesting case of minimal 2-tori. On the other hand, special cases of the results of the previous section have been used to account for certain minimal 2-tori in S^4 [24], S^n and $\mathbb{C}P^n$ [7] by showing that these tori have lifts which are primitive maps of finite type.

In this section, we present a generalisation of these results which accounts for *all* harmonic 2-tori in a sphere S^n. We begin by recalling some results from the well-developed twistor theory of harmonic maps of a Riemann surface into a sphere.

So let $\phi : M \to S^n$ be a harmonic map of a Riemann surface. Let $T = \phi^{-1}TS^n$ with connection ∇ given by the pull-back of the Levi-Civita connection of S^n. Let z be a local holomorphic co-ordinate on M, set $\nabla' = \nabla_{\partial/\partial z}$, $\nabla'' = \nabla_{\partial/\partial \bar{z}}$ and inductively define $\nabla^i\phi$ by

$$\nabla^1\phi = \partial\phi/\partial z; \qquad \nabla^{i+1}\phi = \nabla'\nabla^i\phi.$$

For $x \in M$, define $W_x^j \subset T_x^{\mathbb{C}}$ by

$$W_x^j = span_{\mathbb{C}}\{\nabla_x^i\phi : 1 \le i \le j\}.$$

Clearly, each W_x^j is defined independently of the choice of holomorphic co-ordinate z.

If ϕ is non-constant and (weakly) conformal then each W_x^1 is isotropic for the complexified metric on $T^{\mathbb{C}}$ and is 1-dimensional off a discrete set of points in M. This motivates, in part, the following definition:

Definition The *isotropy dimension* r of a conformal harmonic map $\phi : M \to S^n$ is given by

$$r = \max\{j : \max_x \dim_{\mathbb{C}} W_x^j = j \text{ and } W_x^j \text{ is isotropic for all } x\}.$$

We make the convention that a non-conformal map has isotropy dimension zero.

The following facts are well known (c.f. [60]): if ϕ has isotropy dimension $r > 0$ then

1. For $1 \le j \le r+1$, there is a bundle W^j of $T^{\mathbb{C}}$ whose fibre at x coincides with W_x^j except at a discrete set of points;
2. Each W^j is stable under ∇'';
3. For $1 \le j \le r$, $rank\, W^j = j$ and $rank\, W^{r+1} \le r+1$.

We therefore distinguish two possibilities: either $W^r = W^{r+1}$ or not[1].

[1]These two possibilities regulate the relationship between the isotropy dimension of ϕ and the *isotropy order* of ϕ [2] described by Wood [59] in this volume: if ϕ has isotropy dimension r then ϕ has isotropy order $2r$ if and only if $W^r = W^{r+1}$ and isotropy order $2r+1$ otherwise.

In the first case, we see that W^r is isotropic and stable under ∇' so that the (complexified) inner products

$$(\nabla^i\phi, \nabla^j\phi) \equiv 0$$

for all $i, j \in \mathbb{N}$. Harmonic maps of this kind are variously called *pseudo-holomorphic* [16], *real isotropic* [22] or *superminimal* [9] and were completely classified by Calabi [16] (see also [3]) who proved that all such were projections of horizontal holomorphic curves in the twistor space of S^n. Indeed, since W^r is stable under both ∇' and ∇'', it is parallel so that

$$(W^r \oplus \overline{W^r}) \cap T$$

is a parallel sub-bundle of T with (real) rank $2r$. It follows that ϕ factors through an equatorial $2r$-sphere in S^n so that, without loss of generality, we may take $2r = n$. Now recall that the *twistor space* Z on S^{2r} may be viewed as the bundle of isotropic r-planes in the complexification of TS^{2r}. This is a complex manifold: in fact, $SO(2r+1)$ acts transitively on each of the two connected components of Z and each component is so realised as the generalised flag manifold $SO(2r+1)/U(r)$. It is clear that W^r defines a map $\psi : M \to Z$ covering ϕ and the condition that W^r be parallel is equivalent to the demand that ψ be holomorphic and horizontal with respect to the twistor fibration $Z \to S^{2r}$. For more on twistor spaces of symmetric spaces and their applications to harmonic maps, the Reader is refered to [15], the surveys [12, 13, 46, 61] and the article by Kobak [34] in this volume.

Remark It is clear that the isotropy dimension r of a map $\phi : M \to S^n$ must satisfy $2r \leq n$. In case that $2r = n$, it follows from the easily verified fact

$$(\nabla^{r+1}\phi, \nabla^i\phi) \equiv 0,$$

for $1 \leq i \leq r$, that $W^r = W^{r+1}$ so that ϕ is superminimal.

Let us now consider the non-superminimal harmonic maps where $W^r \neq W^{r+1}$ or, equivalently,

$$(\nabla^{r+1}\phi, \nabla^{r+1}\phi) \not\equiv 0.$$

It is our contention that ϕ is covered by a primitive map into a suitable k-symmetric space and, moreover, that when M is a 2-torus this primitive map is of finite type.

First we describe the relevant k-symmetric space: for $2r < n$, let $F^r(S^n)$ be the bundle over S^n with fibre

$$F^r_x(S^n) = \{w_1 \subset \cdots \subset w_r \subset (T_xS^n)^{\mathbb{C}} : \text{each } w_j \text{ is an isotropic } j\text{-plane}\}$$

Now $SO(n+1)$ acts transitively on $F^r(S^n)$ with stabilisers conjugate to

$$\overbrace{SO(2) \times \cdots \times SO(2)}^{r \text{ times}} \times SO(n-2r)$$

so that $F^r(S^n)$ is a homogeneous space.

Fix a base-point $f = (w_1 \subset \cdots \subset w_r) \in F^r_x(S^n)$ and orthogonalise to obtain isotropic lines $L_1, \ldots, L_r$ and real subspaces L_{r+1} and $L_0 = span_{\mathbb{C}}\{x\}$ so that

$$(\mathbb{R}^{n+1})^{\mathbb{C}} = L_0 \oplus \sum_{i=1}^{r} (L_i \oplus \overline{L_i}) \oplus L_{r+1}.$$

Take $k = 2r + 2$, let ω be the usual primitive k-th root of unity and define $Q \in O(n+1)$ by

$$Q = \omega^{-i} \quad \text{on } L_i.$$

Then conjugation by Q is an order k automorphism of $SO(n+1)$ and the identity component of its fixed set is the stabiliser of f. Thus $F^r(S^n)$ is a k-symmetric space.

Let $\psi : M \to F^r(S^n)$ be a map with projection $\phi : M \to S^n$. We may view ψ as a flag of isotropic sub-bundles

$$\psi^{(1)} \subset \cdots \subset \psi^{(r)} \subset T^{\mathbb{C}}$$

(here again $T = \phi^{-1}TS^n$) and one can prove

Proposition 3.17 *[11] $\psi : M \to F^r(S^n)$ is primitive if and only if*

(i) $\nabla^1\phi$ is a (local) section of $\psi^{(1)}$;

(ii) each $\psi^{(i)}$ is stable under ∇'';

(iii) if σ is a local section of $\psi^{(i)}$ then $\nabla'\sigma$ is a local section of $\psi^{(i+1)}$.

We know from Theorem 3.7 that if ψ is primitive then ϕ is harmonic but, in the present case, we have a converse. Let $\phi : M \to S^n$ be a non-superminimal harmonic map of isotropy order r. It is clear from Proposition 3.17 that the bundle of flags $W^1 \subset \cdots \subset W^r$ defines a primitive map $\psi : M \to F^r(S^n)$. When is ψ of finite type? We have

Lemma 3.18 *[11] A (local) framing of ψ has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ semisimple at $x \in M$ if and only if*

$$(\nabla^{r+1}\phi, \nabla^{r+1}\phi) \neq 0$$

at $x \in M$.

Since $W^r \neq W^{r+1}$, we deduce that this condition holds on a dense open subset of M and so conclude from Theorem 3.10:

Theorem 3.19 *[11] A non-superminimal harmonic map of a 2-torus into a sphere is covered by a primitive map of finite type.*

Thus harmonic 2-tori in spheres are completely accounted for: either they are superminimal and so arise as projections of holomorphic curves or they are obtained by integrating commuting flows.

Example We have seen that a non-superminimal map into S^n must have isotropy dimension r with $2r < n$. Suppose $n = 2m$ and consider maps of isotropy dimension $m-1$. Now $F^{m-1}(S^{2m})$ is the full flag manifold of $SO(2m+1)$ and its k-symmetric structure is precisely that described in Section 3.4. Moreover, when M is a 2-torus, the primitive maps we have constructed are precisely those with Toda frames.

Again, if $n = 2m - 1$, the maximal isotropy dimension of a non-superminimal map is $m - 1$ and, viewing S^{2m-1} as an equator of S^{2m}, we again conclude that such tori are covered by primitive maps with Toda frames.

To summarise: non-superminimal harmonic 2-tori of maximal isotropy order are all obtained from doubly-periodic Toda fields of finite type. This result was first proved by Bolton–Pedit–Woodward [7] (see also [18]) who called such maps *superconformal*.

Remark A similar analysis can be carried out for harmonic maps into a complex projective space. Again one has a notion of an isotropic or superminimal map (these are the legs of the Frenet frame of a holomorphic curve) and all non-superminimal harmonic maps are covered by primitive maps into flag manifolds with their canonical k-symmetric structure. Moreover, when M is a 2-torus, these primitive maps are of finite type. The Reader is refered to [11] for more details on this and the other results of this section.

4 Loop groups and extended framings

The Adler–Kostant–Symes scheme produces commuting Hamiltonian flows on Lie algebras and we have seen how to apply this scheme to the twisted loop algebras $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ and so obtain loops of flat connections and hence harmonic and primitive maps. However, in Section 1.4, a method of Symes was described for integrating these flows via projection of geodesics. We now show how this method applies in our setting. In so doing, we will introduce an analogue of Uhlenbeck's theory [55] of "extended solutions" (see, also, [27] in this volume) and, incidently, obtain a conceptual proof of Theorem 2.4.

Terminology To avoid treating the case of k-symmetric spaces with $k = 2$ separately, henceforth we shall talk of primitive harmonic maps, conscious of the fact that the primitive condition is vacuous when $k = 2$ and that the harmonic condition is implied by the primitive condition when $k > 2$ (Theorem 3.6).

4.1 Iwasawa decomposition of $\Lambda G_\tau^{\mathbb{C}}$

We begin by introducing the infinite-dimensional Lie groups that correspond to the Lie algebras $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$, $\Lambda\mathfrak{g}_\tau$ and $\Lambda_+\mathfrak{g}_\tau^{\mathbb{C}}$ of Section 2.3. So let G be a compact semisimple Lie group with order k automorphism τ whose fixed set is K. Further, let

$$K^{\mathbb{C}} = KB$$

be the Iwasawa decomposition of $K^{\mathbb{C}}$ corresponding to that of $\mathfrak{k}^{\mathbb{C}}$.

Let $\Lambda G_\tau^{\mathbb{C}}$ be the manifold of loops

$$\Lambda G_\tau^{\mathbb{C}} = \{\gamma : S^1 \to G^{\mathbb{C}} : \gamma(\omega\lambda) = \tau\gamma(\lambda) \text{ for all } \lambda \in S^1\}.$$

Thus $\Lambda G_\tau^{\mathbb{C}}$ is an infinite-dimensional Lie group[2] under point-wise multiplication. Further, define subgroups by

$$\Lambda G_\tau = \{\gamma \in \Lambda G_\tau^{\mathbb{C}} : \gamma : S^1 \to G\};$$

$$\Lambda_+ G_\tau^{\mathbb{C}} = \{\gamma \in \Lambda G_\tau^{\mathbb{C}} : \gamma \text{ extends holomorphically to } \gamma : D \to G^{\mathbb{C}},\ \gamma(0) \in B\}.$$

Clearly, $\Lambda G_\tau^{\mathbb{C}}$ has Lie algebra $\Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ and so on.

We now have the following Iwasawa decomposition:

Theorem 4.1 *[19] Multiplication $\Lambda G_\tau \times \Lambda_+ G_\tau^{\mathbb{C}} \to \Lambda G_\tau^{\mathbb{C}}$ is a diffeomorphism onto.*

This result was deduced by Dorfmeister–Pedit–Wu [19] from the special case $\tau = id$ which was proved by Pressley–Segal [43]. The main idea of Pressley–Segal was that the homogeneous space $\Lambda G_\tau^{\mathbb{C}}/\Lambda_+ G_\tau^{\mathbb{C}}$ can be realised as an infinite-dimensional Grassmannian on which one can show that ΛG_τ acts transitively (c.f. [27] in this volume).

4.2 Extended framings and a Symes formula

Let $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_0 + \lambda\alpha''_{\mathfrak{m}}$ be a $\Lambda\mathfrak{g}_\tau$-valued 1-form on a simply-connected surface M which satisfies the Maurer–Cartan equations. We have seen that, for each $\lambda \in S^1$, there is a smooth map $\Phi_\lambda : M \to G$ framing a primitive harmonic map $\phi_\lambda : M \to G/K$ and satisfying

$$\Phi_\lambda^{-1} d\Phi_\lambda = \alpha_\lambda.$$

Moreover, Φ_λ is unique up to left translation by a constant. We may choose these constants so that $\Phi_\lambda(x)$ depends smoothly on λ for some (and hence every) $x \in M$ and then we may view the Φ_λ as a single smooth map $\hat{\Phi} : M \to \Lambda G_\tau$ such that the following diagram commutes:

$$\begin{tikzcd} & \Lambda G_\tau \arrow[d, "ev_\lambda"] \\ M \arrow[ur, "\hat{\Phi}"] \arrow[r, "\Phi_\lambda"'] & G \end{tikzcd}$$

[2] We shall not worry about the topology of $\Lambda G_\tau^{\mathbb{C}}$: in fact, it suffices to work with the (Fréchet) C^∞-topology [43].

where $ev_\lambda : \Lambda G_\tau \to G$ is given by evaluating the loop at $\lambda \in S^1$.

We call such maps $\hat{\Phi}$ *extended framings*. Thus $\hat{\Phi}$ is an extended framing if and only if

$$(\hat{\Phi}^{-1} d\hat{\Phi})_\lambda = \alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_0 + \lambda\alpha''_{\mathfrak{m}},$$

where $\alpha'_{\mathfrak{m}}$ α_0 and $\alpha''_{\mathfrak{m}}$ do not depend on λ. Any extended framing gives rise to a loop of primitive harmonic maps. Moreover, any primitive harmonic map of a simply connected M gives rise to a loop of such maps which are covered in this way by an extended framing. (In the case, $k = 2$ these results are due to Rawnsley [45].)

We can find examples of extended framings by recourse to Theorem 4.1 and the method of Symes. For this, let $\Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}} \subset \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ consist of those elements $\eta \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ for which $\lambda\eta$ extends holomorphically to D. Thus $\eta \in \Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}}$ if and only if

$$\eta = \sum_{n \geq -1} \lambda^n \eta_{-n}.$$

Observe that if $g \in \Lambda_+ G_\tau^{\mathbb{C}}$ and $\eta \in \Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}}$ then $Ad\, g\, \lambda\eta = \lambda\, Ad\, g\, \eta$ also extends holomorphically to D so that $\Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}}$ is invariant under the adjoint action of $\Lambda_+ G_\tau^{\mathbb{C}}$.

Now choose $\eta_o \in \Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}}$ and define the (complex) geodesic $g : \mathbb{R}^2 \to \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$ by

$$g(z) = \exp(z\eta_o).$$

Using Theorem 4.1, we find maps $a : \mathbb{R}^2 \to \Lambda G_\tau$, $b : \mathbb{R}^2 \to \Lambda_+ G_\tau^{\mathbb{C}}$ such that

$$g = ab$$

and we now have

Theorem 4.2 *a is an extended framing.*

Proof. We argue as in Section 1.4 to see that

$$\eta_o\, dz = dg\, g^{-1} = da\, a^{-1} + Ad\, a(db\, b^{-1})$$

so that

$$a^{-1}da = \pi_{\mathcal{K}}(Ad\, a^{-1}\eta_o\, dz).$$

Set $\eta = Ad\, a^{-1}\eta_o$ and observe that since

$$Ad\, g(z)\, \eta_o = Ad \exp(z\eta_o)\, \eta_o = \eta_o,$$

we also have

$$\eta = Ad\, b\, \eta_o$$

so that η takes values in $\Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}}$. This means that $\eta = \sum_{n \geq -1} \lambda^n \eta_{-n}$ so that, by (2.3) we have

$$a^{-1}da = \pi_{\mathcal{K}}(\eta\, dz) = \lambda^{-1}\eta_1\, dz + (\eta_0\, dz)_{\mathfrak{k}} + \lambda\overline{\eta_1}\, d\overline{z} \tag{4.1}$$

and a is an extended framing. □

The extended framings corresponding to finite type primitive harmonic maps all arise in this way: let $d \equiv 1 \bmod k$ and fix an initial condition $\xi_o \in \Lambda_d$. Then $\eta_o = \lambda^{d-1}\xi_o \in \Lambda^{(1)}\mathfrak{g}_\tau^{\mathbb{C}}$ and we apply Theorem 4.2 to get an extended framing a. Now set $\xi = Ad\, a^{-1}\xi_o = \lambda^{1-d}\eta$. Again we have $\xi = Ad\, b\, \xi_o$ so that, since $\lambda^d\xi_o$ extends holomorphically to the disc, $\lambda^d\xi$ does also and it follows that $\xi : \mathbb{R}^2 \to \Lambda_d$. Now

$$d\xi = [\xi, a^{-1}da]$$

while, from (4.1) and (2.5), we have

$$a^{-1}da = \pi_{\mathcal{K}}(\lambda^{d-1}\xi\, dz) = (\lambda^{-1}\xi_d + r(\xi_{d-1}))\, dz + (\lambda\xi_{-d} + \overline{r(\xi_{d-1})})\, d\bar{z}.$$

We therefore conclude that

1. ξ is the solution to the Hamiltonian equations (2.7);
2. the 1-form (2.8) constructed from ξ is precisely $a^{-1}da$ (and so solves the Maurer–Cartan equations so that the proof of Theorem 2.4 is complete);
3. a is precisely the extended framing for the finite type primitive harmonic map corresponding to ξ.

This gives an alternative characterisation of the finite type condition in terms of the extended framing which will be useful below. To be precise, we have proved the following theorem:

Theorem 4.3 *A primitive harmonic map $\phi : \mathbb{R}^2 \to G/K$ into a k-symmetric space is of finite type if and only if, for some $d \equiv 1 \bmod k$, there exists $\xi_o \in \Lambda_d$ such that, if $a : \mathbb{R}^2 \to \Lambda G_\tau$, $b : \mathbb{R}^2 \to \Lambda_+ G_\tau^{\mathbb{C}}$ are defined by*

$$\exp(z\lambda^{d-1}\xi_o) = a(z)b(z),$$

for $z \in \mathbb{R}^2$, then a is an extended framing for ϕ so that

$$\phi = \pi \circ ev_1 \circ a,$$

where $\pi : G \to G/K$ is the coset projection.

Remark These simple results are the starting point of a loop-group theoretic analysis of the harmonic map equations which, to some extent, parallels that of the KdV equation by Segal–Wilson [51]. We shall return to this elsewhere. For a similar approach to Toda fields, the Reader is refered to the contribution of McIntosh in this volume [38] as well as [20, 39, 63].

5 Another approach

To date, the most comprehensive results concerning the construction of harmonic maps from commuting Hamiltonian flows are to be found in [14]. The approach adopted in that paper is similar to the one we have been describing but differs from it in a number of important respects. Among these are:

1. The Hamiltonian flows arise from the r-matrix formalism described in Section 2.5 and not from the Adler–Kostant–Symes scheme.

2. The main objects of study are harmonic maps into a Lie group G rather than into an arbitrary (k-)symmetric space. Since Lie groups are parallelisable, this means that one can treat such maps directly without recourse to a framing. Maps into k-symmetric spaces are then viewed as particular maps into G via a Cartan embedding (see Section 5.3 below).

In the remaining sections of this article, we describe the theory of [14] and its extension to the k-symmetric case in [11]. We will show how it relates to the theory we have been developing above and, in particular, we will see how Theorems 3.4 and 3.9 can be read off a corresponding result in [14].

5.1 Hamiltonian flows in the based loop algebra

We begin by rehearsing Uhlenbeck's zero-curvature reformulation of the harmonic map equations for maps into a Lie group G [55] and the method of Burstall–Ferus–Pedit–Pinkall [14] for producing solutions to those zero-curvature equations. The Reader will find a full account of these results in Wood's contribution to this volume [59] and so we shall be brief here, referring the Reader to [59] or [14] for more details.

So let $\phi : M \to G$ be a map of a Riemann surface into a compact Lie group. As usual, set $\alpha = \phi^{-1}d\phi$. We may view G as the homogeneous space $G/\{e\}$ so that ϕ is its own framing and it then follows from Section 3.2 that ϕ is harmonic if and only if α is co-closed:

$$d^*\alpha = 0. \tag{5.1}$$

Indeed, in this case, $\alpha = \alpha_{\mathfrak{m}}$ in (3.3) while $[\alpha \wedge *\alpha]$ always vanishes.

Combining (5.1) with the Maurer–Cartan equations for α, one concludes with Uhlenbeck that the loop of 1-forms given by

$$A_\lambda = \frac{1-\lambda^{-1}}{2}\alpha' + \frac{1-\lambda}{2}\alpha'' \tag{5.2}$$

satisfies the Maurer–Cartan equations for each $\lambda \in S^1$. Conversely, given a loop of flat connections (5.2) on a simply connected surface M, we integrate to find a harmonic map $\phi : M \to G$ with $\phi^{-1}d\phi = A_{-1}$.

Observe that the loop A_λ in (5.2) satisfies $A_1 \equiv 0$ and so may be viewed as a 1-form with values in the *based* loop algebra $\Omega\mathfrak{g}$ given by

$$\Omega\mathfrak{g} = \{\xi : S^1 \to \mathfrak{g} : \xi(1) = 0\}.$$

View $\Omega\mathfrak{g}$ as a subalgebra of the free loop algebra $\Lambda\mathfrak{g}$ (which is $\Lambda\mathfrak{g}_\tau$ for $\tau = id$) and set $\Omega_d = \Lambda_d \cap \Omega\mathfrak{g}$. We shall produce flat $\Omega\mathfrak{g}$-valued 1-forms (5.2) on $\mathbb{R}^2$ by integrating commuting flows on Ω_d. For this, fix $d \in \mathbb{N}$ and define vector fields X_1, X_2 on Ω_d by

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(\lambda^{-1} - 1)\xi_d].$$

As explained in Wood's contribution [59], there is a Poisson structure on $\Omega\mathfrak{g}$ which arises from an r-matrix as in Section 2.5. Moreover, with respect to this Poisson structure, the vector fields X_i are Hamiltonian vector fields of Ad-invariant functions and so commute. Being of Lax type, they are also complete. Thus, if we fix an initial condition $\xi_o \in \Omega_d$, we may simultaneously integrate the X_i to get a unique map $\xi : \mathbb{R}^2 \to \Omega_d$ satisfying

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(\lambda^{-1} - 1)\xi_d]; \qquad \xi(0) = \xi_o.$$

Then, defining an $\Omega\mathfrak{g}$-valued 1-form by

$$A_\lambda = 2i(\lambda^{-1} - 1)\xi_d \, dz - 2i(\lambda - 1)\xi_{-d} \, d\overline{z}$$

produces a solution to the Maurer–Cartan equations and hence a harmonic map $\phi : \mathbb{R}^2 \to G$ with $\phi^{-1}\partial\phi/\partial z = -4i\xi_d$.

Again we shall call the maps so obtained *harmonic maps of finite type*.

Notation Consistency with our previous development compels us to use slightly different notation from that in Wood's contribution [59] and, indeed, [14]. In those papers, the spectral parameter λ is the reciprocal of ours while the coefficients ξ_d differ by a sign.

Remark One should note that the finite-dimensional subspaces Ω_d are not Poisson submanifolds for the Poisson structure on $\Omega\mathfrak{g}$: they are only invariant for the Hamiltonian flows of Ad-invariant functions. Thus, while the ordinary differential equations that we are interested in evolve on a finite-dimensional space, these spaces are not Poisson manifolds in their own right. This is in contrast with the spaces $\Lambda_d \subset \Lambda\mathfrak{g}_\tau$ discussed above.

When is a harmonic map $\phi : \mathbb{R}^2 \to G$ of finite type? Given a harmonic map ϕ with associated $\Omega\mathfrak{g}$-valued 1-form A_λ, we may reformulate the finite type condition as the demand that, for some $d \in \mathbb{N}$, there is a map $\xi : \mathbb{R}^2 \to \Omega_d$ satisfying

$$d\xi = [\xi, A_\lambda]; \tag{5.3}$$
$$\alpha' = -4i\xi_d \, dz. \tag{5.4}$$

As before, a necessary condition for the existence of such a ξ is that $\phi^{-1}\partial\phi/\partial z$ takes values in a single $Ad\, G^{\mathbb{C}}$-orbit in $\mathfrak{g}^{\mathbb{C}}$. The principal result of [14] is that a partial converse is true for doubly periodic harmonic maps:

Theorem 5.1 *[14] A doubly periodic harmonic map $\phi : \mathbb{R}^2 \to G$ is of finite type if $\phi^{-1}\partial\phi/\partial z$ takes values in a single semisimple $Ad\, G^{\mathbb{C}}$-orbit.*

We shall see later that Theorems 3.4 and 3.9 are corollaries of this theorem. Meanwhile, let us see how it is proved. The first step is to find a formal solution to (5.3) and (5.4), that is, a formal Laurent series

$$Y = \sum_{j \geq -1} \lambda^j Y_{-j}$$

with each $Y_j : \mathbb{R}^2 \to \mathfrak{g}^{\mathbb{C}}$ such that $\alpha' = -4iY_1\, dz$ and

$$dY = [Y, A_\lambda] \tag{5.5}$$

coefficient-wise. Using the semisimplicity hypothesis (and specifically that fact that $\eta \in \mathfrak{g}^{\mathbb{C}}$ is semisimple if and only if $ad\,\eta$ is invertible on its image) one can find an explicit recursion formula for the Y_j and so construct Y. It is an immediate consequence of the recursion formula that if A_λ is doubly periodic then so is each Y_j.

The second step is where we make essential use of the double periodicity hypothesis. It follows from (5.5) that each Y_j is a Jacobi field for ϕ and thus is a solution of a linear elliptic equation. When α is doubly periodic, we view each Y_j as being defined on a torus and the compactness of the torus together with standard elliptic theory now implies that the Y_j span a *finite-dimensional* space. Using this, it is easy to construct a solution $\xi : \mathbb{R}^2 \to \Omega_d$, for some d, from some of the Y_j.

5.2 Extended solutions and the Symes method

A variant of the Symes method of Section 4.2 is available for integrating the flows on Ω_d. This is based on the following loop group decomposition: set

$$\Lambda G^{\mathbb{C}} = \{\gamma : S^1 \to G^{\mathbb{C}}\}$$

and define subgroups

$$\Omega G = \{\gamma \in \Lambda G^{\mathbb{C}} : \gamma : S^1 \to G \text{ and } \gamma(1) = e\};$$

$$\Lambda_+ G^{\mathbb{C}} = \{\gamma \in \Lambda G^{\mathbb{C}} : \gamma \text{ extends holomorphically to } \gamma : D \to G^{\mathbb{C}}\}.$$

We have

Theorem 5.2 *[43] Multiplication $\Omega G \times \Lambda_+ G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}$ is a diffeomorphism onto.*

Remark This is very similar to the Iwasawa decomposition in Section 4.1 the main difference being that the uniqueness of the decomposition is ensured by basing the real loops at 1 rather than basing the holomorphic loops at 0.

Consider now a loop of 1-forms on a simply connected surface M:

$$A_\lambda = \frac{1-\lambda^{-1}}{2}\alpha' + \frac{1-\lambda}{2}\alpha'$$

which solve the Maurer–Cartan equations. For each $\lambda \in S^1$, we integrate to get maps $\Psi_\lambda : M \to G$ such that

$$\Psi_\lambda^{-1} d\Psi_\lambda = A_\lambda.$$

In particular, since $A_1 \equiv 0$, Ψ_1 is constant and we choose the constants of integration so that $\Psi_1 \equiv e$ and the Ψ_λ form a smooth map $\hat{\Psi} : M \to \Omega G$. Such $\hat{\Psi}$ are the *extended solutions* of Uhlenbeck [55] (c.f., [27, 59] in this volume). It is clear from Section 5.1 that if $\hat{\Psi}$ is an extended solution, the map ϕ defined by the commuting diagram

$$\begin{tikzcd} & \Omega G \arrow[d, "ev_{-1}"] \\ M \arrow[ur, "\hat{\Psi}"] \arrow[r, "\phi"] & G \end{tikzcd}$$

is harmonic and all harmonic maps $M \to G$ of a simply connected M are covered by an extended solution in this way.

The method of Symes used in Section 4.2 can be adapted to this setting. Firstly, if $\eta_o \in \Lambda\mathfrak{g}^{\mathbb{C}}$ is such that $\lambda\eta_o \in \Lambda_+\mathfrak{g}^{\mathbb{C}}$ (i.e., $\lambda\eta_o$ extends holomorphically to D) then we define $a : \mathbb{R}^2 \to \Omega G$, $b : \mathbb{R}^2 \to \Lambda_+ G^{\mathbb{C}}$ by

$$\exp(z\eta_o) = a(z)b(z),$$

for $z \in \mathbb{R}^2$, and we have

Proposition 5.3 *a is an extended solution.*

Secondly, if $\xi_o \in \Omega_d$, put $\eta_o = 2i\lambda^{d-1}\xi_o$ and define a as above. Then the map $\xi : \mathbb{R}^2 \to \Omega\mathfrak{g}$ given by

$$\xi = Ad\, a^{-1}\xi_o$$

has the following properties:

1. ξ takes values in Ω_d;
2. $a^{-1}da = 2i(\lambda^{-1}-1)\xi_d\, dz - 2i(\lambda - 1)\xi_{-d}\, d\overline{z}$;
3. $d\xi = [\xi, a^{-1}da]$.

Thus ξ is the solution of our Hamiltonian flows on Ω_d with initial condition ξ_o and a is the extended solution for the corresponding harmonic map of finite type.

The proofs of these assertions are similar to those in Section 4.2 and consequently left to the Reader.

5.3 Maps into k-symmetric spaces and Cartan embeddings

To make contact with the results of Section 3 and, in particular, to obtain primitive harmonic maps into k-symmetric spaces from the constructions of Section 5.1, we introduce a certain embedding of a k-symmetric space G/K into its group of isometries G. This will enable us to view primitive harmonic maps into G/K as maps into G to which the methods of Section 5.1 can be applied.

So let G/K be a k-symmetric space with involution τ and Maurer–Cartan form $\beta : T(G/K) \to \mathfrak{g}$. Define a map $\iota : G/K \to G$ by

$$\iota(g \cdot o) = \tau(g)g^{-1}.$$

It is clear that ι is well-defined. In fact, it is an immersion, as can be seen from the following

Lemma 5.4 *[11] Let θ be the Maurer–Cartan form of G. Then, for $x \in G/K$,*

$$\iota^*\theta_x = \tau_x\beta_x - \beta_x,$$

where $\tau_{g\cdot o} = Ad\, g \circ \tau \circ Ad\, g^{-1}$.

Moreover, if K is the fixed set of τ, then ι is an embedding. We call ι the *Cartan embedding* of G/K into G.

When $k = 2$, the Cartan embedding is well-known to be totally geodesic [17] so that if $\phi : M \to G/K$ is harmonic then $\iota \circ \phi : M \to G$ is also. Thus the methods of Section 5.1 apply to $\iota \circ \phi$. This is the approach to harmonic maps into symmetric spaces adopted in [14]. When $k > 2$, ι is no longer totally geodesic but all is not lost: a primitive map $\phi : M \to G/K$ still gives rise to a loop A_λ of the form (5.2) from which ϕ may be recovered.

For this, let $\phi : M \to G/K$ be a primitive harmonic map into a k-symmetric space and set $\delta = \phi^*\beta$. The equations (3.7) for a framing of ϕ can be written in terms of δ as

$$d\delta' - [\delta' \wedge \delta''] = d\delta'' - [\delta' \wedge \delta''] = 0.$$

However, these are precisely the coefficients of $(\lambda^{-1} - 1)$ and $(\lambda - 1)$ in the loop of 1-forms

$$A_\lambda = (\lambda^{-1} - 1)\delta' + (\lambda - 1)\delta''. \tag{5.6}$$

Thus a primitive map ϕ gives rise to a loop of flat 1-forms A_λ of the form (5.2). To recover $\psi = \iota \circ \phi$ from A_λ, observe that from Lemma 5.4 we get

$$\psi^*\theta = (\omega^{-1} - 1)\delta' + (\omega - 1)\delta''$$

whence

$$\psi^{-1}d\psi = A_\omega.$$

Example Let $k = 2$. Then $\omega = -1$ and $\alpha = \psi^{-1}d\psi = -2\delta$ so that the loop A_λ is just

$$\frac{1-\lambda^{-1}}{2}\alpha' + \frac{1-\lambda}{2}\alpha''$$

which is, of course, the loop associated to the harmonic map $\psi = \iota \circ \phi$ by Uhlenbeck.

We have seen in Section 5.1 how to produce loops (5.2) of flat 1-forms from commuting flows on Ω_d. When does this procedure give rise to primitive harmonic maps? This is a question with a pleasantly simple answer: it is just a matter of choosing the right initial condition for the flows as the following theorem shows [11] (see [14] for the case $k = 2$).

Theorem 5.5 *Fix $d \equiv 1 \bmod k$ and choose $\xi_o \in \Omega_d \cap \Lambda\mathfrak{g}_\tau$. Let $\xi : \mathbb{R}^2 \to \Omega_d$ satisfy*

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(\lambda^{-1} - 1)\xi_d]; \qquad \xi(0) = \xi_o$$

and let $\psi : \mathbb{R}^2 \to G$ be the map satisfying

$$\psi^{-1}\frac{\partial \psi}{\partial z} = 2i(\omega^{-1} - 1)\xi_d \qquad \psi(0) = 0.$$

Then $\psi = \iota \circ \phi$ for a primitive harmonic map $\phi : \mathbb{R}^2 \to G/K$.

It would be natural to call such maps primitive harmonic maps of finite type. However, this terminology has already been reserved for primitive harmonic maps which possess a framing which arises from commuting flows on $\Lambda_d \subset \Lambda\mathfrak{g}_\tau$. A priori, this is a different condition so, for now, we call the maps provided by Theorem 5.5 *primitive harmonic maps of G-finite type*. Later we shall see that G-finite type maps are indeed of finite type in the sense of Section 3.

Meanwhile, in view of the relation

$$\psi^{-1}\partial\psi/\partial z = (\omega^{-1} - 1)\delta'(\partial/\partial z),$$

a semisimple orbit condition on $\delta'(\partial/\partial z)$ implies one for $\psi^{-1}\partial\psi/\partial z$ and a simple adaptation of the proof of Theorem 5.1 then gives:

Theorem 5.6 *[11] Let $\phi : \mathbb{R}^2 \to G/K$ be a doubly periodic primitive harmonic map such that $\delta'(\partial/\partial z)$ takes values in a single semisimple $Ad\,G^{\mathbb{C}}$-orbit. Then ϕ is of G-finite type.*

If Φ is a framing for such a map, we have

$$Ad\,\Phi\alpha_{\mathfrak{m}} = \delta$$

so again, an orbit condition on δ' is implied by one on $\alpha'_{\mathfrak{m}}$and we therefore deduce the following analogue of Theorems 3.4 and 3.9:

Theorem 5.7 *Let $\phi : \mathbb{R}^2 \to G/K$ be a doubly periodic primitive harmonic map into a k-symmetric space. Suppose that some (and hence every) framing of ϕ has $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ taking values in a single semisimple $Ad\,K^{\mathbb{C}}$-orbit in $\mathfrak{g}_{-1}$. Then ϕ is of G-finite type.*

5.4 Finite vs. G-finite type

In this section, we shall show that a primitive harmonic map $\phi : \mathbb{R}^2 \to G/K$ of G-finite type is in fact of finite type. Together with Theorem 5.7, this will provide a proof of Theorems 3.4 and 3.9.

Before embarking on this however, let us pause to consider why this rather indirect method is needed to prove Theorems 3.4 and 3.9. The problem is that the approach of Section 3 deals with framings of ϕ and, moreover, framings of a rather special kind: for ϕ to be of finite type, the map $\xi : \mathbb{R}^2 \to \Lambda_d$ produces a framing with several restrictions:

1. $\alpha'_{\mathfrak{m}}(\partial/\partial z) = \xi_d$ so that, as we have seen, $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ takes values in a single $Ad\, K^{\mathbb{C}}$-orbit.
2. $\alpha_{\mathfrak{k}}(\partial/\partial z) = r(\xi_{d-1})$ so that $\alpha_{\mathfrak{k}}(\partial/\partial z)$ takes values in $\bar{\mathfrak{n}} \oplus \mathfrak{h}$.
3. In view of the equations for ξ we have
$$\frac{\partial \xi_d}{\partial z} = [\xi_d, (r-1)\xi_{d-1}]$$
which amounts to a differential relation between $\alpha'_{\mathfrak{m}}$and $\alpha'_{\mathfrak{k}}$.

These conditions are quite stringent: for example, for the primitive maps into flag manifolds discussed in Section 3.4 they are equivalent to the demand that the framing be a Toda frame.

Now suppose that ϕ does have such a framing with α doubly periodic. The argument of Theorem 5.1 can easily be adapted to produce the necessary $\xi : \mathbb{R}^2 \to \Lambda_d$. Indeed, this is the approach used in [7, 24] to treat primitive maps of flag manifolds since, in this case, by Corollary 3.15, a doubly periodic map with Toda frame has doubly periodic Toda frame. However, in the general case, we know of no direct method to equip a doubly periodic primitive harmonic map with a doubly periodic frame of the right kind. Hence the need to proceed indirectly.

This said, our proof is a simple adaptation of the Symes method we have been developing. So let $\phi : \mathbb{R}^2 \to G/K$ be a primitive harmonic map of G-finite type with $\psi = \iota \circ \phi : \mathbb{R}^2 \to G$. We know from Sections 5.2 and 5.3 that the G-finite type condition means that, for some $d \equiv 1 \bmod k$, there is $\xi_o \in \Omega_d \cap \Lambda\mathfrak{g}_\tau$ so that the map $g : \mathbb{R}^2 \to \Lambda G^{\mathbb{C}}$ $z \mapsto \exp(2i\lambda^{d-1}\xi_o)$ has a unique factorisation

$$g = ab \tag{5.7}$$

$a : \mathbb{R}^2 \to \Omega G$, $b : \mathbb{R}^2 \to \Lambda_+ G^{\mathbb{C}}$ with

$$a_\omega = \psi.$$

On the other hand, we may view g as a map $\mathbb{R}^2 \to \Lambda G^{\mathbb{C}}_\tau$ and use the Iwasawa decomposition of Theorem 4.1 to write

$$g = \tilde{a}\tilde{b}$$

with $\tilde{a} : \mathbb{R}^2 \to \Lambda G_\tau$, $\tilde{b} : \mathbb{R}^2 \to \Lambda_+ G_\tau^{\mathbb{C}}$. We know from Theorem 4.3 that $\tilde{a}$ is an extended framing of a primitive harmonic map of finite type given by

$$\pi \circ ev_1 \circ \tilde{a}$$

and it is our contention that this map is ϕ. For this, it suffices to show that

$$\iota \circ \pi \circ ev_1 \circ \tilde{a} = \psi.$$

From the definition of the Cartan embedding, we get

$$\iota \circ \pi \circ ev_1 \circ \tilde{a} = \tau(\tilde{a}_1)\tilde{a}_1^{-1} = \tilde{a}_\omega \tilde{a}_1^{-1}$$

where we have also used the symmetry $\tilde{a}_{\omega\lambda} = \tau \circ \tilde{a}_\lambda$. However, by the uniqueness of the decomposition (5.7), we have

$$a = \tilde{a}\tilde{a}_1^{-1}; \qquad b = \tilde{a}_1 \tilde{b}$$

so that

$$\tilde{a}_\omega \tilde{a}_1^{-1} = a_\omega = \psi.$$

We have therefore proved:

Theorem 5.8 *A primitive harmonic map $\mathbb{R}^2 \to G/K$ of G-finite type is of finite type.*

5.5 Coda: finite type vs. finite uniton number

There is another interesting class of harmonic maps from a Riemann surface to a Lie group G: these are the harmonic maps with finite uniton number discovered by Uhlenbeck [55]. The theory of these maps is fundamental to the twistorial approach to harmonic 2-spheres in symmetric spaces (see [15, 55, 62] and [27, 59] in this volume).

We conclude this article with a simple example which demonstrates that the class of finite type harmonic maps is essentially disjoint from that of maps with finite uniton number.

First let us recall Uhlenbeck's theory: let $\phi : M \to G$ be a harmonic map of a Riemann surface and suppose that ϕ admits an extended solution $\Phi : M \to \Omega G$ (which is certainly true if M is simply connected). The extended solution is unique up to left multiplication by an constant element of ΩG and we say that ϕ *has finite uniton number* if this constant can be chosen so that Φ takes values in the subspace of (Laurent) polynomial loops in ΩG of some fixed degree:

$$\Phi_\lambda(z) = \sum_{|m| \leq d} \lambda^m T_m(z), \tag{5.8}$$

for $z \in M$. Of course, to make sense of (5.8) we must view G as a matrix group.

Such polynomial loops admit a decomposition into "linear" factors which gives rise to a decomposition of the harmonic map into flag factors as described by Wood [59] in this volume.

The main result of Uhlenbeck is that any extended solution on a *compact* M can be normalised to take values in such a space of polynomial loops. As a consequence, any harmonic map $S^2 \to G$ has finite uniton number.

The situation for finite type harmonic maps is completely different: one can prove

Theorem 5.9 *A finite type harmonic map $\phi : \mathbb{R}^2 \to G$ with $\phi^{-1}\partial\phi/\partial z$ semisimple does not have finite uniton number.*

Corollary 5.10 *A doubly periodic harmonic map $\phi : \mathbb{R}^2 \to G$ with $\phi^{-1}\partial\phi/\partial z$ taking values in a semisimple orbit does not have finite uniton number.*

Note that a harmonic map $\phi : S^2 \to G$ has $\phi^{-1}\partial\phi/\partial z$ *nilpotent* by the usual holomorphic differentials argument.

The proof of Theorem 5.9 would take us too far afield so we shall content ourselves with presenting a simple example which turns out to contain the heart of the matter.

Let $A \in \mathfrak{g}^{\mathbb{C}}$ and suppose that $[A, \overline{A}] = 0$ (from which it follows that A is semisimple). Set $\xi_A = (\lambda^{-1} - 1)A + (\lambda - 1)\overline{A} \in \Omega_1$ and consider the geodesic $g : \mathbb{R}^2 \to \Lambda G^{\mathbb{C}}$ given by

$$g(z) = \exp(z\xi_A).$$

It is easy to see that the extended solution obtained by factorising g is given by

$$a(z) = \exp((\lambda^{-1} - 1)zA + (\lambda - 1)\overline{zA})$$

so that the corresponding harmonic map $\phi : \mathbb{R}^2 \to G$ of finite type is given by

$$\phi(z) = \exp(-2zA - 2\overline{zA}).$$

Thus ϕ is a product of geodesics with commuting generators. The corresponding map $\xi : \mathbb{R}^2 \to \Omega_1$ is constant so that we have a stationary point for the Hamiltonian flows.

Suppose now that ϕ has finite uniton number. Then there is a constant $\gamma \in \Omega G$ for which $\gamma a = \Phi$ has polynomial dependence on λ. But fixing $z_1 \neq z_2 \in \mathbb{R}^2$, this implies that

$$\Phi(z_1)^{-1}\Phi(z_2) = a(z_1)^{-1}a(z_2)$$

is a Laurent polynomial in λ. But

$$a(z_1)^{-1}a(z_2) = \exp((\lambda^{-1} - 1)(z_2 - z_1)A + (\lambda - 1)\overline{(z_2 - z_1)A})$$

which has an essential singularity at 0.

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