A Family of Singly-Periodic Minimal Surfaces Invariant Under a Screw Motion

Michael Callahan*
Mathematical Institute
Oxford University
Oxford OX1 3JP, England

David Hoffman†
Department of Mathematics
University of Massachusetts
Amherst, MA 01003, USA

Hermann Karcher‡
Math. Institut
Universitat Bonn
D-53115 Bonn, Germany

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Introduction

In [2], two of the authors and W. Meeks found examples of translation-invariant, embedded minimal surfaces with an infinite number of topological ends. For each $k > 0$, a surface M_k was constructed, which was invariant with respect to a translation parallel to the x_3-axis, and under a rotation group of order $k+1$ around the x_3-axis. The method of construction was generalized in [3] to obtain the first known examples of embedded, singly-periodic minimal surfaces with an infinite number of topological ends, invariant under screw-motions (with a nontrivial rotational component). For each integer $k > 0$ and angle θ, with $|\theta| < \frac{\pi}{k+1}$, there exists an embedded surface $M_{k,\theta}$ whose orientation-preserving symmetry group contains a rotation of order $k+1$ around the x_3-axis and a screw motion—a unit translation in the x_3-direction, followed by a 2θ rotation around it. See Figure 0. Although the surfaces $M_{k,\theta}$ were conceived

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as smooth deformations of the singly-periodic examples M_k, the proof in [3] did not construct these deformations. In fact, it left open the following questions:

1. Is the family $M_{k,\theta}$ smooth in θ? If so, are they deformations of M_k?

2. Are the surfaces $M_{k,\theta}$ unique?

3. Is there a surface $M_{k,\pi/(k+1)}$? To be more precise: the symmetry groups of the $M_{k,\theta}$ have a single limit as $\theta \to \pi/(k+1)$ and $\theta \to -\pi/(k+1)$. This group contains a translation but is different from the symmetries of $M_{k,0}$. The question becomes, is there a surface with this symmetry group and obvious properties generalizing the $M_{k,\theta}$?

A more down-to-earth question concerned the appearance of these surfaces. The existence proof in [3] used a minimax argument involving unstable minimal annuli spanning a fixed boundary. This technique provides no procedure for producing a fair numerical approximation. (See Section 2.1.) In [2], the conjugate surface method was used to construct the translation-invariant examples M_k. (See Section 2.2.) This technique, however, requires reflective symmetries, which the surfaces $M_{k,\theta}$ do not possess in general. Among the methods currently in use, only the Weierstrass Representation is left. This paper explains, among other things, how we were able to make the pictures of these surfaces, which appear at the beginning of Section 2.

The use of this method in computations that support and guide theoretical investigations in the study of embedded minimal surfaces is well-documented. ([1, 5, 6, 7, 11, 12, 13, 14, 17, 19, 28]. However, for the surfaces $M_{k,\theta}$, we were faced with a computational problem of a greater degree of difficulty. The reasons for this are two-fold. First, while the Gauss map g of a translation-invariant minimal surface descends to a meromorphic function on the quotient surface, the quotient by a twist-motion only allows $\frac{dg}{g}$ to descend. To produce a Gauss map on the quotient, suitable for integration in a generalized Weierstrass Representation, one has to integrate $\mu := \frac{dg}{g}$ and then use the multivalued function $\tilde{g} = \exp \int \mu$. This adds another level of complexity to the computational problem, and makes the associated period problem more difficult. Second, although the generalized Weierstrass Representation has been known for some years [17, 21], it has not been used frequently to compute examples. It was not clear, a priori, that everything would work smoothly; complications and obstacles could arise that would require significant modification of our computational techniques.

One of our goals was to expand our computational techniques to include regular use of the generalized Weierstrass Representation of the $M_{k,\theta}$. There are other situations in which existence of surfaces with very few symmetries is not known; this representation, coupled with numerical searches and exploratory graphics, could contribute a great deal to theoretical understanding. The current situation seemed like a good first test case for our experimental methods because we had very simple, regular end-behavior (all ends are planar) and we had an existence proof in hand. Thus we were: i) not likely to encounter unexpected theoretical difficulties, and; ii) assuming we did so, likely to actually find some examples. Of course we would also get computer-generated images of the surfaces themselves. In this paper, we describe these computations and present some of the calculated images of the surfaces $M_{k,\theta}$.

In Section 1, we give a quick survey of the known singly-periodic embedded minimal surfaces. In Section 2, we discuss the techniques used in their construction. Section 3 is devoted to the generalized Weierstrass Representation for minimal surfaces with planar ends, which are invariant under a screw motion. In Section 4 we derive the representation for the surfaces $M_{k,\theta}$. The numerical computations are described in Section 5. In regard to question 1: our work confirms, beyond reasonable doubt, that there is a smooth family of embedded, singly-periodic minimal surfaces that deform the surface M_k and are invariant under a screw-motion. (We have not carried out the details of a degree theory proof showing that the period problem can be solved.) In regard to question 2: the numerical evidence suggests that there is only one surface for each angle θ and, in fact, the parameters describing the surfaces are monotonic in θ. We note that while embeddedness of the surfaces $M_{k,\theta}$ comes free with the minimax method of [3], we do not know, a priori, that the surfaces computed here are the same ones, and so we must also prove embeddedness. However, a continuity argument from [8] can be used to show that the surfaces we construct, whose images clearly indicate they are embedded, are in fact embedded.

Question 3 is considered in Section 6. Here, we must report that we were initially misled by numerical results and computer graphics. For values of θ very near to $\frac{\pm\pi}{k+1}$, we were able to solve the period problem numerically. This seemed to give evidence for the existence of $M_{k,\frac{\pi}{k+1}}$. The computed $M_{k,\frac{\pi}{k+1}}$ appeared to have vertical planes of symmetry. Under this additional *assumption,* there is a much simpler parameterization using the traditional Weierstrass representation (that is, one where g is well-defined on the quotient). The period problem here is one dimensional, with one free parameter, and the period is much easier to compute. Computations here suggested that the period did *not* change sign, but did approach zero asymptotically for large values of the parameter. In fact, we found out that we could prove that the period problem was *not* solvable. We do this in Section 6. The question of the existence of a surface $M_{k,\frac{\pi}{k+1}}$ without reflectional symmetry is still open.

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1 The examples

1.1 The three types of singly-periodic properly embedded minimal surfaces

Singly-periodic, embedded minimal surfaces $S \subset \mathrm{R}^3$ are naturally classified by their behavior at infinity; that is by their number of topological ends. There are three possibilities: one, two, or infinitely-many ends. A typical example with *one topological end* is Scherk's singly-periodic surface, which is invariant under a translation. See Figure 1.1. Another, even older, example with one topological end is the helicoid, which is invariant under a 1-parameter group of screw motions. (To see that these surfaces have one end, just observe that the portion of the complete surface not drawn is connected; this is true for the complement of any compact portion of the surface.) This class of single-ended, singly-periodic, properly embedded minimal surfaces has been studied a great deal recently and many new examples have been found [9, 10, 17, 18, 21]. For every such example, there is a translation or a screw motion symmetry such that the symmetry subgroup generated by this motion has finite index in the total symmetry group. (Here we explicitly exclude embedded doubly-periodic surfaces, which are invariant under a lattice of translations.)

There is but one embedded minimal surface with *two topological ends* that is invariant under an infinite symmetry group, namely the catenoid. [24] The catenoid is the unique nonplanar minimal surface of rotation. (See [4, 14, 22].)

The third class, the one that will concern us here, is the class of properly embedded, singly-periodic minimal surfaces with more than two ends or, if you like, more than one end and *not* the catenoid. The surfaces in this class all have an *infinite number of ends* and, furthermore, any end that is topologically an annulus must be *asymptotic to the exterior of a compact set in a flat plane*. Such an end is called *planar*. The basic appearance of such surfaces is governed by the following structure theorem.

Theorem 1.1 ([3]) *Suppose S is a properly embedded minimal surface in R^3 with an infinite symmetry group and more than one topological end. Then, if S is not the*

Figure 0: Surfaces $M_{k,\theta}$ for $k = 1$ and increasing values of θ.

Figure 0 (continued): Basic pieces of $M_{1,\theta}$, corresponding to the shaded region in the parameterization domain (explained in Section 4).

Figure 1.1: Scherk's singly-periodic surface, which has one topological end, and a twisted deformation found by Karcher and Pitts.

catenoid,

1. *The symmetry group of S contains an infinite cyclic subgroup, generated by a screw motion, s, which has finite index;*

2. *M has an infinite number of ends. Any end which is topologically annular is planar. If the screw motion s has nontrivial rotational part, the translation is orthogonal to the planar ends;*

3. *$\Sigma = S/s$ has finite total curvature if an only if Σ has finite topology, in which case Σ is conformally a compact Riemann surface $\overline{\Sigma}$ punctured at a finite number r of points. Moreover,*

$$\int_{\Sigma} K dA = 2\pi(\chi(\Sigma) - r)\,.$$

1.2 Riemann's examples

The classical examples, and for many years the only ones known of this type, were the surfaces $\mathcal{R}$ of Riemann. They form a one parameter family, and each $\mathcal{R}$ is fibred by round circles. That is, the intersection of $\mathcal{R}$ with any plane parallel to a planar

Figure 1.2: A Riemann example and its symmetry. Dark lines lie in the surface and alternate with light lines, normal to the surface, about which the surface rotates into itself.

end is a round circle, the only exception being those planes actually asymptotic to the ends. These planes intersect $\mathcal{R}$ in straight lines. All these lines are parallel and lie in a single plane P. The surfaces possess a single vertical plane of symmetry V. The orientation-preserving-translation subgroup of the symmetry group is generated by a vector T in the direction of $P \cap V$, whose length is equal to twice the distance between successive lines on $\mathcal{R}$. (Translation by $\frac{1}{2}T$ is orientation-reversing.)

By the Schwarz Reflection Principle, a minimal surface that contains a line is symmetric under rotation by π about that line. Rotation about two successive lines generates T. Together with reflection in V, this gives all the evident symmetries of $\mathcal{R}$. However, there is one additional symmetry: The lines in P parallel to the lines in $P \cap \mathcal{R}$ and halfway between them meet $\mathcal{R}$ orthogonally, and rotation by π about these lines is a symmetry of $\mathcal{R}$. (Such a line is referred to as a *normal symmetry line.*) See Figure 1.2.

Modulo the translation T, $\mathcal{R}$ is a genus one surface with two ends. As a consequence of Theorem 1.1, 3., $\mathcal{R}/T$ must be, conformally, a torus punctured in two points. Because of all its inherited reflectional symmetry, it is a rectangular torus.

1.3 Modern translation invariant examples

For over one hundred years, these were the only known examples. Recently, an infinite sequence M_k of new examples was constructed [2]. See Figure 1.3.1. After normalization, these examples are invariant under a vertical translation T of length equal to 1. They share the following properties:

(1.3.1.) M_k has flat ends asymptotic to horizontal planes at integral and half-integral heights;

(1.3.2.) Horizontal planes intersect M_k in closed Jordan curves, except those at integral or half-integral height. These meet M_k in $k+1$ lines. These lines intersect in a single point and make equal angles there;

(1.3.3.) In each plane at quarter and three-quarter integral height, there are $k+1$ equally-spaced normal symmetry lines;

(1.3.4.) M_k possesses $k+1$ vertical planes of reflective symmetry;

(1.3.5.) Horizontal planes at quarter or three-quarter integral height are planes of reflective symmetry of M_k.

The quotient surface M_k/T has genus $2k+1$ and two planar ends. These examples have been generalized by inserting "Neovius handles" at the half integral levels. The resulting surfaces have all the properties of M_k, except that the quotient by T produces a surface of genus $4k+1$ with 2 planar ends [15]. See Figure 1.3.2.

Remark 1.1 *These examples, together with the Riemann examples were* all *the known examples of singly-periodic, embedded minimal surfaces with more than one end. To get a feeling for the property of examples note, for instance that all of these examples have odd genus. Recently, F. Wei has discovered a family $\mathcal{W}$ of embedded singly-periodic minimal surfaces that have genus-two quotients [25, 27, 26]. These generalize the Riemann example $\mathcal{R}$. Morphologically, they look like the surfaces $\mathcal{R}$ but with i) alternating interplanar distances ii) in the narrower of the two slabs the tube grows a handle as in the Hoffman-Wohlgemuth examples in Figure 1.3.2.*

2 The methods of construction

The surfaces M_k can be constructed in three different ways: by a minimax procedure using a sequence of least area annuli; by solving the Plateau Problem for an

Figure 1.3.1: The surface M_1 and its symmetry lines and planes.

Figure 1.3.2: Hoffman-Wohlgemuth examples constructed by adding Neovius handles to the M_ks.

appropriate polygonal boundary; and by the Weierstrass Representation Theorem. In [2], all these methods are discussed.

In [3] the existence of screw-motion-invariant embedded minimal surfaces is established:

Theorem 2.1 ([3]) *For every positive integer k and angle θ, $0 < |\theta| < \frac{\pi}{k+1}$, there exists a properly embedded minimal surface $M_{k,\theta}$ invariant under a screw motion of the form*

$$s \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} R_{2\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2.1}$$

where $R_{2\theta}$ is rotation in the (x_1, x_2)-plane by 2θ. These surfaces satisfy (1.3.1), (1.3.2), (1.3.3).

2.1 The minimax procedure

The existence of these surfaces is proved by a minimax procedure, generalizing the one used for the surfaces M_k. The idea is to produce a properly immersed minimal annulus bounded by $k+1$ straight lines in $\{x_3 = 0\}$ and $\{x_3 = \frac{1}{2}\}$.

Lemma 2.1 *Let L_k be a set of $k+1$ lines, contained in the (x_1, x_2)-plane, that meet in equal angles at the origin, $k \geq 1$. Let $L_{k,\theta}$ denote the image of L_k under the composition of a rotation around the x_3-axis by θ and a vertical translation by $(0, 0, \frac{1}{2})$. Then $L_k \cup L_{k,\theta}$ is the boundary of a properly immersed minimal annulus $A_{k,\theta}$ satisfying:*

1. *$A_{k,\theta} - \{(0,0,0), (0,0,\frac{1}{2})\}$ is an embedded surface in the slab $\{0 \leq x_3 \leq \frac{1}{2}\}$.*
2. *$A_{k,\theta}$ is invariant under rotation around the x_3-axis by $\frac{2\pi}{k+1}$.*
3. *$A_{k,\theta}$ is invariant by rotation by π around the lines in $\{x_3 = \frac{1}{4}\}$ obtained from L_k by first rotating around the x_3-axis by $\frac{\pi}{k+1} + \frac{\theta}{2}$ and then translating vertically by $(0, 0, \frac{1}{4})$.*

The desired surface $M_{k,\theta}$ is produced from $A_{k,\theta}$ by Schwarz reflection around its line boundaries.

The method of proof involves first finding least-area annuli bounded by "bowties" of the sort in Figure 2.1 separated by a distance $t > 0$. This solution is then rescaled so that its maximum Gauss curvature is 1 in absolute value. In this family it is

Figure 2.1

shown that one can choose a sequence of solutions with $t_i \to 0$, as well as rescaling and translation factors so that there is a subsequence that converges to the desired surface $A_{k,\theta}$.

However, this gives no means of producing an image of $A_{k,\theta}$. While it is possible to solve numerically the boundary value problem for the bowties in Figure 2.1, and then let the size of the bowtie expand, this is not the same procedure and the limit surface is *not* $A_{k,\theta}$. Thus, these solutions are not close to the desired surface. In fact, it is not hard to show that the limit surface extends by Schwarz reflection to the surface in Figure 1.1, which does not even have planar horizontal ends. A key difference is that the basic building block of this surface is stable, whereas $A_{k,\theta}$ is *not* stable.

Remark 2.1 *This method proves existence, but uniqueness is usually extremely difficult to establish for surfaces found by this technique. Moreover, the smooth dependence of $A_{k,\theta}$ (and hence $M_{k,\theta}$) on θ has not been proved, although this would follow from uniqueness of the $A_{k,\theta}$. One of our motivations in carrying out the numerical experiments described below is to demonstrate, experimentally, the smooth dependence of $M_{k,\theta}$ on θ.*

Remark 2.2 *The existence proof in [3] fails when the twist angle for $A_{k,\theta}$ is equal to $\frac{\pi}{k+1}$ in absolute value. For this and other reasons, we believed that the surface $M_{k,\frac{\pi}{k+1}}$ did not exist and that as $|\theta|$ approached $\frac{\pi}{k+1}$, the surfaces $M_{k,\theta}$ drifted off to some degenerate limit. The computations described in Section 5 were inconclusive, but did suggest that if the surface existed in the family we constructed, then it would regain reflectional symmetry. In Section 6, we prove this is* not *the case.*

Figure 2.2: A basic piece of the surface M_1 (see Figure 1.3.1), bounded by curves lying in planes of reflective symmetry, and its conjugate surface, bounded by line segments and rays.

2.2 The conjugate surface method

Because the translation invariant surfaces M_k possess many reflective symmetry planes, it is possible to decompose them into simply-connected pieces, bounded by planar geodesic principal curvature lines. This means that the conjugate surface to the basic piece is bounded by line segments and rays. See Figure 2.2. By solving the Plateau problem with this conjugate boundary, it is possible to prove the existence of the surfaces M_k, and such a method could be used to produce images of M_k. (See [2] and [14].) For the twisted surfaces $M_{k,\theta}$, $0 < |\theta| < \frac{\pi}{k+1}$, there are *no* planes of reflective symmetry and this process cannot even be started. Moreover, the symmetries of $M_{k,\theta}$ cannot be used to find a subdomain that is stable, and which will produce the surface by Euclidean motions.

2.3 The Weierstrass Representation

In subsequent sections we will: develop the theory for a generalized Weierstrass Representation for screw-motion-invariant, singly-periodic minimal surfaces with planar ends (Section 3); describe how to apply this theory to the surfaces $M_{k,\theta}$ (Section 4);

and then carry out the numerical experiments, describing the techniques and results (Section 5). As necessary background for this, we will briefly review the standard theory as applied to the untwisted examples M_k.

Suppose $\overline{\Sigma}$ is a compact Riemann surface. One should imagine $\overline{\Sigma}$ as, for example, the conformal compactification of the quotient M_k/T where T is the group of translational symmetries of M_k. Specify in addition $\mathcal{E} = \{e_1 \dots e_r\} \subset \overline{\Sigma}$ an even number of points, g a meromorphic function and dh a holomorphic one form on $\overline{\Sigma}$ satisfying the following properties:

i) $g(e_i) = 0, \infty$, alternating on the e_i, $1 \leq i \leq r$, with branching order $m_i \geq 2$ at e_i;

ii) dh has: a zero of order $m_i - 2$ at e_i if $m_i > 2$; a zero of order $|n|$ wherever g has a pole or zero of order n on $\Sigma = \overline{\Sigma} - \mathcal{E}$; and is regular everywhere else.

Then

$$X(p) = \mathrm{Re} \int_{p_0}^{p} \Phi$$

where

$$\Phi = (\frac{1}{2}(g^{-1} - g)dh\,, \frac{i}{2}(g^{-1} + g)dh\,, dh)\,, \tag{2.3.2}$$

defines a conformal, minimal, multi-valued immersion of Σ into R^3. That is, it defines a conformal, minimal immersion of the universal cover $\tilde{\Sigma}$ of Σ into R^3.

We would like this map to descend to a particular covering $\Sigma' \to \Sigma$ corresponding to the covering $M_k \to M_k/T$. Here, $\Sigma = \Sigma'/\mathbf{T}$, where $\mathbf{T}$ is some infinite cyclic group of conformal diffeomorphisms acting freely on Σ'. Such a covering can be specified by giving an element Δ in $H^1(\Sigma, \mathrm{Z})$, i.e. a homomorphism $H_1(\Sigma, \mathrm{Z}) \to \mathrm{Z}$. Here, if α is a closed curve in Σ, the end-points of a lift of α to Σ' are related by $T^{\Delta([\alpha])}$, where T is a generator of $\mathbf{T}$. In fact, since the action of the translations on M_k extends to the compactification of M_k obtained by filling in points at the ends, we require that $\Sigma' \to \Sigma$ extend to a covering $\overline{\Sigma'} \to \overline{\Sigma}$, where T again acts freely on $\overline{\Sigma'}$. Hence the covering is specified by an element Δ in $H^1(\overline{\Sigma}, \mathrm{Z})$. Note that for Σ' to be connected, Δ must be primitive.

The map X defines a homomorphism $\delta X \colon H_1(\overline{\Sigma}, \mathrm{Z}) \to \mathrm{R}^3$, whose image is a group of translational symmetries of the minimal surface $X(\tilde{\Sigma})$. We want X to descend to Σ', where the deck transformations of the covering $\Sigma' \to \Sigma$ are generated by a vertical translation. Hence the translational symmetries of $X(\tilde{\Sigma})$ should be just those we desire $X(\Sigma')$ to have, and no others. That is, if α is a closed curve in Σ, and α'

a lift to $\tilde{\Sigma}$, we require that the endpoints of $X(\alpha')$ differ by a vertical translation by the vector $(0,0,\Delta([\alpha]))$. This is the condition

$$\delta X = (0,0,\Delta). \tag{2.3.3}$$

In fact, this is nothing more than a restatement of the Weierstrass-Osserman Representation Theorem for complete minimal surfaces of finite total curvature, with one assumption, and one modification. The assumption is that all the ends of the quotient surface in R^3/T are flat. The modification is the requirement that, instead of all periods of (2.3.2) being zero, there is one vertical period. For details and proofs, see [7], [20], or [22]. Because we wish to produce an embedded surface with planar ends, the orientation of the ends must alternate and g must be branched at the ends. This is condition (2.3.1)i). Condition (2.3.1)ii) is the same as in the finite total curvature case.

In the next section, we will develop this representation in the more general setting of a screw-motion-invariant surface. Here we have described the special case when the twist is just a translation. The critical difference—and the one that creates all the difficulty—is the following: given a screw-motion-invariant minimal surface the Gauss map $g = \sigma \circ N$ is *not* well defined on the quotient. Thus, we cannot specify a meromorphic function to be integrated as in (2.3.1). However, $\frac{dg}{g}$ *does* descend to the quotient as a meromorphic one-form, which we will label μ. We can then try to reproduce S as a multivalued map from the quotient by using $\tilde{g} = \exp \int \mu$ in place of g. However $\tilde{g}$ is itself multivalued and subject to period problems.

3 The generalized Weierstrass Representation for screw-motion invariant surfaces

Suppose S is a complete minimal surface in R^3 that is invariant under a screw-motion of the form (2.1), where $R_{2\theta}$ is rotation by 2θ in the first two coordinates. We will condense our notation, writing $p \in \mathrm{R}^3$ as $p = (z, x_3)$, $z = x_1 + i\,x_2$ and

$$s(z, x_3) = (e^{2i\theta} z, x_3 + 1)\,. \tag{3.1}$$

Suppose S is embedded. Then by Theorem 1.1, $\Sigma\colon = S/s$ has finite topology if and only if it has finite total curvature, and then Σ is conformally a compact Riemann surface $\overline{\Sigma}$ punctured in a finite number of points $\mathcal{E} = \{e_1, \ldots, e_r\}$. If we assume further that S has more than one topological end, then all the annular ends of $S \subset \mathrm{R}^3$, and therefore of $\Sigma \subset \mathrm{R}^3/s$, are planar. A planar end is, conformally, a punctured disk, asymptotic to a plane $x_3 = c$, and representable as a graph of the form

$$x_3 = c + \frac{a\,x_1 + b\,x_2}{|(x_1, x_2)|^2} + o^2(|(x_1, x_2)|)\,.$$

Our assumptions of vertical translation in s (a convenient convention) and embeddedness dictate that the limit tangent plane is horizontal. (We note that for singly-periodic embedded minimal surfaces invariant under a screw-motion, it is always the case that the translational part of s is orthogonal to the ends, provided the rotational part of s is nonzero. See Theorem 1.1. If S is translation-invariant, this need not be the case, as the examples of Riemann show. See Figure 1.2 and [2].)

In contrast to what happens for minimal surfaces invariant under *translation*, the Gauss map of S does not in general descend to the compactified quotient $\overline{\Sigma}$. However, certain properties of the Gauss map on S persist in the quotient. For example, if the Gauss map $g = \sigma \circ N$ is vertical at a point $q \in S$, then $g(q) = 0, \infty$ and $g(s^i p) = 0, \infty$ for all $j \in Z$. Hence we may speak of points of Σ as having a vertical normal, even though the Gauss map does not descend to Σ as a mapping to S^2. We will label this collection of points by $\mathcal{V}$. Note that the points of $\mathcal{E} \subset \overline{\Sigma}$, corresponding to the planar ends, are vertical points of g on $\overline{\Sigma}$. Similarly, we may speak about the *order of* g at a point of Σ.

The Gauss map g is meromorphic on S and must be branched at a planar end. Let m_i be the branching order at $e_i \in \mathcal{E}$. We will use the convention that the order of g at a zero is positive and at a pole negative. Because S is embedded, g must alternate between 0 and ∞ on the ends of S, ordered by height in R^3. Therefore, the number of ends of $\overline{\Sigma}$ is even and the ends can be ordered so that $m_1, \ldots, m_r$ alternate in sign. Because on S

$$g(s^k p) = e^{2ik\theta} g(p), \qquad k \in \mathrm{Z},$$

it follows that $\mu := \frac{dg}{g}$ is well-defined on $\overline{\Sigma}$. At vertical points of $\overline{\Sigma}$, μ has simple poles. The residue of μ is the order of the pole or zero of g. At a branch point of g on Σ, μ has a zero whose order is equal to the absolute value of the branching order of g at the point in question.

The height function on S is harmonic and its complex differential is holomorphic,[1] with zeros precisely at the vertical points $p \in S$, of order equal to n , where $\pm n$ is the order of the zero/pole of g at p. We will denote this height function by h and its *complex differential* by dh. At a planar end, dh has a zero of order equal to $|m| - 2$,

[1]By the complex differential of a real-valued harmonic function f on S, we mean $df + idf^*$ where f^* is the, locally defined, harmonic conjugate of f; df^* is globally defined.

where m is the order of the zero or pole of g at the end. The one-form dh is clearly invariant under the action of S and hence defines a holomorphic one-form on $\overline{\Sigma}$ with zeros at the vertical points $\mathcal{V} \subset \Sigma$, of order $|n_j|$, and zeros at ends $e_i \in \mathcal{E}$ of order $|m_i| - 2$.

To summarize, let $\overline{\Sigma} = \overline{S}/s$, $\Sigma = S/s$, $\mu = \frac{dg}{g}$ on $\overline{\Sigma}$, and dh the complex differential of the height function on $\overline{\Sigma}$:

The poles of μ on $\overline{\Sigma}$ are all simple and all the residues are integers. All the points of $\mathcal{E}$ are poles of μ and the signs of the residues alternate. That is, if m_i is the residue of μ at $e_i \in \mathcal{E}$, $m_i m_{i+1} < 0$. The other poles of μ on Σ are called vertical points $\mathcal{V} = \{v_1, \ldots, v_s\}$ with residues n_j, $j = 1, \ldots, s$. (3.2)

The one-form dh is holomorphic on $\overline{\Sigma}$ and its zeros on Σ consist precisely of the vertical points $\mathcal{V}$ with order $|n_j|$, $j = 1, \ldots, s$. At any end-point $e_i \in \mathcal{E}$ where $|m_i| \geq 3$, dh has a zero of order $|m_i| - 2$. (3.3)

Furthermore,

$$\Sigma_1^s |n_j| + \Sigma_1^r |m_i| = 2(r + k - 1)\,, \tag{3.4}$$

where $k = genus\,\overline{\Sigma}$. This last formula is simply the Euler characteristic of $\overline{\Sigma}$, computed by summing the zeros of the holomorphic one-form dh.

Because we have normalized the vertical translational part of s to have length 1, a closed curve $\alpha \subset \Sigma$ must satisfy

$$\int_\alpha dh \in \mathrm{Z}$$

and it is straightforward to verify that the vertical displacement map

$$\Delta\colon H_1(\Sigma, \mathrm{Z}) \to \mathrm{Z}, \qquad \Delta([\alpha]) = \int_\alpha dh \tag{3.5}$$

is well-defined and an additive homomorphism.

Locally, $g = \exp \int \frac{dg}{g}$ on S. Suppose α is a closed curve on Σ. Then

$$F(\alpha) := \exp \int_\alpha \frac{dg}{g}$$

is the ratio of the values of g at the end-points of any lift of α to S. End-points are related by the action of s^k for some $k \in \mathrm{Z}$, and so $F(\alpha)$ is unitary. It is easily seen that F defines a multiplicative homomorphism:

$$F\colon H_1(\Sigma, \mathrm{Z}) \to S^1\,,$$

and

$$F([\alpha]) = e^{2i\Delta([\alpha])\theta}\,, \tag{3.6}$$

where 2θ is the twist angle of s. This is because the endpoints of the lift of α must differ in height by $\Delta([\alpha])$. Therefore the lifts differ by the action of $s^{\Delta([\alpha])}$.

The metric on S is given by:

$$ds = \frac{1}{2}(|g| + |g|^{-1})\,|dh|\,.$$

Completeness of S is equivalent to the condition that

$$\int_{\widetilde{\alpha}} (|g| + |g|^{-1})|dh| = \infty\,, \tag{3.7}$$

where α is any path in Σ with $\lim\limits_{t\to\infty} \alpha(t) \in \mathcal{E} \subset \overline{\Sigma}$.

Theorem 3.1 *Suppose S is a complete embedded minimal surface in R^3 that is invariant under the screw motion $s(z,t) = (e^{2i\theta}z, t+1)$. Suppose further that S has more than one topological end and $\Sigma = S/s$ has finite topology. Then all the ends of S are planar (asymptotic to horizontal planes), and Σ is a compact Riemann surface $\overline{\Sigma}$ punctured in a finite number of points (one for each end). Σ has finite total curvature equal to $-4\pi(k+r-1)$, where k is the genus of $\overline{\Sigma}$ and r is the number of endpoints.*

Let $g\colon S \to \mathrm{C}\cup\{\infty\}$ be the stereographic projection of the Gauss map of S and let h be the height function on S. Then $\mu = \frac{dg}{g}$ is a meromorphic one-form and dh is a holomorphic one-form on S. They descend to a meromorphic and a holomorphic one-form on $\overline{\Sigma}$. These forms satisfy the conditions (3.2), (3.3), (3.4), (3.5) and (3.6).

The Gauss map g of S may be realized on $\overline{\Sigma}$ by

$$\widetilde{g} = e^{\int \mu}\,, \tag{3.8}$$

which is a multivalued meromorphic mapping from $\overline{\Sigma}$ to $\mathrm{C}\cup\{\infty\}$, and may be used to reconstruct S as follows:

The mapping $X\colon \Sigma \to \mathrm{R}^3$ defined by the Weierstrass representation

$$\begin{aligned} X(p) &= Re\int_{p_0}^{p} \Phi, \\ \Phi &= ((\widetilde{g}^{-1} - \widetilde{g})\frac{dh}{2},\ i(\widetilde{g}^{-1} + \widetilde{g})\,\frac{dh}{2},\ dh) \end{aligned} \tag{3.9}$$

is a multivalued conformal minimal embedding whose range is S. Specifically, if α is a closed curve on Σ, α' is a lift of α to $\tilde{\Sigma}$ (the universal cover of Σ) with endpoints p_0, p_1, then

$$X(p_1) = s^{\Delta([\alpha])}p_0. \tag{3.10}$$

Note that this relation implies that the minimal immersion $\tilde{\Sigma} \to \mathrm{R}^3$ descends to a minimal immersion $\Sigma' \to \mathrm{R}^3$, where Σ' is the covering specified by the cohomology class Δ; (3.9) is analogous to (2.3.3) in the translational case.

Conversely, suppose one has a compact Riemann surface $\overline{\Sigma}$;

$$\mathcal{E} = \{e_1 \dots e_r\} \subset \overline{\Sigma};$$
$$\mathcal{V} = \{v_1 \dots v_s\} \subset \Sigma = \overline{\Sigma} - \mathcal{E}\,;$$

integers m_i, n_j, $1 \le i \le r$, $1 \le j \le s$; a meromorphic one-form μ; a holomorphic one-form dh, a cohomology class $\Delta \in H^1(\Sigma, \mathrm{Z})$, and an angle θ satisfying (3.2), (3.3), (3.4), (3.5) and (3.6). Then the multivalued function $\tilde{g}$ defined by (3.8) defines a multivalued, conformal, minimal immersion X as in (3.9). That is, it defines a conformal minimal immersion $\tilde{\Sigma} \to \mathrm{R}^3$ where $\pi\colon \tilde{\Sigma} \to \Sigma$ is the universal cover of Σ. The image $S = X(\tilde{\Sigma})$ is invariant under s, all of its annular ends are planar ends, its vertical points are $X(\pi^{-1}\mathcal{V})$ and its ends are $X(\pi^{-1}\mathcal{E})$.

The immersion X will descend to Σ', where Σ' is the covering specified by Δ, provided (3.10) holds. In that case, $X\colon \Sigma \to \mathrm{R}^3/s$ is well-defined and proper, and a punctured neighborhood of a point $e \in \mathcal{E} = \overline{\Sigma} - \Sigma$ is mapped into a planar end of S/s.

Remark 3.1 *If all the conditions in the second part of Theorem 3.1 are satisfied, X is an embedding of suitably small punctured neighborhoods of the points in $\mathcal{E}$. However it is not necessarily true that $X(\Sigma)$ is one-to-one; the surface $X(\Sigma) \subset \mathrm{R}^3/s$ may fail to be embedded.*

4 The generalized Weierstrass Representation of the surfaces $M_{k,\theta}$

We wish to find the Weierstrass representation for the surfaces $M_{k,\theta}$ described in Section 2, Theorem 2.1. For the moment, we assume the surface, as described, exists. We will use its geometric properties to deduce its Weierstrass representation. Once this is done, we must verify that this representation, with appropriate choice of parameters, actually produces a surface with *all* the required properties.

4.1 Symmetry and the underlying Riemann surface

The screw motion

$$S(z,t) := (e^{2i\theta}z, t+1) \tag{4.1}$$

acts on the minimal surface $M_{k,\theta}$ in such a way that the quotient is a surface $\Sigma := M_{k,\theta}/S$ of genus $2k+1$, with two planar ends in the space form $\mathbb{R}^3/S$. The surface has further symmetries that descend to the quotient Σ:

i) the rotation ρ around the x_3-axis by an angle $\frac{2\pi}{k+1}$;

ii) the $180°$ rotations around the other (horizontal) normal symmetry lines. There are $k+1$ of them at each level half-way between neighboring ends, i.e., at heights $\frac{1}{4}+\frac{1}{2}\mathbb{Z}$;

iii) the $180°$ rotations about the lines on the surface, of which there are $k+1$ at each half-integer height. (Each $(k+1)$-tuple of lines meets on the x_3-axis, and these are the only points of the surface on the x_3-axis.)

The generalized Weierstrass data $\frac{dg}{g}$ and dh are invariant under the rotations around the x_3-axis, and therefore pass to the quotient $\overline{\Sigma}/\rho$. Note that the Weierstrass representation produces the minimal surface by integration of differential forms; if these forms are lifts from a quotient surface, then we might as well determine and integrate these forms on the quotient. This is a simplification, since the Riemann-Hurwitz formula

$$2-2(2k+1)=\chi(\overline{\Sigma})=(k+1)\chi(\overline{\Sigma}/\rho)-4k$$

implies that the quotient $T := \overline{\Sigma}/\rho$ is a torus, independent of k. Moreover, this torus is rectangular, as the following symmetry argument shows.

The surface $M_{k,\theta}$ has $k+1$ horizontal straight lines through each point where it meets the x_3-axis. (See iii) above.) Each such $(k+1)$-tuple of lines is identified to one line in T by the rotation ρ. The $180°$ rotations of $M_{k,\theta}$ around these lines pass to an orientation-reversing involution of the torus T. The vertical points of $M_{k,\theta}$ project to two points on $\overline{\Sigma}$ that are fixed by ρ, and hence they project to two points on T. Therefore, the lines on $M_{k,\theta}$ project to two disjoint components of the fixed-point set of an orientation-reversing involution of T. This is only possible on a rectangular torus. There, the involution can be visualized as reflections in, say, a horizontal line on the fundamental rectangle for the torus. Our symmetry considerations have identified the relevant Riemann surfaces on which the Weierstrass differential forms have to be constructed; rectangular tori form a one parameter family and this will be a free parameter in the representation.

Note that on each horizontal line in the fixed point set of the aforementioned involution of T, there is a point corresponding to the vertical normal on Σ and a point corresponding to the planar end. Together with the conformal parameter determining the rectangular torus, this would give five real parameters. However, further symmetry considerations will show that there are only three independent real parameters.

Consider rotations around the horizontal normal-symmetry lines at quarter-integral heights $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$. These descend to a single, orientation-preserving involution, r, of T that fixes four points and interchanges the lines described above. This involution must be rotation by π about he four fixed points in the torus, which we can choose, without loss of generality, to be the half-period points. The horizontal lines discussed in the previous paragraph must pass through *quarter-period points.* They are indicated by the dashed lines in Figure 4.2. Since the involution interchanges the ends, choosing the position of the end on one of the horizontal lines determines its position on the other. The same is true of the points with vertical normal. Hence, there are only three independent real parameters that determine the conformal structure, the position of the ends, and the vertical points.

We observe that, by Riemann-Hurwitz, the torus modulo this involution T/r is a sphere. This fact will be important later on.

4.2 The differential forms dh and $\frac{dg}{g}$

The Gauss map of $M_{k,\theta}$ has order k at the vertical points and order $k+2$ at the planar ends. This means that dh on $\overline{\Sigma}$ has zeros of order k at these points. Since the branching order of $\overline{\Sigma} \to T$ is $k+1$ at these points, this shows that dh descends to a *holomorphic* form on T. Hence there are no further choices: dh is a constant multiple of the translation-invariant, standard differential form on the torus: $dh = c\,du\,,\ u \in T = \mathbb{C}/\Gamma$, where c is a nonzero complex constant. The magnitude of c is irrelevant (it rescales the surface). However, the real part of dh must be zero when applied to tangent vectors of the horizontal lines in T that correspond to the lines in the surface. This is because $x_3 = \mathrm{Re}\int dh$. Hence c must be purely imaginary. Without loss of generality, we assume that $c = i$:

$$dh = i du\,. \tag{4.2}$$

As observed in Section 3, the one-form $\frac{dg}{g}$ on $\bar{\Sigma}$ has a simple pole at the points with vertical normal vector (including the ends). The residue of $\frac{dg}{g}$ is the order of the pole or zero of g. In our case, the residue at a point with vertical normal in Σ is $\pm k$ and the residue at an end is $\pm(k+2)$. As observed in Section 4.1, $\frac{dg}{g}$ passes to the quotient T. It has simple poles at the points corresponding to the vertical normals

Figure 4.2. Portrait of $\frac{dg}{g}$. The poles occur at the vertical points v_1, v_2 and the ends e_1, e_2. Lines of $M_{k,\theta}$ lie above dashed lines, on which $\frac{dg}{g}$ is real.

and the ends. Since the branching order of $\bar{\Sigma} \to T$ is $k+1$ at these points, the residues become $\pm\frac{k}{k+1}$ and $\pm\frac{k+2}{k+1}$ respectively. Using the information in this paragraph, we will now write down a formula for $\frac{dg}{g}$, in terms of elliptic functions on T.

4.3 The elliptic function Z

In section 4.1, we noted that T/r was conformally S^2. After we identify S^2 with $\mathrm{C} \cup \{\infty\}$ by choosing three points in S^2 to be mapped to $0, -1, \infty$, we may consider the projection $T \to T/r = S^2$ to be an elliptic function of degree two. We will choose this identification as follows. The projection of the vertex A of the rectangle goes to ∞, and the projection of the mid-point B of the vertical edge goes to zero. The mid-points M_1, M_2 between these points project to a single point on S^2. This point is identified with -1. Now

$$Z\colon T^2 \to S^2 = \mathrm{C} \cup \{\infty\}$$

is a well-defined elliptic function.

Consider the symmetries of Z. Reflection of the torus in the horizontal (resp. vertical) line through B is antiholomorphic; it fixes B and A with values $0, \infty$ and it interchanges (resp. fixes) M_1, M_2 with value $Z(M_i) = -1$. This implies:

$$Z(R(P)) = \overline{Z(P)}\,,$$

where R is reflection in either the horizontal or the vertical line through B. The values of Z are therefore real on the fixed-point sets of these two torus reflections; i.e. Z is real on the edges and the symmetry axis of the fundamental rectangle. In particular, the other two branch values of Z are real. (They occur at the midpoint of the rectangle and the midpoint of its horizontal edge.)

Figure 4.3. Values of the elliptic function Z on the fundamental rectangle. Branch values are $0, \infty, \lambda \leq \frac{1}{\lambda}$. Z is real on solid lines, *unitary* on dashed lines.

Similarly, reflection in the horizontal line through M_1 (which is the same involution of T as reflection in the horizontal line through M_2) interchanges B and A and fixes the M_i. Recall that $Z(M_i) = -1$. $Z(A) = \infty$ and $Z(B) = 0$. Hence

$$Z(\widetilde{R}(P)) = (\overline{Z}(P))^{-1},$$

where $\widetilde{R}$ is this reflection of T. The lines are the fixed point set of these reflections. It follows that they are mapped by Z to the unit circle. In particular, the other branch values are also symmetric with respect to the unit circle, and we name them $\lambda < \frac{1}{\lambda} \in \mathrm{R}^+$.

Recall that this second reflection of T is the same involution to which the $180°$ rotations of $M_{k,\theta}$, around the horizontal lines on this surface, descend under the quotient mapping (first by the screw motion S, then by the vertical rotation ρ). We summarize the properties of Z:

i) the composition

$$M_{k,\theta} \to M_{k,\theta}/S = \overline{\Sigma} \to \overline{\Sigma}/\rho = T \xrightarrow{Z} S = \mathrm{C} \cup \{\infty\}$$

is a meromorphic map on $M_{k,\theta}$;

ii) Z is unitary on all the horizontal lines in $M_{k,\theta}$;

iii) The $4(k+1)$ fixed points, under the $180°$ rotations of $\overline{\Sigma}$ around the horizontal symmetry normals, are mapped to the four (real) branch values of Z, which are $0, \infty, \lambda \leq \frac{1}{\lambda}$, $\lambda \in \mathrm{R}^+$.

The function Z is well-adapted (although not perfectly, as we shall see) to the symmetry of $M_{k,\theta}$. We remark that the Weierstrass $\wp$-function on T has the same branch points as Z. It is normalized by its Mittag-Leffler expansion, while we wish Z to be adapted to our minimal surface. Because they both have the same double pole, $Z = c\wp + d$, for some complex constants c, d.

Notice that the logarithmic derivative $\frac{Z'}{Z}$ is again an elliptic function; it has two simple poles at the double zero and double pole of Z and two simple zeros at the other branch points of Z. The functions $(\frac{Z'}{Z})^2$ and $Z + \frac{1}{Z} - \lambda - \frac{1}{\lambda}$ are therefore proportional. The proportionality factor is real, and positive, since both functions are positive on the horizontal line through B. This factor is irrelevant to our discussion, and by scaling the fundamental rectangle it can be made to equal 1. Then we have:

$$\left(\frac{Z'}{Z}\right)^2 = Z + \frac{1}{Z} - \lambda - \frac{1}{\lambda}; \ Z'^2 = Z^3 - \left(\lambda + \frac{1}{\lambda}\right) Z^2 + Z, \ \ \lambda \in \mathrm{R}.$$

This is the classical description of a (rectangular) torus as a cubic curve (with conformal parameter λ). We consider a choice of λ as specifying a particular torus and the differential equation as defining a doubly-periodic function (in fact a family $Z_a(u) := Z(u + a)$).

4.4 The one-form $\frac{dg}{g}$ in terms of Z

Recall that on $M_{k,\theta}$ the Gauss map has k-fold zeros and poles at the finite vertical points and $(k+2)$-fold zeros and poles at the punctures. Therefore, the differential of $\frac{dg}{g}$ on both $M_{k,\theta}$ and on $M_{k,\theta}/S = \overline{\Sigma}$ has simple poles with residues $\pm k, \mp(k+2)$. On $\overline{\Sigma}$, each puncture is joined to one of the (two) vertical points by $(k+1)$ horizontal lines and the residues there have opposite sign, since the Gauss map points in opposite directions. We noted that the projection of the vertical points of Σ, and the projection of the planar ends in $\overline{\Sigma}$ both lie on the projection of the lines that lie in $\overline{\Sigma}$, on which Z is unitary. We label in T the image of the vertical points by v_1, v_2, and the image of the ends by e_1, e_2. We also noted in Section 4.2 that these points are symmetric with respect to r, which is order-two rotation about the half-periods. We may specify by two unitary numbers, e, v, these geometrically determined points:

$$Z(e_1) = Z(e_2) = e; \ Z(v_1) = Z(v_2) = v\,.$$

From the differential equation for Z and the relationship $Z \circ r = Z$, we have

$$Z'(e_1) = -Z'(e_2) = \pm\sqrt{e^2(e + e^{-1} - (\lambda + \lambda^{-1}))}\,,$$

$$Z'(v_1) = -Z'(v_2) = \pm\sqrt{v^2(v + v^{-1} - (\lambda + \lambda^{-1}))}\,.$$

Knowing the zeros, poles and residues of $\frac{dg}{g}$ allows us to determine what $\frac{dg}{g}$ must be in terms of Z. Let u be the standard coordinate on $\mathrm{C}/\Gamma = T$. Then $dZ = Z'du$, and $du = \frac{dZ}{Z'}$ has neither zeros nor poles on T. We must have

$$\frac{dg}{g} = \left(\frac{-(k+2)}{k+1} \frac{Z'(e_1)}{Z-e} + \frac{k}{k+1} \frac{Z'(v_1)}{Z-v} + c \right) \frac{dZ}{Z'} . \tag{4.3}$$

We now have the generalized Weierstrass representation for our minimal surfaces:

$$g = \exp \int \frac{dg}{g} ; \tag{4.4}$$

$$X(p) = \int_{p_0}^{p} (\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1) dh , \tag{4.5}$$

where dh is as in (4.2). These data contain three real parameters; $\lambda \in \mathrm{R}, e, v \in e^{i\mathrm{R}}$, as well as one complex parameter c.

4.5 Geometric properties of the derived minimal surfaces

The horizontal straight lines on T are mapped to *level lines* of the image $X(T)$, since the third component of (4.5) is constant along these lines. On the horizontal straight lines, which correspond to the straight lines on the minimal surface in R^3, the differential $\frac{dg}{g}$ must have real values (on the tangent vectors to these lines). This is because g on these lines takes values on a line through 0 in $\mathrm{C} \cup \{\infty\}$. But it is not clear that our expression has this property. On the other hand, numerical experimentation indicates that the values *do* come out real! It turns out that the $\wp$-function, which was our guide for the normalization of Z, was not a perfect choice. To see this property, we should have mapped the preimages of those horizontal lines to R rather than $e^{i\mathrm{R}}$. We correct this with the Möbius transformation

$$m(z) := \frac{z - e^{i\alpha}}{1 - e^{i\alpha} z} ,$$

which interchanges R and $e^{i\mathrm{R}}$:

$$m(0) = -e^{i\alpha}; \quad m(\infty) = -e^{-i\alpha}; \quad m(1) = 1 .$$

The most symmetric situation is achieved if we send the other two branch values, $\lambda, \frac{1}{\lambda}$, to $e^{\pm i\alpha}$. This would require α to satisfy

$$e^{i\alpha}(1 - e^{i\alpha}\frac{1}{\lambda}) = \frac{1}{\lambda} - e^{i\alpha} ,$$

Figure 4.5. Values of the elliptic function $\mathfrak{J}$ on the fundamental rectangle. Branch values are $\pm e^{\pm i\alpha}$. $\mathfrak{J}$ is *real* on dashed lines, *unitary* on vertical solid lines.

which is the case if we choose α such that $\cos\alpha = \lambda$. The following Jacobi-type elliptic function turns out to be better adapted to this symmetry of the minimal surface:

$$\mathfrak{J} := m \circ Z\,.$$

It has branch values $\pm e^{\pm i\alpha}$, $\cos\alpha = \lambda$. We define $E := m(e), V := m(v) \in \mathrm{R}$ to be the special values of $\mathfrak{J}$, and we can rewrite the equation for $\frac{dg}{g}$ in (4.2):

$$\frac{dg}{g} = \left(\frac{-\frac{k+2}{k+1}\,\mathfrak{J}'(e_1)}{\mathfrak{J} - E} + \frac{\frac{k}{k+1}\,\mathfrak{J}'(v_1)}{\mathfrak{J} - V} + c \right) \frac{d\,\mathfrak{J}}{\mathfrak{J}'}\,. \tag{4.6}$$

Along the lines on T in question, $\frac{d\mathfrak{J}}{\mathfrak{J}'} = du$ is real (on the tangent vectors), $\mathfrak{J}$ is real and the constants E, V, $\mathfrak{J}'(e_1)$, $\mathfrak{J}'(v_1)$ are also real. Therefore, we have to choose c to be *real*. With these choices $g = \exp(\int \frac{dg}{g})$ has values on a fixed meridian; i.e. reflection in the special lines on T is an *isometry* of the Riemannian metric

$$ds = (|g| + \frac{1}{|g|})|dh|\,.$$

Therefore, these lines are geodesics on the surface. The second fundamental form is given by the real part of the quadratic form $\frac{dg}{g} \cdot dh$, which takes on values in $i\mathrm{R}$ on the tangent vectors to these geodesics. Therefore, these geodesics have no normal curvature (they are "asymptotic lines") and must be straight lines in R^3, as desired.

Next, we choose c so that the horizontal period of $\frac{dg}{g}$ is also imaginary. We may do this explicitly in terms of elliptic integrals. We have no more parameters to adjust this imaginary period to $\frac{2\pi}{k+1}i$, but the residues help us: if we move the horizontal generator towards a symmetry line then, since $\mathfrak{J}$ is real on them (as is du), we pick up *half the residues* of the two poles at e_1, v_1 (resp., e_2, v_2) of $\frac{dg}{g}$. Thus the period is:

$$\frac{\pm 2\pi i}{2} \left(\frac{k+2}{k+1} - \frac{k}{k+1} \right) = \pm \frac{2\pi i}{k+1}\,,$$

as needed.

The screw motion S and the rotation ρ of $M_{k,\theta}$ do not change $|g|$, but only the phase of g ($g \to e^{i\theta}g$ and $g \to e^{\frac{i2\pi}{k+1}}g$, respectively). Therefore, the period of $\frac{dg}{g}$ should be $i\theta$ on a vertical generator of T and $\frac{2\pi i}{k+1}$ on a horizontal generator. Now $du = \frac{d\mathfrak{J}}{\mathfrak{J}'}$ is *imaginary* on the vertical generator and $\mathfrak{J}$ has complex conjugate values at points symmetric to the symmetry lines $\mathfrak{J}^{-1}(\mathrm{R})$. The vertical period of $\frac{dg}{g}$ is therefore imaginary. (Note that c has already been chosen to be real.) The precise value, $i\theta$, of this period depends on $\lambda = \cos\alpha$. E and V will be determined, as functions of λ, in the next section. We do not need to specify this dependence. We now know that the Weierstrass data (4.1)–(4.3) will produce a surface with the desired symmetries, provided we can solve the period problem.

4.6 The period problem

In Sections 4.1–4.5 we have derived a three parameter family of minimal surfaces that have all the desired symmetry properties of the family $M_{k,\theta}$. What remains to be shown is the existence of parameters for which the Weierstrass representation (3.8), which is multivalued on T, defines a mapping on $\overline{\Sigma}$ that has a single period, corresponding to the desired screw motion S. (See Theorem 3.1.)

Half the horizontal generator of T, which joins the branch points where $Z = 0$ and $Z = \lambda$, is mapped to a level-line on the minimal surface. At the endpoints of this level-line, the surface normals are symmetry normals which meet at an angle $\frac{\pi}{k+1}$ at a point we will call $Q \in \mathrm{R}^3$. The $180°$ rotation around these normals continues the surface analytically, and they continue the level line to a *closed* "waist" curve on the minimal surface. There is no period problem here.

Next we consider the image of a wider and wider band around this level line. Near the point v_1, the boundary level-lines of the band converge to two half-lines that meet at a vertical point with an angle $\frac{\pi}{k+1}$. This point must be *vertically above* the point Q where the symmetry normals of the waist line meet; if this is the case, it is also true for pieces of the surface obtained by symmetry operations from the first one. Therefore, we have a 2-dimensional period problem; the parameters e, v have to be chosen in such a way that the horizontal lines meet vertically above Q. The existence proof for solving this period problem is conceptually easy. The previous formulaes allow us to find an approximate solution (e_1, v_1) on the computer. Then one has to take a (possibly large) circle around (e_1, v_1) in the parameter domain and prove that the corresponding intersection points of the horizontal lines form a curve with nonzero winding number, with respect to $0 \in \mathrm{R}^2$. Then 0 is in the image of our parameter map and the period problem is solved. In some situations, such an argument can be pushed through. In this case, we can only see that the accurately computed image curve encloses 0; we have not proved by hand that it does.

5 The numerical computations

The numerical attack on the problem consists of two steps. First, we use a program to calculate parameter values (λ, e, v) for which the period, as described above, vanishes. Then we use these parameter values to obtain pictures of the surface itself.

Both steps require performing the integration in the Weierstrass representation (4.1)–(4.4). This is done using a straightforward Gaussian quadrature routine. As is the case for other, simpler examples, the integrands involve functions defined on a branched cover of S^2, rather than S^2 itself. Care must be taken to stay on the same branch of Z' (which is defined in Section 4.3). This example differs from earlier ones in that g is itself defined by an integral, rather than as an algebraic combination of Z and Z'. The routine that computes g integrates $\frac{dg}{g}$ from the last calculated point; therefore, one must be careful to start all the integration paths from points where the value of g is already known.

To calculate the period for particular values of (λ, e, v), we integrate first from $Z = 0$ to $Z = \lambda$. We may assume that g, which is unitary at $Z = 0$, has the value 1 there. This integral fixes the point Q, defined in Section 4.6, to lie at the origin. We can then integrate from $Z = 0$ to two pairs of points on either side of v on the unit-circle $|Z| = 1$. These pairs of points fix the locations of the lines on the surface. As described above in Section 4.6, the period problem is solved when the intersection of these lines lies above Q.

To solve the period problem, we used a program that minimized $x^2+y^2+(\theta-\theta_0)^2$, where (x, y) is the period vector, θ is the calculated twist-angle, and θ_0 is a desired twist angle, set beforehand. The minimization algorithm we used was the downhill simplex method, which is described in [23]. Of course, it is necessary to check that the calculated minimum actually produces a zero period.

Once parameter values corresponding to zero periods were found, we used Jim Hoffman's MESH program [16] to produce a fundamental piece of the surface. MESH lays out a mesh of triangles on a domain in the plane so as to produce a mesh with similar-sized triangles in R^3. MESH takes as input a definition of the planar domain and a function that, given two points in the domain, returns the difference vector in R^3 between the corresponding points in the range. Here, the domain is $\{|Z| \leq$

$1\} - [0,1]$. (MESH can handle branch cuts.) This corresponds to a fundamental symmetric piece, multiple copies of which can be sewn together along symmetry lines to produce the whole surface. The only new difficulty presented by this example is the need to know g nearby in order to calculate it at a point. However, this was easily accomplished by having MESH keep track of the value of g at the points it has already calculated and having it start at $Z = 0$, for which g can be assumed to be 1.

The results for $k = 1$ are shown in Figure 5.1, where λ, e and v are plotted as functions of the twist angle. In the graph, the conformal parameter λ becomes close to 1 as $\theta_0 \to \pi/2$. In fact, we do not trust our numerical computations when the twist angle is near the limit; the theoretically correct value of λ should approach 1. We observed that many of the runs failed to converge to a minimum in a reasonable amount of time for extreme twisting angles. When we increased the precision of the calculations by taking more integration steps, we found parameter values that seemed to kill the periods with λ close to 1. (These are the isolated points that appear in the graph.) Note that as $\lambda \to 1$, the conformal structure of the torus T degenerates; at the same time, it becomes more and more difficult to calculate the relevant integrals precisely, with the result that numerical errors become larger. Our first runs failed because the errors dominated the calculation, and even the more accurate calculations found values of λ that were too large, because of the difficulty of computing these integrals on a nearly singular Riemann surface.

From our own preconceptions, reinforced by the computer graphics of the surfaces for θ near $\pi/k+1$, we expected the surfaces $M_{k,(\pi/k+1)}$ to have reflection symmetries in vertical planes. One interpretation of our data is that the underlying torus is degenerating as $\theta \to \pi/k+1$, implying that the limit-angle is not achievable. Under the assumption of the existence of reflection symmetry in vertical planes, we prove that, in fact, it is not achievable.

6 Nonexistence of the surfaces $M_{k,(\pi/k+1)}$ with reflectional symmetry

In this section, we will prove that the examples $M_{k,(\pi/k+1)}$ do not exist; the assumed symmetry of the surface leads to Weierstrass data for which the period problem is not solvable.

Figure 5.1: Numerical results for $k = 1$. Left: calculated values of λ graphed against θ. Right: the unitary numbers e and v graphed against θ. The left-hand scale applies to the lower curve and gives the argument of e; the right-hand scale applies to the upper curve, which is the argument of v. Note that in the limit $\theta \to \pi/2$, e and v tend toward complex conjugates of one another.

6.1 The Weierstrass data

In Section 4, we showed that $\Sigma := M_{k,\theta}/S$, where S is the screw motion defined in (4.1), is a $(k+1)$-sheeted covering of a rectangular torus, T, the cover being defined by a normal rotation of order $k+1$ about the vertical axis. Rotation by π about the horizontal lines in $M_{k,\theta}$, at heights $x_3 = p$ and $x_3 = p + \frac{1}{2}$, $p \in \mathrm{Z}$, induces an orientation-reversing involution of T. We may assume that T is the torus C/Γ, where Γ is the lattice $\{m\tau + ni \mid m, n \in Z\}$ for some real $\tau > 0$, and that the involution of T is given by complex conjugation; the lines on Σ will lie above the lines $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \frac{1}{2}$, considered to lie in T. As in Section 4, dh descends to T and must be given by

$$dh = i\,dz\,. \tag{6.1}$$

Note this differs from what we did in Section 4, where we placed the lines over $\operatorname{Im} z = \frac{1}{4}$ and $\operatorname{Im} z = \frac{3}{4}$.

The normal-rotational symmetries about horizontal lines at height $x_3 = \frac{1}{4} + \frac{1}{2}p \in \mathrm{Z}$, descend to T as an orientation-preserving involution that interchanges $\operatorname{Im} z = 0$ with $\operatorname{Im} z = \frac{1}{2}$. As in Section 4, we know the fixed points of the involution form a translated half-period lattice, about which this involution is 180^0 rotation. Because this rotation interchanges $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \frac{1}{2}$, this lattice lies on the lines $\operatorname{Im} z = \frac{1}{4}$

Figure 6.1.1. Symmetries of the surface in the parameterization domain. Fixed points of normal rotation are at points in $\mathcal{D}$, indicated by a $\bullet$. Horizontal lines on $M_{k,\frac{\pi}{k+1}}$ lie over the dashed horizontal lines of the torus. Planar lines of symmetry lie over solid vertical lines.

and $\operatorname{Im} z = \frac{3}{4}$. After a translation of T, if necessary, we may assume that these points lie on the diagonals of T, and we label them $\mathcal{D}$ as in Figure 6.1.1. The ends e_1, e_2 and the points v_1, v_2 of Σ on the vertical axis lie on the horizontal lines and we label them so that $\{e_1, v_1\}$ lies on the lines over $\operatorname{Im} z = \frac{1}{2}$ and $\{e_2, v_2\}$ lies on the lines over $\operatorname{Im} z = 0$. The ends e_1, e_2 are interchanged by rotation by π about the points in $\mathcal{D}$, as are the vertical points v_1, v_2.

Our imagined surface $M_{k,\frac{\pi}{k+1}}$ has two special properties not shared with the family $M_{k,\theta}$. The first property is an *assumption* of additional symmetry not possible for $\theta \neq \frac{\pi}{k+1}$: namely, the existence of $k+1$ of vertical planes of symmetry, as was the case for $M_k = M_{k,0}$. See Figure 6.1.2.

Reflection in these planes descends to Σ, and then to the torus T as a single orientation-reversing involution. This involution of T must correspond to reflection in a line that itself is invariant under reflection in $\operatorname{Im} z = 0, \frac{1}{2}$; such a line must be vertical. The fixed point set of this reflection contains the set $\{e_1, e_2, v_1, v_2\}$, and so must consist of a pair of vertical lines symmetrically placed with respect to 180^0 rotation around $\mathcal{D}$. Thus the involution is reflection in the lines $\operatorname{Re} z = 0, \frac{\tau}{2}$, and the set $\{e_1, e_2, v_1, v_2\}$ consists of the half-period points of T. Without loss of generality, we may assume that e_1 lies at the center point of T; symmetry with respect to $\mathcal{D}$ dictates the placement of e_2, v_1 and v_2.

The second distinctive property of $M_{k,\frac{\pi}{k+1}}$ is that g^{k+1} is well-defined on T. (This is the case even without the reflectional symmetry assumption.) On $M_{k,\frac{\pi}{k+1}}$ and hence on $\Sigma = M_{k,\frac{\pi}{k+1}}/S$, the Gauss map is vertical at the points over e_i, v_i and has branching order k at the points over v_1 and v_2, and branching order $k+2$ at the points over e_1 and e_2. The $(k+1)$-fold covering $\Sigma \to T$ is branched over these points, so g^{k+1} has

Figure 6.1.2: Symmetries of the imagined surface $M_{1,\pi/(k+1)}$.

Figure 6.1.3: The pole structure of g^{k+1}.

these same branching orders at the points e_1, e_2, v_1, v_2. If we orient the surface so that $g^{k+1}(e_1) = 0$, we arrive at the zero-pole structure of g^{k+1} as indicated in Figure 6.1.3; we already know that there are no other zeros or poles. Thus g^{k+1} is determined up to a nonzero multiplicative constant. Because g must be unitary at the fixed points of the normal rotation on Σ, g^{k+1} is unitary on $\mathcal{D} \subset T$. Multiplication of g by a unitary number $e^{i\phi}$ is equivalent to a rotation by ϕ around the vertical axis, which is irrelevant to the geometry of the surface. This means that the Gauss map of $M_{k,\frac{\pi}{k+1}}$ will be completely determined by insisting that g^{k+1} be unitary on $\mathcal{D}$.

6.2 The period problem

From Section 6.1, we now know that the only freedom we have in the Weierstrass data is the conformal type of the rectangular torus. We will now show that there is

Figure 6.2. The path I.

only one period to worry about; the *assumed* symmetries of $M_{k,\frac{\pi}{k+1}}$ make the period problem one-dimensional.

The horizontal lines of Σ over $\operatorname{Im} z = 0$ and the horizontal lines over $\operatorname{Im} z = \frac{1}{2}$ lie directly over one another. The horizontal normal-rotation axis bisects the angle between the successive lines on the surface above and below it; when $\theta = \frac{\pi}{k+1}$, the points on the surface above $\mathcal{D} \subset T$ (i.e. the points where the horizontal normal-rotation axes meet the surface) must lie in the same vertical plane as the lines on the surface, which are aligned vertically as they were on $M_k = M_{k,0}$.

We can express this in terms of the Weierstrass representation as follows. Rotate the surface around the vertical axis so that one of the horizontal lines above $\operatorname{Im} z = 0$ is parallel to the x_2-axis; this is equivalent to requiring g to be real along this line. Then we must have

$$\operatorname{Re} \int_I (g^{-1} - g)dh \left(= \operatorname{Re} \int_I (g^{-1} - g)idu \right) = 0\,, \tag{6.2}$$

where I is the vertical segment beginning on $\operatorname{Im} z = \frac{1}{2}$, and ending at $\frac{3}{4}\tau + \frac{3}{4}i \in \mathcal{D}$ (see Figure 6.2). This condition assures that the horizontal symmetry normal through the end point of I lies in the $x_1 = 0$ plane; the built-in symmetry will insure that this is true everywhere. The reflectional symmetries also insure that there is no period on a horizontal generator of the torus; in fact the existence of normal-rotational symmetries already implies this, as was observed in Section 4.6.

We have one real period condition (6.2) and one free parameter, τ, describing the rectangular torus; one would expect a solution. However, we will prove in Section 6.4 that there is no solution.

6.3 Expression of g in terms of Jacobi-type elliptic functions

To show that (6.2) is not solvable for any rectangle, we will utilize some of the basic, degree-two, elliptic functions introduced in [17], [9], Section 3. The functions

Figure 6.3.1 (left) The translated half-period lattice $\mathcal{E}$ indicated by solid squares. Modulo 180^0 rotation, r_E, about $\mathcal{E}$ we get a sphere T/r_E. Assigning three complex values to points on this sphere determines $\mathfrak{J}_E$. (right) Values of $\mathfrak{J}_E$ are real along horizontal dashed lines, imaginary along vertical solid lines, unitary on the vertical dotted lines. Branch values $\pm E^{\pm 1} \in i\mathrm{R}$ are also indicated.

we will use are called $\mathfrak{J}_E$ and $\mathfrak{J}_F$. We will construct $\mathfrak{J}_E$ directly, and then express $\mathfrak{J}_F$ in terms of $\mathfrak{J}_E$.

We construct $\mathfrak{J}_E$ as follows. Translate the half-period lattice by $\frac{i}{4}$, producing the lattice points $\mathcal{E}$ indicated by small squares in Figure 6.3.1. Rotation by π about these points (which we will denote by r_E) is an involution of the torus T; the quotient T/r_E is a sphere, twice-covered by T. Once we identify this sphere with $\mathrm{C} \cup \{\infty\}$, we will have a degree-two elliptic function, branched at the points of $\mathcal{E}$.

We completely determine this identification by specifying the values of three points of the sphere. We do this with the following in mind: we want $\mathfrak{J}_E$ to have simple zeros and poles on the half-period lattice and—in particular—a zero at the center; we want $\mathfrak{J}_E$ to be real on the horizontal bisector of T. The first requirement plus the fact that $\mathfrak{J}_E \circ r_E = \mathfrak{J}_E$ force the placement of the zeros and poles. We choose to meet the second by specifying 1 as the complex value associated to the points on the sphere T/r_E as indicated in Figure 6.3.1. Now $\mathfrak{J}_E$ is completely determined.

Observe that reflection ρ in the vertical lines $\mathrm{Re}\, z = 0, \frac{\tau}{2}$ is an orientation-reversing involution that leaves $\mathrm{Im}\, z = 0, \frac{1}{2}$ invariant. Along these lines, $\mathfrak{J}_E$ is real. Hence ρ induces reflection in the imaginary axis; $\mathfrak{J}_E \circ \rho = -\overline{\mathfrak{J}}_E$. In particular, $\mathfrak{J}_E$ is imaginary on the lines $\mathrm{Re}\, z = 0, \frac{\tau}{2}$. Similar reasoning shows that $\mathfrak{J}_E$ is unitary on $\mathrm{Re}\, z = \frac{i}{4}, \frac{3i}{4}$, and that $\mathfrak{J}_E$ is imaginary on $\mathrm{Im}\, z = \frac{1}{4}, \frac{3}{4}$. It then follows that $\mathfrak{J}_E = \pm i$ on the points of $\mathcal{D}$. Correct choice of orientation dictates the value $+i$ as in Figure 6.3. In Figure 6.5, we label points of T by their $\mathfrak{J}_E$ values, determined by the symmetries discussed above. In particular, $\mathfrak{J}_E$ has imaginary values at its branch points $\mathcal{E}$.

We denote by $\pm E^{\pm 1} \in i\mathrm{R}$ the values of $\mathfrak{J}_E$ on $\mathcal{E}$. The value of E is determined by

Figure 6.3.2. The function $\mathfrak{I}_F$. Values are imaginary along vertical solid lines, real along dashed lines, and unitary on horizontal dashed lines.

the parameter, τ, that specifies the rectangular lattice.

In order to express g^{k+1} using $\mathfrak{I}_E$, we need one other elliptic function that we will produce from $\mathfrak{I}_E$ by a translation $\mathbf{t}$ and a Möbius transformation of the sphere. Specifically, the translation we want is $\mathbf{t}(z) = z + \frac{1}{4}(\tau + i)$, which moves $\mathcal{E}$ to the lattice $\mathcal{F}$ indicated in Figure 6.3.2. The Möbius transformation is $z \to \frac{-z+i}{z+i}$, which takes $0 \to 1$, $\infty \to -1$, $i \to 0$, and $\mathrm{R} \to e^{i\mathrm{R}}$, $i\mathrm{R} \to \mathrm{R}$, $e^{i\mathrm{R}} \to i\mathrm{R}$. We define $\mathfrak{I}_F$ by the relation

$$\mathfrak{I}_F \circ \mathbf{t} = \frac{-\mathfrak{I}_E + i}{\mathfrak{I}_E + i}. \tag{6.3}$$

Note that the branch values of $\mathfrak{I}_F$, which we denote by $\pm F^{\pm 1}$, are real.

We are now in a position to express g^{k+1} in terms of $\mathfrak{I}_E$ and $\mathfrak{I}_F$. We simply note that the function $\mathfrak{I}_E^{k+1} \cdot \mathfrak{I}_F$ has the same zeros and poles, to the same order, as g^{k+1} and is unitary on $\mathcal{D}$. By the results of Section 6.1, we may assert that

$$g^{k+1} = \mathfrak{I}_E^{k+1} \cdot \mathfrak{I}_F\,, \tag{6.4}$$

and

$$g = \mathfrak{I}_E \sqrt[k+1]{\mathfrak{I}_F}\,, \tag{6.5}$$

Here g lives on Σ, which is a $(k+1)$-fold cover of T. We may, however, compute locally and consider points of T as corresponding to an appropriate lifting to Σ.

6.4 The obstruction to the solution of the period problem

Because $\mathfrak{I}_F$ has no zeros on the interval I defined in Section 6.2 (see Figure 6.3) and is equal to 1 at one end point of I, it follows that F, its value at the other end point, is positive. Choose the branch of $\sqrt[k+1]{\mathfrak{I}_F}$ that is real and positive at this point. Then this branch is real from v_1 to e_1. The function $\mathfrak{I}_E$ is real on this line and it follows from (6.5) that g is real along this line. We already know that its image is a horizontal line, and, since g is real along it, it must be parallel to the x_2-axis. Now we may use the period condition (6.2). On I,

$$dh = idz = -dt \text{ and } g = \sqrt[k+1]{\mathfrak{J}_F} \cdot \mathfrak{J}_E = f(t)e^{i\phi(t)} ,$$

where $f(t)$ is the positive $(k+1)$-st root of $\mathfrak{J}_F$ at $I(t) := \frac{3}{4}\tau + it, \frac{1}{2} \leq t \leq \frac{3}{4}$, and $e^{i\phi(t)}$ is the value of $\mathfrak{J}_E$ at $I(t)$. We know that $\phi(t)$ is monotonic with $\phi(\frac{1}{2}) = 0$ and $\phi(\frac{3}{4}) = \frac{\pi}{2}$. Because $\mathfrak{J}_F$ has no branch points in I°, $\mathfrak{J}_F$ is not equal to 1 anywhere on I° (and, for that matter, $F \neq 1$). Hence, the expression

$$\begin{aligned} \operatorname{Re}\{g^{-1} - g\} &= \operatorname{Re}\{(f^{-1} - f)\cos\phi - i(f^{-1} + f)\sin\phi\} \\ &= (f^{-1} - f)\cos\phi \end{aligned}$$

never changes sign on I, and is nonzero on I°. It follows that

$$\operatorname{Re}\int_I (g^{-1} - g)dh = -\int_{t=\frac{1}{2}}^{t=\frac{3}{4}} \operatorname{Re}\left\{g^{-1}(\frac{3}{4}\tau + it) - g(\frac{3}{4}\tau + it)\right\} dt$$

can never be zero, no matter what the value of τ. Hence, there is no rectangular torus for which the period problem (6.2) can be solved; the surface $M_{k,\frac{\pi}{k+1}}$ does not exist with reflectional symmetry.

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