# DISCRETIZING CONSTANT CURVATURE SURFACES VIA LOOP GROUP FACTORIZATIONS: THE DISCRETE SINE- AND SINH-GORDON EQUATIONS

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## 1. Introduction

In recent years mathematical physicists have been studying discrete (in space and time) analogs of integrable non-linear field models motivated by questions arising in statistical mechanics (spin models) and quantum field theory [10]. Perhaps it is necessary to explain what we mean by *integrable*. We call a non-linear field equation integrable if it arises as the flatness (zero-curvature) condition of a connection with values in a loop Lie algebra. A standard example for this is the non-linear $\sigma$-model, i.e., the harmonic map equation for maps of a surface into a symmetric space [14, 19, 18, 7].

Many problems in classical surface geometry (minimal surfaces, constant curvature surfaces, Willmore surfaces) give rise to well known field equations: the Liouville euqation, sine- and sinh-Gordon equations and more generally the Toda field equations [6, 4], all of which are harmonic map equations. The essential requirement for a discrete version of such equations is that the discretization is also *integrable*. By this we clearly mean that the discretized equation is the zero curvature equation of a loop Lie group valued connection over discretized space-time (the lattice $\mathbb{Z}^2$ in the simplest case) where the dependence on the loop parameter ("spectral parameter") should be the same as in the smooth case. Whether, given a certain integrable field equation, such a discretization exists and whether it is unique is by no means clear. Thus, rather than deriving integrable discrete analogs from first principles, they are found by ad hoc considerations.

In this note we focus on the sine- and sinh-Gordon equations which arise as the integrability conditions for constant negative Gauss curvature ($K$-surfaces) and constant mean curvature (CMC) surfaces. The unit normal map of either surface is a harmonic map into the 2-sphere $S^2$ and thus gives rise to an extended frame (c.f. section 2) into an appropriate loop group of $\mathbf{SU}(2)$. This exhibits the sine- and sinh-Gordon equations naturally as integrable equations in the above mentioned sense. The past few years have seen substantial progress in understanding such equations from a differential geometric viewpoint: the methods suggested by mathematical physicists to construct solutions, the $R$-matrix and A-K-S scheme, loop group factorizations and dressing action, have successfully been applied to classify and parametrize a large class of surfaces [6]. In many cases

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the dressing orbit though a trivial or *vacuum* solution accounts for all solutions one is interested in (e.g. doubly periodic solutions) [9, 5]. It is precisely this construction which we will use to find discrete versions of the sine- and sinh-Gordon equations. Since the vacuum solution, a 2-parameter subgroup, can be easily discretized (c.f. sections 3 and 4) we obtain via dressing discrete extended frames, i.e., maps from the lattice $\mathbb{Z}^2$ into a suitable loop group of $\mathbf{SU}(2)$, for discrete analogs of $K$- and CMC-surfaces. The compatibility equations for the existence of such a map, i.e., the products of the values of the extended frame evaluated along any given quadrilateral equals the identity, give integrable discretized versions of the sine- and sinh-Gordon equations. Geometrically the extended frame describes the unit normal map of the constant curvature surface so that we obtain natural definitions of discrete harmonic maps from the lattice $\mathbb{Z}^2$ into the 2-sphere for both, Lorentzian and Euclidean discrete space-time. This is carried out in some detail for $K$-surfaces (c.f. section 3) and we make contact to the work of Bobenko-Pinkall [2]. The case of CMC-surfaces is more involved and the reader is referred to [2] for an exhaustive treatment, including explicit parametrizations of solutions in terms of theta functions. The investigation of discrete analogs of minimal surfaces in $\mathbb{R}^3$ (Liouville equation), Willmore surfaces (Toda field equations for $\mathbf{SO}(5)$) and more generally harmonic surfaces in a compact symmetric space will be forthcoming.

Finally, from a practical point of view the existence of exact discrete analogs of certain surface classes also is advantageous when performing computer experiments: to run an alogrithm based on an exact discrete theory avoids most of the problems arising when smooth theories are put into algorithms.

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## 2. Loop Group Factorization and Dressing Action

In this section we recall the basic ideas how to apply loop group factorizations and the corresponding dressing actions to obtain solutions to soliton equations. In particular, this approach applies to the sine-Gordon equation and sinh-Gordon equation which describe constant negative Gauss curvature surfaces ($K$-surfaces) and constant mean curvature surfaces (CMC-surfaces). Using these methods we will derive integrable discrete versions of the sine- and sinh-Gordon equations, which will give rise to discrete $K$-surfaces and discrete CMC-surfaces.

Let $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a $K$-surface or CMC-surface, where $D$ is a simply connected region containing the origin. Then its Gauss map $N\colon D \to S^2$ is harmonic if $D$ is given the appropriate metric: for CMC-surfaces this is the induced metric of $f$, and for $K$-surfaces it is the metric given by the $2^{nd}$ fundamental form of $f$ (which is a Lorentz metric since $K < 0$, see [13]).

For a $K = -1$ surface $f\colon D \to \mathbb{R}^3$, let $\omega\colon D \to (0, \pi)$ be the angle between the asymptotic curves on the surface. Then there exists a unique frame

$$F\colon D \to \mathbf{SU}(2)$$

of the Gauss map $N\colon D \to S^2$ (i.e., $N = \pi \circ F$, where $\pi\colon \mathbf{SU}(2) \to S^2 = \mathbf{SU}(2)/S^1$ is the natural projection) satisfying

$$\begin{aligned} \partial_x F &= F \cdot \frac{i}{2} \begin{pmatrix} -\partial_x \omega & 1 \\ 1 & \partial_x \omega \end{pmatrix} = F \cdot (\partial_x \Omega + B)\,, \\ \partial_y F &= F \cdot \frac{i}{2} \begin{pmatrix} 0 & -e^{-i\omega} \\ -e^{i\omega} & 0 \end{pmatrix} = F \cdot e^{-\Omega}(-B)e^{\Omega}\,, \\ F(0) &= I\,, \end{aligned} \tag{2.1}$$

where $x, y$ are coordinates on the surface consisting of arc-length parameters of the asymptotic curves, $\Omega = \frac{i}{2}\left(\begin{smallmatrix} -\omega & 0 \\ 0 & \omega \end{smallmatrix}\right)$ and $B = \frac{i}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The integrability condition

$$\partial_y(\partial_x \Omega + B) + \partial_x(e^{-\Omega} B e^{\Omega}) = [\partial_x \Omega + B, -e^{-\Omega} B e^{\Omega}] \tag{2.2}$$

for the existence of $F$ (i.e., the Gauss equation for the surface) is the sine-Gordon equation

$$\partial_x \partial_y \omega = \sin \omega\,. \tag{2.3}$$

It is easy to check that (2.2) is invariant if we insert a spectral parameter $\lambda \in \mathbb{R}^*$ into (2.1) in the following way:

$$\begin{aligned} \partial_x \Phi &= \Phi \cdot (\partial_x \Omega + B\lambda)\,, \\ \partial_y \Phi &= \Phi \cdot e^{-\Omega}(-B)e^{\Omega}\lambda^{-1}\,, \\ \Phi(0, -) &= I\,. \end{aligned} \tag{2.4}$$

Thus, $\Phi\colon D \times \mathbb{R}^* \to \mathbf{SU}(2)$ is a family of frames $F_\lambda = \Phi(-, \lambda)\colon D \to \mathbf{SU}(2)$ for the harmonic maps $N_\lambda = \pi \circ F_\lambda\colon D \to S^2$. In the sequel we shall call such $\Phi$ an *extended frame* (of $N$). The corresponding family of $K$-surfaces is given by Sym's formula [16, 13]

$$f_\lambda = \left(\frac{d}{d\lambda}\Phi\right)\Phi^{-1}\colon D \to \mathbf{su}(2) = \mathbb{R}^3\,. \tag{2.5}$$

Constructing solutions to the sine-Gordon equation is equivalent to finding extended frames $\Phi$ solving (2.4) for some $\Omega = \frac{i}{2}\left(\begin{smallmatrix} \omega & 0 \\ 0 & -\omega \end{smallmatrix}\right)$ with $\omega\colon D \to [0, \pi)$. Notice that the sine-Gordon equation has the vacuum solution $\omega \equiv 0$ with corresponding extended frame

$$\Phi^B = \exp\big((x\lambda - y\lambda^{-1})B\big)\,. \tag{2.6}$$

To obtain other solutions we apply the dressing action, for which we introduce the following loop spaces:

$$L\mathbf{SU}(2) = \{g\colon \mathbb{R}^* \to \mathbf{SU}(2)\,;\, g(-\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) g(\lambda) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$$

is an infinite dimensional Lie group with Lie algebra

$$L\mathbf{su}(2) = \{\xi\colon \mathbb{R}^* \to \mathbf{su}(2)\,;\, \xi(-\lambda) = Ad\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \xi(\lambda)\}\,.$$

The symmetry condition in the definition of $L\mathbf{SU}(2)$ is equivalent to the following: if $g(\lambda) = \sum_{k\in\mathbb{Z}} g_k\lambda^k \in L\mathbf{SU}(2)$ is the Fourier series expansion then even coefficents are diagonal whereas odd coefficents are off-diagonal. $L\mathbf{SU}(2)$ has two Lie subalgebras

$$L^+\mathbf{su}(2) = \{\xi \in L\mathbf{su}(2)\,;\, \xi(\lambda) = \sum_{k\geq 0} \xi_k\lambda^k\}\,,$$
$$L^-\mathbf{su}(2) = \{\xi \in L\mathbf{su}(2)\,;\, \xi(\lambda) = \sum_{k<0} \xi_k\lambda^k\}\,,$$

whose direct sum is $L\mathbf{su}(2)$. The corresponding Lie subgroups of $L\mathbf{SU}(2)$ are

$$L^+\mathbf{SU}(2) = \{g \in L\mathbf{SU}(2)\,;\, g(\lambda) = \sum_{k\geq 0} g_k\lambda^k\}\,,$$
$$L^-\mathbf{SU}(2) = \{g \in L\mathbf{SU}(2)\,;\, g(\lambda) = I + \sum_{k<0} g_k\lambda^k\}\,.$$

The following result is proven in [8].

**Theorem 2.1.** *Multiplication $L^-\mathbf{SU}(2) \times L^+\mathbf{SU}(2) \to L\mathbf{SU}(2)$ is a diffeomorphism onto the open and dense Lie subgroup $L^-\mathbf{SU}(2)\cdot L^+\mathbf{SU}(2) \subset L\mathbf{SU}(2)$.*

With this at hand, let

$$G = L\mathbf{SU}(2) \times L\mathbf{SU}(2)\,,$$
$$G_1 = \operatorname{diag} G\,,$$
$$G_2 = L^-\mathbf{SU}(2) \times L^+\mathbf{SU}(2)$$

with Lie algebras $\mathcal{G}, \mathcal{G}_1$, and $\mathcal{G}_2$, respectively. Theorem 2.1 implies [17] that multiplication

$$G_1 \times G_2 \to G \tag{2.7}$$

is a diffeomorphism onto the open and dense Lie subgroup $G_1 \cdot G_2$. For $g \in G_1\cdot G_2$ we denote its component in $G_i$ by $g_{G_i}$. On the Lie algebra level this decomposition is explicitly given by

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2, \quad (\xi,\eta) = (\xi_+ + \eta_-, \xi_+ + \eta_-) + (\xi_- - \eta_-, \eta_+ - \xi_+)\,,$$

where $\xi = \xi_- + \xi_+ \in L^-\mathbf{su}(2) \oplus L^+\mathbf{su}(2)$. Now let $\Phi\colon D\times\mathbb{R}^* \to \mathbf{SU}(2)$ be an extended frame. Due to the specific form of the coefficient matrices in (2.4) we can view $\Phi\colon D \to L\mathbf{SU}(2)$. Proofs of the following lemma in similar contexts can be found in [11, 5, 9]

**Lemma 2.1.** *Let $h = (h_-, h_+) \in G_2$ and $\Phi\colon D \to L\mathbf{SU}(2)$ be an extended frame. Then $\Psi\colon \widetilde{D} \to L\mathbf{SU}(2)$ defined by*

$$(\Psi,\Psi) = (h_-\Phi, h_+\Phi)_{G_1}$$

*is also an extended frame on some $\widetilde{D} \subseteq D$. Moreover,*

$$h\#\Phi\colon = \Psi$$

*defines an action, the so called* dressing action, *on extended frames.*

Applying this construction to the vacuum solution $\Phi^B$ given by (2.6) yields the dressing orbit

$$G_2 \# \Phi^B$$

through $\Phi^B$, which provides an infinite-dimensional space of (local) solutions to the sine-Gordon equation (2.3).

We now turn to the case of CMC-surfaces. The extended frame $\Phi$ of a harmonic Gauss map $N\colon \mathbb{R}^2 \to S^2$ satisfies

$$\begin{aligned} \partial_x \Phi &= \Phi \cdot \frac{1}{2} \begin{pmatrix} i\partial_y \omega & e^{-\omega}\lambda + e^{\omega}\lambda^{-1} \\ -e^{\omega}\lambda - e^{-\omega}\lambda^{-1} & -i\partial_y \omega \end{pmatrix}, \\ \partial_y \Phi &= \Phi \cdot \frac{i}{2} \begin{pmatrix} -\partial_x \omega & e^{-\omega}\lambda - e^{\omega}\lambda^{-1} \\ -e^{\omega}\lambda + e^{-\omega}\lambda^{-1} & \partial_x \omega \end{pmatrix}, \\ \Phi(0,-) &= I\,, \end{aligned} \tag{2.8}$$

where $x, y$ are the principal curvature coordinates on the CMC-surface (we allow no umbilic points), $e^{-2\omega}$ is the conformal factor of the induced metric $ds^2 = e^{-2\omega}(dx^2 + dy^2)$ and the spectral parameter $\lambda \in S^1 \subset \mathbb{C}$ [9, 3, 4]. The integrability condition for the system (2.8) is the (elliptic) sinh-Gordon equation

$$\partial_x^2 \omega + \partial_y^2 \omega + 2\sinh 2\omega = 0\,. \tag{2.9}$$

As before, $F_\lambda = \Phi(-,\lambda)\colon \mathbb{R}^2 \to \mathbf{SU}(2)$ is a family of frames of the harmonic maps $N_\lambda = \pi \circ F_\lambda\colon \mathbb{R}^2 \to S^2$. Sym's formula [3, 13] retrieves (a family of) two CMC-surfaces (parallel to each other) of mean curvature $H = \frac{1}{2}$ via

$$f_\lambda^{\pm} = \left(\frac{\partial}{\partial\lambda}\Phi\right)\Phi^{-1} \pm N_\lambda\colon \mathbb{R}^2 \to \mathbf{su}(2) = \mathbb{R}^3\,.$$

The following loop spaces will be relevant for the dressing procedure: the loop group

$$\Lambda\mathbf{SL}(2,\mathbb{C}) = \{g\colon S^1 \to \mathbf{SL}(2,\mathbb{C})\,;\, g(-\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) g(\lambda) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$$

with Lie algebra

$$\Lambda\mathbf{s}\ell(2,\mathbb{C}) = \{\xi\colon S^1 \to \mathbf{s}\ell(2,\mathbb{C})\,;\, \xi(-\lambda) = Ad\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \xi(\lambda)\}\,.$$

$\Lambda\mathbf{SL}(2,\mathbb{C})$ has two Lie subgroups

$$\begin{aligned} \Lambda\mathbf{SU}(2) &= \{g \in \Lambda\mathbf{SL}(2,\mathbb{C})\,;\, g(\lambda) \in \mathbf{SU}(2) \text{ for } \lambda \in S^1\}\,, \\ \Lambda_A^+\mathbf{SL}(2,\mathbb{C}) &= \{g \in \Lambda\mathbf{SL}(2,\mathbb{C})\,;\, g \text{ extends holomorphically to } |\lambda| < 1 \text{ and } g(0) \in A\} \end{aligned}$$

where $A = \{\left(\begin{smallmatrix} \rho & 0 \\ 0 & \frac{1}{\rho} \end{smallmatrix}\right)\,;\, \rho \in \mathbb{R}^+\} \subset \mathbf{SL}(2,\mathbb{C})$ is the imaginary torus. The following global factorization result is proven in [15].

**Theorem 2.2.** *The multiplication map*

$$\Lambda\mathbf{SU}(2) \times \Lambda_A^+\mathbf{SL}(2,\mathbb{C}) \to \Lambda\mathbf{SL}(2,\mathbb{C})$$

*is a diffeomorphism. In particular, every $g \in \Lambda\mathbf{S}L(2,\mathbb{C})$ has a unique decomposition $g = g_u g_+$ where $g_u \in \Lambda\mathbf{SU}(2)$ and $g_+ \in \Lambda_A^+\mathbf{SL}(2,\mathbb{C})$.*

Notice that an extended frame can be viewed as a map $\Phi\colon \mathbb{R}^2 \to \Lambda\mathbf{SU}(2)$. Now Theorem 2.2 gives rise to the dressing action of $\Lambda_A^+\mathbf{SL}(2,\mathbb{C})$ on extended frames [11, 5]: for every $h \in \Lambda_A^+\mathbf{SL}(2,\mathbb{C})$,

$$h\#\Phi\colon = (h\Phi)_u \tag{2.10}$$

is also an extended frame. As before, (2.8) has the vacuum solution

$$\Phi^B = \exp\bigl(xB(\lambda+\lambda^{-1}) + iyB(\lambda-\lambda^{-1})\bigr) \tag{2.11}$$

corresponding to $\omega \equiv 0$, and the dressing orbit

$$\Lambda_A^+\mathbf{SL}(2,\mathbb{C})\#\Phi^B$$

through $\Phi^B$ yields an infinite-dimensional space of solutions to the sinh-Gordon equation. In fact, all finite-type solutions, in particular, all doubly periodic solutions, are contained in this orbit [9, 5].

## 3. Discrete Sine-Gordon Equation and Discrete $K$-Surfaces

In this section, we use the dressing action (cf. lemma 2.1) to derive an integrable discrete version of the sine-Gordon equation. Using the Sym formula (2.5) we then give the discrete analog of a $K$-surface, which coincides with the geometric definition given in [2].

We start deriving a discrete analogue of an extended frame $\Phi\colon \mathbb{R}^2 \to L\mathbf{SU}(2)$ using the dressing action in lemma 2.1. For this it suffices to discretize the vacuum solution

$$\Phi^B = \exp\left((x\lambda - y\lambda^{-1})B\right) .$$

A natural discretization can be obtained by solving the following system over $\mathbb{Z}^2$ with meshsize $\delta > 0$:

$$\begin{aligned}
\Phi_{n+1,m} &= \Phi_{n,m} \cdot \frac{1}{\Delta_+}(I + \delta B\lambda)\,,\\
\Phi_{n,m+1} &= \Phi_{n,m} \cdot \frac{1}{\Delta_-}(I - \delta B\lambda^{-1})\,,\\
\Phi_{0,0} &= I\,,
\end{aligned}$$

where $\Delta_\pm = \sqrt{\det(I \pm \delta B\lambda^{\pm 1})} = \sqrt{1 + \frac{1}{4}\delta^2\lambda^{\pm 2}}$. This system is motivated by the naive discretization of $d\Phi = \Phi\cdot(B\lambda dx - B\lambda^{-1}dy)$ scaled by the factors $\Delta_\pm^{-1}$ to ensure that the solution

$$\Phi^B_{n,m} = \frac{1}{\Delta_+^n\Delta_-^m}(I+\delta B\lambda)^n(I-\delta B\lambda^{-1})^m \tag{3.1}$$

takes values in $\mathbf{SU}(2)$ for $\lambda \in \mathbb{R}^*$. Moreover,

$$\Phi^B_{n,m}(-\lambda) = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\Phi^B_{n,m}(\lambda)\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$$

so that we have a map

$$\Phi^B\colon \mathbb{Z}^2 \to L\mathbf{SU}(2)\,.$$

Now let $h = (h^-, h^+) \in G_2$ and consider

$$\Phi\colon = h\#\Phi^B\colon \mathbb{Z}^2 \to L\mathbf{SU}(2)\,,$$

where, as in the smooth case (cf. lemma 2.1), $(h\#\Phi^B)_{n,m} = \left((h,h)(\Phi^B_{n,m},\Phi^B_{n,m})\right)_{G_1}$. Then, $h(\Phi^B,\Phi^B) = (\Phi,\Phi)g$, with $g=(g^-,g^+)\colon \mathbb{Z}^2 \to G_2$, in particular,

$$g^+(0)\colon \mathbb{Z}^2 \to S^1 \subset \mathbf{SU}(2)\,.$$

Thus, there exists a unique function $u\colon \mathbb{Z}^2 \to [0,\pi)$ such that

$$g^+(0) = \begin{pmatrix} e^{\frac{i}{2}u} & 0 \\ 0 & e^{-\frac{i}{2}u} \end{pmatrix}.$$

Strictly speaking $\Phi$ and $g$ may not be defined on all of $\mathbb{Z}^2$ since the loop group factorization (2.7) is only defined on the open and dense subset $G_1 \cdot G_2$. However, it will be a consequence of the next theorem that $\Phi$ and $g$ are defined on $\mathbb{Z}^2$.

**Theorem 3.1.** $\Phi\colon \mathbb{Z}^2 \to L\mathbf{SU}(2)$ *satisfies the equations*

$$\begin{aligned} \Phi_{n+1,m} &= \Phi_{n,m}\Omega_{n,m}\,, \\ \Phi_{n,m+1} &= \Phi_{n,m}\Theta_{n,m}\,, \\ \Phi_{0,0} &= I\,, \end{aligned} \tag{3.2}$$

*where*

$$\Omega_{n,m}(\lambda) = \frac{1}{\Delta_+}\left(\begin{pmatrix} e^{-\frac{i}{2}(u_{n+1,m}-u_{n,m})} & 0 \\ 0 & e^{\frac{i}{2}(u_{n+1,m}-u_{n,m})} \end{pmatrix} + \delta B\lambda\right), \tag{3.3}$$

$$\Theta_{n,m}(\lambda) = \frac{1}{\Delta_-}\left(I - \frac{\delta i}{2}\begin{pmatrix} 0 & e^{\frac{i}{2}(u_{n,m+1}+u_{n,m})} \\ e^{-\frac{i}{2}(u_{n,m+1}+u_{n,m})} & 0 \end{pmatrix}\lambda^{-1}\right). \tag{3.4}$$

*The compatibility equation*

$$\Theta_{n,m}\Omega_{n,m+1} = \Omega_{n,m}\Theta_{n+1,m} \tag{3.5}$$

*unravels to*

$$\begin{aligned} &\frac{4}{\delta^2}\sin\frac{u_{n+1,m+1}-u_{n+1,m}-u_{n,m+1}+u_{n,m}}{4} \\ &\quad = \sin\frac{u_{n+1,m+1}+u_{n+1,m}+u_{n,m+1}+u_{n,m}}{4}\,, \end{aligned} \tag{3.6}$$

*which is a discrete version of the sine-Gordon equation.*

*Remark.* (i) The system (3.2) is an exact discrete analogue of the continuous equations (2.4) for an extended frame in the sense that it has similar $\lambda$-dependence.

(ii) Equation (3.6) is known as the Hirota equation [12]. It recursively defines for each set of prescribed Cauchy data a unique solution $u\colon \mathbb{Z}^2 \to [0,\pi)$ and hence, using (3.3) and (3.5), a unique solution $\Phi\colon \mathbb{Z}^2 \to L\mathbf{SU}(2)$ by (3.2). This shows that $\Phi$ is indeed defined on $\mathbb{Z}^2$.

*Proof.* Since

$$(\Phi, \Phi) = (h^-, h^+)(\Phi^B, \Phi^B)(g^-, g^+)^{-1}$$

we obtain two expressions for $\Omega_{n,m}$:

$$\begin{aligned}\Omega_{n,m} = \Phi_{n,m}^{-1}\Phi_{n+1,m} &= g_{n,m}^-(\Phi_{n,m}^B)^{-1}(h^-)^{-1}h^-\Phi_{n+1,m}^B(g_{n+1,m}^-)^{-1}\\ &= g_{n,m}^-\frac{1}{\Delta_+}(1+\delta B\lambda)(g_{n+1,m}^-)^{-1}\\ &= g_{n,m}^+\frac{1}{\Delta_+}(1+\delta B\lambda)(g_{n+1,m}^+)^{-1}\,.\end{aligned}$$

After cancelling $\frac{1}{\Delta_+}$, the latter expression takes values in $L^+\mathbf{SU}(2)$, while the former expression has a simple pole at $\lambda = \infty$ and thus

$$\Delta_+\Omega_{n,m} = \Omega_{n,m}^{(0)} + \Omega_{n,m}^{(1)}\lambda\,.$$

Since $g^-(\infty) = 1$ and $g^+(0) = \left(\begin{smallmatrix} e^{\frac{i}{2}u} & 0\\ 0 & e^{-\frac{i}{2}u}\end{smallmatrix}\right)$ we conclude

$$\Omega_{n,m}^{(1)} = \delta B$$

and

$$\Omega_{n,m}^{(0)} = \begin{pmatrix} e^{-\frac{i}{2}(u_{n+1,m}-u_{n,m})} & 0\\ 0 & e^{\frac{i}{2}(u_{n+1,m}-u_{n,m})}\end{pmatrix}.$$

A similar calculation gives the expression for $\Theta_{n,m}$. That the compatibility equation (3.5) is equivalent to the discrete sine-Gordon equation follows by direct computation. □

To give Theorem 3.1 geometric content, we define the discrete analogue of a $K$-surface using the Sym formula (2.5) and derive the basic properties of such a surface. It turns out that these surfaces are precisely the ones introduced and studied in [2]. Let

$$\Phi\colon \mathbb{Z}^2 \to L\mathbf{SU}(2)$$

be a discrete extended frame, i.e., $\Phi$ satisfies (3.2) for some $u\colon \mathbb{Z}^2 \to [0,\pi)$, and define

$$\begin{aligned} &f\colon \mathbb{Z}^2 \to \mathbb{R}^3 = \mathbf{su}(2)\,,\\ &f = \left(\frac{d}{d\lambda}\Big|_{\lambda=1}\Phi\right)\Phi^{-1}(1)\,.\end{aligned} \tag{3.7}$$

We will call $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$ the *discrete $K$-surface* corresponding to the solution $u\colon \mathbb{Z}^2 \to [0,\pi)$ of the discrete sine-Gordon equation (3.6). Then

$$f_{n+1,m} - f_{n,m} = Ad\Phi_{n,m}\frac{4\delta}{4+\delta^2}\begin{pmatrix} 0 & \frac{i}{2}e^{-\frac{i}{2}(u_{n+1,m}-u_{n,m})}\\ \frac{i}{2}e^{-\frac{i}{2}(u_{n+1,m}-u_{n,m})} & 0\end{pmatrix}, \tag{3.8}$$

$$f_{n,m+1} - f_{n,m} = Ad\Phi_{n,m}\frac{4\delta}{4+\delta^2}\begin{pmatrix} 0 & \frac{i}{2}e^{\frac{i}{2}(u_{n,m+1}+u_{n,m})}\\ \frac{i}{2}e^{-\frac{i}{2}(u_{n,m+1}+u_{n,m})} & 0\end{pmatrix}, \tag{3.9}$$

where we have evaluated $\Phi_{n,m}$ at $\lambda = 1$. The following geometric properties [2] can now be easily derived using the above formulas.

**Theorem 3.2.** *Let $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete $K$-surface.*
*(i) The edges have constant length, i.e.,*

$$\|f_{n+1,m} - f_{n,m}\| = \|f_{n,m+1} - f_{n,m}\| = \frac{4\delta}{4+\delta^2}\,.$$

*(ii) The angles between the edges eminating from a common vertex are given by*

$$\begin{aligned}
\angle(f_{n+1,m} - f_{n,m}, f_{n,m+1} - f_{n,m}) &= \frac{u_{n+1,m} + u_{n,m+1}}{2}\,,\\
\angle(f_{n,m+1} - f_{n,m}, f_{n-1,m} - f_{n,m}) &= \pi - \frac{u_{n,m+1} + u_{n-1,m}}{2}\,,\\
\angle(f_{n-1,m} - f_{n,m}, f_{n,m-1} - f_{n,m}) &= \frac{u_{n-1,m} + u_{n,m-1}}{2}\,,\\
\angle(f_{n,m-1} - f_{n,m}, f_{n+1,m} - f_{n,m}) &= \pi - \frac{u_{n,m-1} + u_{n,m}}{2}\,.
\end{aligned}$$

*In particular, their sum is $2\pi$ and thus the four edges eminating from a common vertex are coplanar.*

*(iii) Let $P(p,q,r)$ denote the plane spanned by the vertices $p,q,r$ and denote by*

$$\begin{aligned}
\alpha &= \angle(P(f_{n,m}, f_{n+1,m}, f_{n,m-1}), P(f_{n+1,m-1}, f_{n+1,m}, f_{n,m-1}))\,,\\
\widetilde{\alpha} &= \angle(P(f_{n+1,m}, f_{n,m}, f_{n+1,m-1}), P(f_{n,m-1}, f_{n,m}, f_{n+1,m-1}))\,,
\end{aligned}$$

*then*

$$\sin\frac{\alpha}{2}\sin\frac{\widetilde{\alpha}}{2} = \frac{4-\delta^2}{4+\delta^2}\,,$$

*which is an equivalent formulation of the discrete sine-Gordon equation (*3.6*).*

From (ii) of theorem 3.2 we see that the discrete analogue of the angle $\omega$ between the asymptotic curves on a $K$-surface, i.e., the angle $\omega_{n,m}$ between the edges $f_{n+1,m} - f_{n,m}$ and $f_{n,m+1} - f_{n,m}$, is given by

$$\omega_{n,m} = \frac{u_{n+1,m} + u_{n,m+1}}{2}\,. \tag{3.10}$$

Notice that the angle $\omega_{n,m}$ uniquely determines the remaining angles in the quadrilateral with vertices $f_{n,m}, f_{n+1,m}, f_{n,m+1}, f_{n+1,m+1}$: from theorem 3.2(ii) we see that the angle at the vertex $f_{n+1,m+1}$ is also $\omega_{n,m}$ and the angles at the vertices $f_{n+1,m}, f_{n,m+1}$ are equal, say $\widetilde{\omega}_{n,m}$. But from the discrete sine-Gordon equation (3.6) we obtain

$$\widetilde{q}_{n,m} = \frac{q_{n,m} - k}{1 - kq_{n,m}}$$

where $q = e^{i\omega}, \widetilde{q} = e^{i\widetilde{\omega}}, k = \frac{\delta^2}{4}$. Moreover, the four angles $\omega_{n,m}, \widetilde{\omega}_{n-1,m}, \omega_{n-1,m-1}, \widetilde{\omega}_{n,m-1}$ around the vertex $f_{n,m}$ add up to $2\pi$ (c.f. theorem 2.3(ii)) so that we obtain the following geometric version of the discrete sine-Gordon equation [12, 2, 1]:

$$q_{n,m}q_{n-1,m-1} = \frac{1 - kq_{n-1,m}}{q_{n-1,m} - k} \cdot \frac{1 - kq_{n,m-1}}{q_{n,m-1} - k}\,.$$

The Gauss map of a $K$-surface $f\colon D \to \mathbb{R}^3$ with extended frame $\Phi\colon D \to L\mathbf{SU}(2)$ is given by [13, 3]

$$N = Ad\Phi(1)\left(\begin{smallmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{smallmatrix}\right) : D \to S^2 \subset \mathbf{su}(2) = \mathbb{R}^3 .$$

Thus, it is natural to define the discrete Gauss map $N\colon \mathbb{Z}^2 \to S^2$ by

$$N_{n,m}\colon = Ad\Phi_{n,m}(1)\left(\begin{smallmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{smallmatrix}\right).$$

From (3.8), (3.9) one sees that $N_{n,m}$ is perpendicular to the plane spanned by the vertex star eminating from $f_{n,m}$ (c.f. theorem 3.2(ii)). The harmonic map condition for $N$, namely

$$d(*dN \times N) = 0$$

holds in the following precise sense for the discrete map $N_{n,m}$: the discrete 1-form

$$\frac{N_p + N_q}{2} \times \frac{N_p - N_q}{2}$$

where $e = \vec{pq}$ is an edge, is co-closed. For further discussions of these issues we refer the reader to [2].

## 4. Discrete sinh-Gordon equation and discrete CMC-surfaces

Applying the same procedure as in the previous section we will derive an integrable discrete analog of the sinh-Gordon equation, which will give rise to discrete CMC-surfaces.

We begin by deriving a discrete analog of an extended frame $\Phi\colon \mathbb{R}^2 \to \Lambda\mathbf{SU}(2)$ using the dressing action (2.10). Again it suffices to discretize the vacuum solution (2.11)

$$\Phi^B(x, y, \lambda) = \exp(xB(\lambda + \lambda^{-1}) + iyB(\lambda - \lambda^{-1})) ,$$

where $B = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. As before, we obtain

$$\Phi^B_{n,m}(\lambda) = \frac{1}{\Delta_+^n \Delta_-^m}(I + \delta B(\lambda + \lambda^{-1}))^n (I + i\delta B(\lambda - \lambda^{-1}))^m \in \mathbf{SU}(2) , \tag{4.1}$$

where

$$\Delta_+ = \sqrt{\det(I + \delta B(\lambda + \lambda^{-1}))} = \sqrt{1 + \frac{1}{4}\delta^2(\lambda + \lambda^{-1})^2} = \sqrt{1 + \delta^2 \cos^2 \alpha} ,$$
$$\Delta_- = \sqrt{\det(I + i\delta B(\lambda - \lambda^{-1}))} = \sqrt{1 - \frac{1}{4}\delta^2(\lambda - \lambda^{-1})^2} = \sqrt{1 + \delta^2 \sin^2 \alpha} ,$$

for $\lambda = e^{i\alpha}$. In addition we have

$$\Phi^B_{n,m}(-\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\Phi^B_{n,m}(\lambda)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) ,$$

and hence (4.1) gives the discrete vacuum solution

$$\Phi^B : \mathbb{Z}^2 \to \Lambda\mathbf{SU}(2) .$$

Now let $h \in \Lambda_A^+\mathbf{SL}(2, \mathbb{C})$ and consider

$$\Phi = h\#\Phi^B : \mathbb{Z}^2 \to \Lambda\mathbf{SU}(2)$$

where, as in the smooth case (2.10),

$$(h\#\Phi^B)_{n,m} = (h\Phi^B_{n,m})_u\,.$$

Then $h\Phi^B = \Phi g$ with $g\colon \mathbb{Z}^2 \to \Lambda^+_A \mathbf{SL}(2,\mathbb{C})$. In particular we have

$$g(0)\colon \mathbb{Z}^2 \to A\,,$$

where $A = \{\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{smallmatrix}\right); \rho \in \mathbb{R}^+\} \subset \mathbf{SL}(2,\mathbb{C})$ is the imaginary torus. Thus, there exists a unique map

$$\omega\colon \mathbb{Z}^2 \to \mathbb{R}$$

such that

$$g(0) = \left(\begin{smallmatrix} e^{\frac12 \omega_{n,m}} & 0 \\ 0 & e^{-\frac12 \omega_{n,m}}\end{smallmatrix}\right).$$

**Theorem 4.1.** *The extended frame $\Phi\colon \mathbb{Z}^2 \to \Lambda\mathbf{SU}(2)$ satisfies the following equations:*

$$\begin{aligned} \Phi_{n+1,m} &= \Phi_{n,m}\Omega_{n,m} \\ \Phi_{n,m+1} &= \Phi_{n,m}\Theta_{n,m} \\ \Phi_{0,0} &= I \end{aligned} \tag{4.2}$$

*where*

$$\begin{aligned} \Omega_{n,m} = \frac{1}{\Delta_+}\Big(&\frac{\delta}{2}\begin{pmatrix} 0 & e^{\frac12(\omega_{n+1,m}+\omega_{n,m})} \\ -e^{-\frac12(\omega_{n+1,m}+\omega_{n,m})} & 0 \end{pmatrix}\lambda^{-1} + \begin{pmatrix} \alpha_{n,m} & 0 \\ 0 & \overline{\alpha_{n,m}}\end{pmatrix} \\ &+ \frac{\delta}{2}\begin{pmatrix} 0 & e^{-\frac12(\omega_{n+1,m}+\omega_{n,m})} \\ -e^{\frac12(\omega_{n+1,m}+\omega_{n,m})} & 0 \end{pmatrix}\lambda\Big) \\ \Theta_{n,m} = \frac{1}{\Delta_-}\Big(&-\frac{i\delta}{2}\begin{pmatrix} 0 & e^{\frac12(\omega_{n,m+1}+\omega_{n,m})} \\ -e^{-\frac12(\omega_{n,m+1}+\omega_{n,m})} & 0 \end{pmatrix}\lambda^{-1} + \begin{pmatrix} \beta_{n,m} & 0 \\ 0 & \overline{\beta_{n,m}}\end{pmatrix} \\ &- \frac{i\delta}{2}\begin{pmatrix} 0 & -e^{-\frac12(\omega_{n,m+1}+\omega_{n,m})} \\ e^{\frac12(\omega_{n,m+1}+\omega_{n,m})} & 0 \end{pmatrix}\lambda\Big) \end{aligned}$$

*and*

$$\begin{aligned} |\alpha_{n,m}|^2 &= 1 - \delta^2 \sinh^2\Big(\frac{\omega_{n+1,m}+\omega_{n,m}}{2}\Big)\,, \\ |\beta_{n,m}|^2 &= 1 - \delta^2 \sinh^2\Big(\frac{\omega_{n,m+1}+\omega_{n,m}}{2}\Big)\,. \end{aligned}$$

*The compatibility condition*

$$\Omega_{n+1,m}\Theta_{n,m} = \Theta_{n+1,m}\Omega_{n,m} \tag{4.3}$$

*unravels to*

$$\begin{aligned} \alpha_{n,m+1} &= \frac{\overline{\alpha_{n,m}}}{\Delta}\cosh\frac{\omega_{n+1,m+1}-\omega_{n,m}}{2} + i\frac{\overline{\beta_{n,m}}}{\Delta}\sinh\frac{\omega_{n+1,m+1}-\omega_{n+1,m}+\omega_{n,m+1}-\omega_{n,m}}{2}, \\ i\beta_{n+1,m} &= \frac{\overline{\alpha_{n,m}}}{\Delta}\sinh\frac{\omega_{n+1,m+1}+\omega_{n+1,m}-\omega_{n,m+1}-\omega_{n,m}}{2} + \frac{\overline{\beta_{n,m}}}{\Delta}\cosh\frac{\omega_{n+1,m+1}-\omega_{n,m}}{2}, \end{aligned}$$

*where*

$$\Delta = \cosh\frac{\omega_{n+1,m}-\omega_{n,m+1}}{2}\,,$$

*and*

$$i(\alpha_{n,m+1}\beta_{n,m} - \alpha_{n,m}\beta_{n+1,m}) = \frac{1}{2}\sinh\frac{\omega_{n+1,m+1} + 2\omega_{n+1,m} + \omega_{n,m}}{2} + \frac{1}{2}\sinh\frac{\omega_{n+1,m+1} + 2\omega_{n,m+1} + \omega_{n,m}}{2},$$

*which is a discretized version of the elliptic sinh-Gordon equation.*

*Proof.* By construction we have

$$h\Phi^B = \Phi g$$

for $h \in \Lambda_A^+\mathbf{SL}(2,\mathbb{C})$ and $g\colon \mathbb{Z}^2 \to \Lambda_A^+\mathbf{SL}(2,\mathbb{C})$. Thus, using (4.1), we obtain

$$\Omega_{n,m} = \Phi_{n,m}^{-1}\Phi_{n+1,m} = g_{n,m}(\Phi_{n,m}^B)^{-1}\Phi_{n+1,m}^B g_{n+1,m}^{-1} = g_{n,m}\frac{1+\delta B(\lambda+\lambda^{-1})}{\Delta_+}g_{n+1,m}^{-1}.$$

For $g \in \Lambda\mathbf{SU}(2,\mathbb{C})$ let

$$\overline{g}(\lambda)\colon = (g(1/\overline{\lambda})^*)^{-1}$$

be the conjugation with regard to the real form $\Lambda\mathbf{SU}(2)$. Note that $g \in \Lambda_A^+\mathbf{SL}(2,\mathbb{C})$ if and only if $\overline{g} \in \Lambda_A^-\mathbf{SL}(2,\mathbb{C}) = \{g \in \Lambda\mathbf{SL}(2,\mathbb{C}) | g \text{ extends holomorphically to } |\lambda| > 1, g(\infty) \in A\}$. Then

$$\overline{\Omega}_{n,m} = \Omega_{n,m} = \overline{g}_{n,m}\frac{1+\delta B(\lambda+\lambda^{-1})}{\Delta_+}\overline{g}_{n+1,m}^{-1}$$

so that we have

$$g_{n,m}(1+\delta B(\lambda+\lambda^{-1}))g_{n+1,m}^{-1} = \overline{g}_{n,m}(1+\delta B(\lambda+\lambda^{-1}))\overline{g}_{n+1,m}^{-1}.$$

The left hand side has a simple pole at $\lambda = 0$ with residue

$$\Omega_{n,m}^{(-1)} = g_{n,m}(0)\delta B g_{n+1,m}^{-1}(0) = \frac{\delta}{2}\begin{pmatrix} 0 & e^{\frac{1}{2}(\omega_{n+1,m}+\omega_{n,m})} \\ -e^{-\frac{1}{2}(\omega_{n+1,m}+\omega_{n,m})} & 0 \end{pmatrix},$$

whereas the right hand side has a simple pole at $\lambda = \infty$ with residue

$$\Omega_{n,m}^{(1)} = \overline{g}_{n,m}(0)\delta B\overline{g}_{n+1,m}^{-1}(0) = \frac{\delta}{2}\begin{pmatrix} 0 & e^{-\frac{1}{2}(\omega_{n+1,m}+\omega_{n,m})} \\ -e^{\frac{1}{2}(\omega_{n+1,m}+\omega_{n,m})} & 0 \end{pmatrix}.$$

Hence

$$\Omega_{n,m} = \frac{1}{\Delta_+}(\Omega_{n,m}^{(-1)}\lambda^{-1} + \Omega_{n,m}^{(0)} + \Omega_{n,m}^{(1)}\lambda)$$

with

$$\Omega_{n,m}^{(0)} = \begin{pmatrix} \alpha_{n,m} & 0 \\ 0 & \overline{\alpha}_{n,m} \end{pmatrix}$$

for some $\alpha\colon \mathbb{Z}^2 \to \mathbb{C}$. From $\det\Omega_{n,m} = 1$ we obtain

$$|\alpha_{n,m}|^2 = 1 - \delta^2\sinh^2(\frac{\omega_{n+1,m}+\omega_{n,m}}{2}).$$

Similarly one shows the corresponding statements for $\Theta_{n,m}$. The compatibility equations follow by equating coefficients at powers of $\lambda$ in (4.3). $\square$

One can now proceed as in section 3 and describe the discrete analogues of CMC surfaces obtained from the above formulas. This would go beyond the intention of the present note. We refer the interested reader to [2] where a detailed discussion of this aspect can be found.

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