THE MODULI SPACE OF COMPLETE EMBEDDED CONSTANT MEAN CURVATURE SURFACES

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Abstract. We examine the space of surfaces in $\mathbb{R}^3$ which are complete, properly embedded and have nonzero constant mean curvature. These surfaces are noncompact provided we exclude the case of the round sphere. We prove that the space $\mathcal{M}_k$ of all such surfaces with k ends (where surfaces are identified if they differ by an isometry of $\mathbb{R}^3$) is locally a finite dimensional real analytic variety. When the linearization of the quasilinear elliptic equation specifying mean curvature equal to one has no L^2-nullspace we prove that $\mathcal{M}_k$ is locally the quotient of a real analytic manifold of dimension $3k-6$ by a finite group (i.e. a real analytic orbifold), for $k \geq 3$. This finite group is the isotropy subgroup of the surface in the group of Euclidean motions. It is of interest to note that the dimension of $\mathcal{M}_k$ is independent of the topology of the underlying punctured Riemann surface to which Σ is conformally equivalent. These results also apply to hypersurfaces of $\mathbb{H}^{n+1}$ with nonzero constant mean curvature greater than that of a horosphere and whose ends are cylindrically bounded.

I. Introduction

A fundamental object of study in classical differential geometry is the class of hypersurfaces with constant mean curvature in the simply connected space forms. We focus on the class of those surfaces in $\mathbb{R}^3$ (and hypersurfaces in $\mathbb{H}^{n+1}$ with an additional hypothesis) which, in addition to having constant, nonzero mean curvature, are embedded and complete. We will always use embedded to mean properly embedded. Our aim is to completely understand the moduli space of all such surfaces. In this note we describe the local structure of this space.

The main body of this paper treats the case of surfaces in $\mathbb{R}^3$; the applications to hypersurfaces in $\mathbb{H}^{n+1}$ are discussed in §5. We take the mean curvature of a submanifold to be the sum of the principal curvatures rather than the average, so that the unit sphere in $\mathbb{R}^3$ has constant mean curvature $\mathbf{H} \equiv 2$. A compact, embedded surface of constant mean curvature is necessarily a round sphere, thus the surfaces considered here are all

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noncompact. We define the "ends", E_j, of an embedded surface Σ with finite topology to be the noncompact connected components of the surface near infinity i.e.

$$\Sigma \cap (\mathbb{R}^3 \setminus B_R(0)) = \cup_{j=1}^{k} E_j,$$

where $B_R(0)$ denotes the ball of radius R about the origin and R is chosen sufficiently large so that the number k is constant for all $R' > R$. For surfaces of constant nonzero mean curvature $\mathbf{H}$ in $\mathbb{R}^3$, the sign of $\mathbf{H}$ and its particular value may be changed by a reversal of orientation and a homothety respectively. Thus, we always normalize our surfaces to have mean curvature $\mathbf{H} \equiv 1$. The canonical example of such a surface is the unit cylinder. In 1841, C. Delaunay [1] discovered a one-parameter family of embedded constant mean curvature (CMC) surfaces of revolution. These surfaces are periodic and interpolate between the unit cylinder and the singular surface formed by a string of spheres of radius 2, each tangent to the next along a fixed axis. In particular, he established that every CMC surface of revolution was necessarily one of these "Delaunay surfaces". Examples of embedded, complete CMC surfaces with more complicated topology were finally found in 1987 by N. Kapouleas [4]. Kapouleas produced his surfaces by first constructing surfaces which are approximately CMC and then perturbing them to nearby CMC surfaces by solving the partial differential equation which prescribes the mean curvature to 1. This construction yields embedded, complete surfaces with arbitrary genus and k ends, $k \geq 3$. Kapouleas also constructed, both compact and noncompact, immersed CMC surfaces. K. Grosse-Brauckmann [2] has used conjugate surface constructions to obtain families of symmetric embedded complete CMC surfaces. In particular he has shown that the genus zero surfaces with k ends possessing the maximal symmetry (k-fold dihedral symmetry, interchanging the ends and reflecting across a plane) lie in a one parameter family, realizing each asymptotic Delaunay surface twice except for an extremal asymptotic surface $D_{\bar{\epsilon}}, \bar{\epsilon} < 1$, which is uniquely realized. $\bar{\epsilon}$ is determined by the requirement that the ratio of the minimum 'neck size' $\bar{\epsilon}$ to the maximum 'bulge' (see below) is equal to $1/(k-1)$.

Results of Meeks [9], and Korevaar-Kusner-Solomon [7] establish that a CMC surface with at most two ends is necessarily Delaunay. Furthermore, the main result of [7] implies that every embedded, complete CMC surface is "asymptotically Delaunay". This result is a central ingredient in our analysis of the space of all CMC surfaces and is recalled more carefully in the next section.

We define $\mathcal{M}_k$ to be the space of all complete, embedded CMC surfaces in $\mathbb{R}^3$ with k ends, where two surfaces are considered equivalent if they differ by a rigid motion of $\mathbb{R}^3$. In [5], Korevaar and Kusner examined the degree to which general elements $\Sigma \in \mathcal{M}_k$ 'look like' the surfaces constructed by Kapouleas. In particular, they showed that any $\Sigma \in \mathcal{M}_k$ is contained in a regular neighborhood of a piecewise linear graph in $\mathbb{R}^3$. This neighborhood is given by a union of solid, half-infinite cylinders of radius 6, cylindrical segments of radius 6 and solid balls of radius 21; the number of each type is bounded by the topology of Σ. Consequently they established a priori area and curvature bounds for $\Sigma \in \mathcal{M}_k$. This led to a local compactness result and suggested that $\mathcal{M}_k$ is a finite dimensional real analytic variety, which is what we establish here.

Let $\Sigma \hookrightarrow \mathbb{R}^3$ be a complete, embedded CMC surface with k ends, with unit outward normal ν. One way to obtain nearby surfaces is as follows. Given a function $\phi \in C^\infty(\Sigma)$, we define a new surface Σ_ϕ by

$$\Sigma_\phi = \{x + \phi(x)\nu(x) : x \in \Sigma\}.$$

Σ_ϕ will also be embedded provided that ϕ is sufficiently small in C^2-norm. We need to consider a slightly larger class of variations of Σ than just those arising from normal graphs in order to allow for variations of the directions of the ends of Σ. These other variations are discussed more carefully below. The mean curvature, $\mathbf{H}_\phi$, of Σ_ϕ may be expressed in terms of ϕ and $\Sigma \hookrightarrow \mathbb{R}^3$ as follows

$$N(\phi) \equiv 2(\mathbf{H}_\phi - 1) = \Delta\phi + |\mathbf{A}|^2\phi + Q(\phi, \nabla\phi, \nabla^2\phi), \tag{1.1}$$

where Δ denotes the Laplace-Beltrami operator on Σ, $|\mathbf{A}|^2$ is the squared norm of the second fundamental form $\mathbf{A}$ of Σ, and $Q(\phi, \nabla\phi, \nabla^2\phi)$ is the quadratically vanishing nonlinearity. An explicit calculation of $Q(\phi, \nabla\phi, \nabla^2\phi)$ may be found in [4], Lemma C.2. From this we see that Σ_ϕ will have constant mean curvature $\mathbf{H}_\phi \equiv 1$ if and only if $N(\phi) = 0$. Moreover the linearized (or Jacobi) operator applied to ϕ is simply

$$L\phi = \frac{\partial}{\partial t} N(t\phi)\Big|_{t=0} = \Delta\phi + |\mathbf{A}|^2\phi. \tag{1.2}$$

Analyzing this operator, and a modified one introduced below, allows us to establish the basic results concerning the local structure of $\mathcal{M}_k$; in particular, the nonexistence of a L^2-nullspace for L at $\Sigma \in \mathcal{M}_k$ guarantees that $\mathcal{M}_k$ is locally a real analytic orbifold near Σ.

We now state our main result concerning the local structure of $\mathcal{M}_k$.

Theorem 1.3. *$\mathcal{M}_k$ is locally a finite dimensional real analytic variety. Moreover, if $\Sigma \in \mathcal{M}_k$ satisfies the hypothesis:*

$$\text{If } \phi \in L^2(\Sigma) \text{ and } L\phi = 0, \text{ then } \phi = 0 \tag{1.4}$$

then in a neighborhood of Σ, $\mathcal{M}_k$ is the quotient of a real analytic manifold of dimension $3k-6$ by the finite isotropy subgroup of Σ in the group of Euclidean motions, for $k \geq 3$.

The proof of Theorem 1.3 is presented in Theorems 3.1 and 4.1 below.

Remark 1.5. For many CMC surfaces this isotropy subgroup is trivial; however, there are numerous examples of CMC surfaces for which this subgroup is nontrivial (see [2] and [4]).

Remark 1.6. As noted above, there do not exist CMC surfaces with only one end. When $k = 2$, any such surface is Delaunay and these form a $3k - 5 = 1$ dimensional family. The proof of Theorem 1.3 recovers this fact. It is interesting to note that the construction given by Kapouleas has $3k - 6$ continuous parameters (see [4], Remark

4.6), however it is unknown whether or not these give rise to continuous families of surfaces ([4], Remark 2.4). Theorem 1.3 gives compelling evidence that, in the case that a surface Σ constructed as in [4] satisfies hypothesis (1.4), the $3k-6$ continuous parameters of Kapouleas do yield continuous families of CMC surfaces. In this case the construction given in [4] would produce an open set in $\mathcal{M}_k$.

Remark 1.7. Theorem 1.3 actually is valid in a more general context. The surface Σ need not be embedded but only 'weakly' or 'Alexandrov' embedded, i.e. $\Sigma = F(\partial\Omega)$ where F is an immersion into $\mathbb{R}^3$ of a connected, open domain $\Omega \subset \mathbb{R}^3$. For such surfaces the Alexandrov reflection argument of [7] is still valid and Theorem 2.2 (see below) still holds. This is the main geometric condition that a CMC surface Σ needs to satisfy in order for us to establish Theorem 1.3.

This work arose in part from a previous study by Mazzeo, Pollack and Uhlenbeck [8] of the solutions to the singular Yamabe problem on $\mathbb{S}^n \setminus \Lambda$ where $\Lambda = \{p_1, \dots, p_k\}$ is a set of k points. The problem there was to understand the space $\mathcal{M}_\Lambda$ of all metrics conformal to the standard metric which are complete on $\mathbb{S}^n \setminus \Lambda$ and have constant scalar curvature. The existence of such metrics was established by Schoen [11] and the main result of [8] is that $\mathcal{M}_\Lambda$ is locally a real analytic variety and is a real analytic manifold of dimension k provided a condition analogous to (1.4) is satisfied. The proof of the "good case" of Theorem 1.3, i.e. when L has no L^2-nullspace is completely analogous to the proof of [8, Corollary 5.5], and is sketched below in the proof of Theorem 3.1. The proof in [8] concerning the structure of $\mathcal{M}_\Lambda$ as a real analytic variety does not generalize to the constant mean curvature setting. This forced us to find a different proof, which is given in §4 below. This analysis also applies to the singular Yamabe problem and provides a new, straightforward proof that $\mathcal{M}_\Lambda$ is always locally a real analytic variety.

Many authors have remarked on the strong analogy between constant mean curvature surfaces and constant scalar curvature metrics. In fact, most theorems in one subject have a counterpart in the other. This note reinforces this analogy, however a deep explanation for the relationship between these two problems, one intrinsic and the other extrinsic, is still missing.

II. Background Material

In this section we record some background material, principally results from [7] and [8], which will be used in the proof of Theorem 1.3.

Given a unit vector $\mathbf{a} \in \mathbb{R}^3$ and a smooth function $\rho = \rho(t,\theta)$, for $t \in \mathbb{R}$, and $\theta \in [0, 2\pi)$, the *cylindrical graph* of ρ, about the axis $\{t\mathbf{a} : t \in \mathbb{R}\} \subset \mathbb{R}^3$, is the image of the mapping

$$\mathbf{F}(t,\theta) = t\mathbf{a} + \rho(t,\theta)\omega(\theta),$$

with $\omega(\theta) = \mathbf{b}\cos\theta + \mathbf{c}\sin\theta$, and $(\mathbf{a},\mathbf{b},\mathbf{c})$ a positively oriented orthonormal frame. The *steepness* of the graph is measured by

$$v = \sqrt{1 + \rho_t^2 + (\frac{\rho_\theta}{\rho})^2} = \frac{1}{\nu \cdot \omega},$$

where ν is the unit outward normal. As mentioned in the introduction, the Delaunay surfaces refer to the (embedded) surfaces of revolution with constant mean curvature equal to 1. Such a surface, when expressed as a cylindrical graph, is prescribed by its axis and a function $\rho = \rho(t)$, which satisfies the differential equation

$$\frac{\rho_{tt}}{v^3} - \frac{1}{\rho v} + 1 = 0. \tag{2.1}$$

The positive functions $\rho(t)$ satisfying (2.1) are all periodic and may be distinguished by their minimum value $\epsilon \in (0, 1]$, which we refer to as the Delaunay parameter of the corresponding surface D_ϵ. This parameter is the minimum 'neck size' of the surface. The maximum bulge μ is the maximum of ρ_ϵ, $1 \le \mu < 2$, where $\mu \to 2$ as $\epsilon \to 0$. The main result of [7] is that any complete, embedded CMC surface with finite topology is asymptotically Delaunay.

Theorem 2.2 [7]. *If Σ is a complete, embedded CMC surface of finite topology, then for each end E_j of Σ there exists an embedded Delaunay surface D_{ϵ_j} in $\mathbb{R}^3$ to which E_j converges exponentially. That is, for $x \in \mathbb{R}^3$ with $|x|$ sufficiently large, we may write E_j and D_{ϵ_j} as cylindrical graphs over a fixed axis, $\{t\mathbf{a} : t \in \mathbb{R}\} \subset \mathbb{R}^3$, of functions ρ_{E_j} and $\rho_{D_{\epsilon_j}}$ respectively, with*

$$|\rho_{E_j} - \rho_{D_{\epsilon_j}}| \le C\, e^{-\lambda_j t},$$

as $t \to +\infty$, where C and λ_j are positive constants, and similar estimates hold for all derivatives of ρ_{E_j} and $\rho_{D_{\epsilon_j}}$.

An important ingredient in our analysis is an understanding of the Jacobi fields on the Delaunay surfaces (i.e. solutions of $L\phi = 0$ on D_ϵ). By separating variables in the equation $L\phi = 0$, one may realize all the Jacobi fields in terms of the solutions of the corresponding family of second order ODE's indexed by the eigenvalues k^2, $k = 0, 1, 2, \ldots,$ of $-\partial_\theta^2$ on $\mathbb{S}^1$. There are six geometrically natural Jacobi fields on D_ϵ. The first two correspond to the infinitesimal translations along the Delaunay axis and the infinitesimal changes in the Delaunay parameter, and are denoted by $\phi^{(\epsilon)}_{0,1}$ and $\phi^{(\epsilon)}_{0,2}$ respectively. As functions of the variables (t, θ), these are the only Jacobi fields which are independent of θ, i.e. they are the $k = 0$ solutions. The four solutions corresponding to the first eigenvalue $k = 1$ are the infinitesimal translations orthogonal to the Delaunay axis and the infinitesimal rotations of the Delaunay axis about an orthogonal axis. These are denoted by $\phi^{(\epsilon)}_{1,1,i}$ and $\phi^{(\epsilon)}_{1,2,i}$ respectively, where $i = 1, 2$ denotes the two orthogonal axes. Note that these functions depend on the background Delaunay surface, and in particular on the Delaunay parameter ϵ. In both the $k = 0$ and $k = 1$ cases, the solutions corresponding to translations are bounded and periodic, while the solutions corresponding to changes in the Delaunay parameter and rotations of the axis are of linear growth. As observed in [7], it is easy to see that all solutions corresponding to the higher eigenvalues, $k \ge 2$, grow exponentially either as $t \to \infty$ or $t \to -\infty$.

Using the fact that the CMC surface Σ has an asymptotic axis corresponding to each end, we describe a neighborhood of Σ, containing an open set in $\mathcal{M}_k$, as follows. Some elements of $\mathcal{M}_k$ near to Σ have ends with axes in the same direction as those of Σ. These surfaces can be obtained as normal graphs over Σ, as described earlier. For other nearby surfaces, the directions of their axes may differ by a small rotation from those of Σ. Accordingly, we form a $2k$ dimensional family $\Sigma(\tau)$ of variations of Σ. Here $\tau = (\tau_1, \dots, \tau_k)$, where each $\tau_j = (\tau_{j,1}, \tau_{j,2})$ is a pair of real numbers describing a rotation of the axis for the end E_j, pivoting near ∂E_j, in the two directions orthogonal to this axis, which we index as $1, 2$. Recall that $\partial E_j \subset \partial B_R(0)$ for R sufficiently large. The exact transitions between $\Sigma \cap B_R(0)$ and the ends of each surface in this family are not too important, provided it is done smoothly and 'economically.' However, note that each such surface has constant mean curvature except in a small neighborhood of each ∂E_j, and also that $\Sigma(0) = \Sigma$. Since every CMC surface has asymptotic axes, by Theorem 2.2, we can express any such surface near Σ as a normal graph over some $\Sigma(\tau)$ with τ small.

Weighted Sobolev Spaces and the Deficiency Subspace

The function spaces we use here are exponentially weighted Sobolev spaces based on $L^2(\Sigma)$; these are written $H^s_\gamma(\Sigma)$, or just H^s_γ, for $\gamma, s \in \mathbb{R}$, $s > 3$. The last condition ensures that the spaces behave well under nonlinear operations, such as taking products (e.g. terms like $\phi\nabla^2\phi$ which occur in $Q(\phi, \nabla\phi, \nabla^2\phi)$). To define $H^s_\gamma(\Sigma)$, decompose Σ into the union of the ends $E_1, \dots, E_k$ and a compact piece K. Over K an element $h \in H^s_\gamma$ restricts to an ordinary H^s function. Over E_j, $h = e^{\gamma t}\tilde{h}$, where $\tilde{h} \in H^s([0,\infty) \times \mathbb{S}^1, dt\, d\theta)$. Note that by Theorem 2.2 the measure here is uniformly equivalent to the one induced by Σ for any fixed $\Sigma \in \mathcal{M}_k$. When $\gamma > 0$, functions in $H^s_\gamma(\Sigma)$ are allowed to have exponential growth, such as $e^{\gamma' t}$, for any $\gamma' < \gamma$, while functions in $H^s_{-\gamma}(\Sigma)$ must decay at least as fast as $e^{-\gamma t}$. In particular, $H^s_{-\gamma}(\Sigma) \subset H^s(\Sigma) \subset H^s_\gamma(\Sigma)$.

In analogy with [8], given Σ, a complete, embedded CMC surface with k ends, we use the Jacobi fields above to define a $6k-$dimensional linear space, W, which we shall call the *deficiency subspace*. Theorem 2.2 implies there are k axes and k Delaunay surfaces $\{D_{\epsilon_j}\}, j = 1, \dots, k$, about those axes to which Σ converges as $|x| \to \infty$. We define W to be the linear span of the functions $\phi^{(\epsilon_j)}_{0,1}$, $\phi^{(\epsilon_j)}_{0,2}$, $\phi^{(\epsilon_j)}_{1,1,i}$, and $\phi^{(\epsilon_j)}_{1,2,i}$ for $i = 1, 2$ and $j = 1, \dots, k$, cut off to have support outside of a ball $B_R(0) \in \mathbb{R}^3$ for some R sufficiently large as in Theorem 2.2. We note that $W \subset H^s_\gamma(\Sigma)$ for all $\gamma > 0$; however $W \not\subset H^s(\Sigma)$, since the Jacobi fields defining W are not in $L^2(\Sigma)$. Theorem 2.2 implies that the linearized operator (1.2) on Σ is an asymptotically periodic operator on each end E_j. In particular, on E_j we have

$$L = L_{\epsilon_j} + e^{-\alpha t} F, \tag{2.3}$$

where F is a second order operator with coefficients bounded in C^∞ as functions of (t, θ), L_{ϵ_j} is the Jacobi operator on D_{ϵ_j} and α is a positive constant.

Operators of the form (2.3) have arisen previously in problems in geometry. In particular, we refer to work of C. Taubes [13] where some aspects of the Fredholm theory in a similar asymptotically periodic setting were developed. In [8] a detailed analysis of certain mapping properties for such operators was obtained. The basic Fredholm result established there applies to the linearization L to give the following.

Proposition 2.4 [8]. *There exists an infinite, discrete set of numbers $\Gamma \subset \mathbb{R}$ such that the bounded operator*

$$L : H^{s+2}_{\gamma}(\Sigma) \longrightarrow H^{s}_{\gamma}(\Sigma) \tag{2.5}$$

is Fredholm for all values of the weight parameter $\gamma \notin \Gamma$. In particular, $0 \in \Gamma$, so the map (2.5) is not Fredholm on the ordinary unweighted Sobolev spaces, but is Fredholm for all values of γ sufficiently near, but not equal to zero.

The numbers $\gamma_j \in \Gamma$ are the real parts of the 'indicial roots' of the model operators L_{ϵ_j} on each end E_j. The significance of these indicial roots is explained, to some extent, in the discussion of the relative index theorem below. A more complete description of Γ and the proof of Proposition 2.4 is given in [8].

Understanding the local structure of the space $\mathcal{M}_k$ near Σ is closely related to determining when L is actually injective or surjective, and when not injective, understanding the elements in its nullspace. In general this is a very difficult question; however, using duality and the fact that L is self-adjoint on L^2, i.e. when $\gamma = 0$, we have the following corollary.

Corollary 2.6. *Suppose that L has no L^2-nullspace. Then for all $\delta > 0$ sufficiently small*

$$L : H^{s+2}_{\delta} \longrightarrow H^{s}_{\delta} \quad \textit{is surjective,}$$
$$L : H^{s+2}_{-\delta} \longrightarrow H^{s}_{-\delta} \quad \textit{is injective.}$$

Corollary 2.6 states that if L has no global L^2-nullspace, then we can find a solution $w \in H^{s+2}_{\delta}$ to the equation $Lw = f$ for every $f \in H^{s}_{\delta}$, whenever $\delta > 0$. In particular, this holds whenever $f \in H^{s}_{-\delta}$. Whenever f decays at some exponential rate like this, we expect the solution w to be somewhat better behaved than a general H^{s+2}_{δ} function. It is immediate that w is in H^{s+2}_{δ} for any $\delta > 0$, however we can do even better. This is the subject of what we will call the

Linear Decomposition Lemma 2.7 [8]. *Suppose $f \in H^{s}_{-\delta}$ for some $\delta > 0$ sufficiently small, and $w \in H^{s+2}_{\delta}$ solves $Lw = f$. Then $w \in H^{s+2}_{-\delta} \oplus W$, i.e. w may be decomposed into a sum $v + \phi$, where $v \in H^{s+2}_{-\delta}$ decays at the same rate as f and ϕ is in the deficiency subspace W.*

The space W may also be thought of as the 'parameter space' for the linear problem, since its elements are the potentially occurring parameters (disregarding the exponentially decreasing components) for Jacobi fields, i.e. those v such that $Lv = 0$. These functions are the subject of the next subsection.

The Bounded Nullspace

It remains to understand the nullspace of the Jacobi operator L. When acting on H^{s+2}_{δ} (for $\delta > 0$) the nullspace of L will be the direct sum of the L^2-nullspace and what we call the 'bounded nullspace' $\mathcal{B}$, defined by

$$\mathcal{B} = \{v \in H^{s+2}_{\delta} : Lv = 0, v \notin H^{s+2}_{-\delta}\}. \tag{2.8}$$

It follows immediately from the linear decomposition lemma (2.7) that $\mathcal{B} \subset H^{s+2}_{-\delta} \oplus W$. In addition one can show that (2.3) implies that the nullspace of L in L^2 is the same as the nullspace of L in $H^{s+2}_{-\delta}$, i.e. any solution $v \in L^2$ to $Lv = 0$ decays exponentially at some uniform rate (see Proposition 4.14, [8]). Analyzing the L^2-nullspace of L is a difficult problem which we do not address here. However, the dimension of $\mathcal{B}$ may be calculated by means of a relative index theorem. We define the relative index for any $\gamma_1, \gamma_2 \notin \Gamma$ by

$$\text{rel-ind}\,(\gamma_1, \gamma_2) = \text{ind}\,(\gamma_1) - \text{ind}\,(\gamma_2).$$

where

$$\text{ind}\,(\gamma) = \text{dim ker } L|_{H^s_\gamma} - \text{dim coker } L|_{H^s_\gamma}.$$

From the fact that the adjoint of L on H^s_γ is L on $H^{-s}_{-\gamma}$, and using duality and elliptic regularity, it is easy to see that for $\delta > 0$ sufficiently small we have

$$\text{rel-ind}\,(\delta, -\delta) = 2\dim(\mathcal{B}).$$

This reduces the problem of calculating $\dim(\mathcal{B})$ to that of calculating rel-ind $(\delta, -\delta)$. The latter may be calculated by using a Relative Index Theorem of R. Melrose for metrics with asymptotically cylindrical ends (see [10]). The deformation Σ_t of $\Sigma = \Sigma_0$, to a surface Σ_1 with exactly cylindrical ends may be done in such a way that the corresponding family of operators $L_t = \Delta_{\Sigma_t} + |\mathbf{A}_{\Sigma_t}|^2$ are Fredholm on H^s_δ and $H^s_{-\delta}$ for every $0 \leq t \leq 1$. This is a consequence of the fact that the Jacobi fields corresponding to the $k > 1$ eigenvalues grow or decay exponentially on D_ϵ for all $\epsilon \in (0, 1]$. This means that $\delta \notin \Gamma_t$ for any $0 \leq t \leq 1$, where Γ_t is the set of weights where L_t fails to be Fredholm as in Proposition 2.4. Under such deformations the relative index of the operators L_t is independent of t.

The relative index is computed in terms of asymptotic data on this deformed surface. Since all ends of Σ_1 are cylindrical, the model operator at each end is simply the linearization L on the cylinder, $\partial_t^2 + \partial_\theta^2 + 1$. The asymptotic data we need to compute is the sum of the 'ranks' of the poles along the real line of the 'indicial family' of this model operator, as defined in [10]. The indicial family is obtained by taking the Fourier transform in the t variable. Letting ζ be the dual variable, we obtain the indicial family $I(\zeta) \equiv \partial_\theta^2 + 1 - \zeta^2$. This is a holomorphic family of elliptic operators whose inverses, $I(\zeta)^{-1}$, form a meromorphic family of operators. The poles of this family occur at the elements of

$$\mathcal{P} = \{\pm 1, 0, \pm i\sqrt{k^2 - 1}\} \text{ for } k \geq 2.$$

The only poles along the real line are at ± 1 and at 0. The poles at ± 1 are simple, and have rank 1. This corresponds to the fact that these arise from the eigenvalue $\lambda_0 = 0$ of $-\partial_\theta^2$, which is simple. The pole at 0 arises from the multiplicity two eigenvalue $\lambda_1 = 1$ of $-\partial_\theta^2$. Since this pole is double, the rank at 0 counts each of these eigenvalues twice. Hence, the rank at 0 is 4. Thus the total rank, from all these poles, is 6. Summing over all k ends, we conclude that the relative index is $6k$. Hence

Theorem 2.9. $\dim(\mathcal{B}) = 3k$.

The argument outlined above is exactly analogous to Theorem 4.24 of [8] and we refer there for more details of the proof of Theorem 2.9.

III. The Smooth Case

In this section we indicate how the results of §2 establish that $\mathcal{M}_k$ is a real analytic orbifold locally near Σ under the assumption that the linear operator L has no L^2−nullspace.

Theorem 3.1. *Suppose that $\Sigma \in \mathcal{M}_k$ satisfies hypothesis (1.4). Then there is an open set $\mathcal{U} \subset \mathcal{M}_k$ containing Σ, such that $\mathcal{U}$ is the quotient of a real analytic manifold of dimension $3k - 6$ by the finite isotropy subgroup of Σ in the group of Euclidean motions, for $k \geq 3$. When $k = 2$, $\mathcal{U}$ is a real analytic manifold of dimension $3k - 5 = 1$; indeed $\mathcal{M}_2 \cong (0, 1]$.*

Proof. We need to modify the nonlinear operator defined in (1.1) slightly in order to accommodate normal variations off of the surfaces $\Sigma(\tau)$, with τ small, whose ends are rotations of those of Σ. Consider the operator

$$N(\phi, \tau) \equiv 2(\mathbf{H}_{\phi,\tau} - 1) = \Delta_{\Sigma(\tau)}\phi + |\mathbf{A}(\Sigma(\tau))|^2\phi + Q_{\Sigma(\tau)}(\phi, \nabla\phi, \nabla^2\phi),$$

where $\mathbf{H}_{\phi,\tau}$ is the mean curvature of the surface given by the normal graph of ϕ over $\Sigma(\tau)$. We shall continue to refer to this as the operator N. The set $\mathcal{U}$ is contained in the set of all solutions of $N(\phi, \tau) = 0$.

The main problem is to find spaces on which N acts as a real analytic function and on which its linearization L is surjective. Hypothesis (1.4) coupled with the linear decomposition lemma implies that

$$L : H^{s+2}_{-\delta} \oplus W \to H^s_{-\delta} \tag{3.2}$$

is a surjective map. However, the nonlinear operator when applied to elements of the form $(v + \psi)$ with $v \in H^{s+2}_{-\delta}$ and $\psi \in W$ does not map into $H^s_{-\delta}$. This may be remedied by recalling that the elements of our parameter space W are composed of functions which are the derivatives of actual one parameter families of solutions to $N_{\epsilon_j}(v_\eta(t, \theta)) = 0$ on each asymptotic Delaunay surface D_{ϵ_j} of Σ. Given a Delaunay surface D_ϵ there exist functions $u_{\epsilon+\eta}(t)$, $\eta \in (-\epsilon, 1 - \epsilon]$, which parameterize the other Delaunay surfaces as deformations off of D_ϵ, i.e.

$$D_{\epsilon+\eta} = \{x + u_{\epsilon+\eta}(t)\nu(x) : x \in D_\epsilon\}.$$

In particular, on D_ϵ, $N(u_{\epsilon+\eta}(t)) = 0$ and

$$\phi_{0,2}^{(\epsilon)} \equiv \left. \frac{d}{d\eta} u_{\epsilon+\eta}(t) \right|_{\eta=0}$$

satisfies $L\phi_{0,2}^{(\epsilon)} = 0$. The curve $v_\eta = u_{\epsilon+\eta}(t)$ is a one parameter family of solutions to the nonlinear equation whose tangent vector at $\eta = 0$ is represented by the Jacobi field corresponding to the infinitesimal changes in the Delaunay parameter on D_ϵ. There are similarly defined curves corresponding to the other Jacobi fields $\phi_{0,1}^{(\epsilon)}$, and $\phi_{1,1,i}^{(\epsilon)}$, for $i = 1, 2$. The remaining Jacobi fields $\phi_{1,2,i}^{(\epsilon)}$ in W are obtained as the derivatives of the family of surfaces $\Sigma(\tau)$. This allows us to define a corresponding 6k-dimensional manifold, $\mathbb{W} = \mathbb{W}_0 \times \mathcal{V}$, where $\mathcal{V}$ is a small ball in the τ parameter space and $T_0\mathbb{W}_0$ is the span of $\{\phi_{0,1}^{(\epsilon)}, \phi_{0,2}^{(\epsilon)}, \phi_{1,1,1}^{(\epsilon)}, \phi_{1,1,2}^{(\epsilon)}\}$. $\mathbb{W}$ is composed of curves of solutions which are annihilated by N_{ϵ_j} on E_j. By construction $T_0\mathbb{W} = W$. Furthermore the nonlinear operator

$$N : H^{s+2}_{-\delta} \oplus \mathbb{W} \to H^s_{-\delta}$$

defined by $N(\eta_1, \eta_2, \tau) = N(\eta_1 + \eta_2, \tau)$ is a real analytic map with L as its surjective linearization.

To verify the last claim we proceed as follows. The main step is to write $N(\eta_1+\eta_2, \tau)$ in such a way that it obviously belongs to $H^s_{-\delta}$. On each end E_j we may write N as an exponentially small perturbation of the corresponding operator on the asymptotic Delaunay surface, i.e. $N = N_\epsilon + e^{-\alpha t}\hat{Q}$. We then use a common formula for the remainder in Taylor's theorem to obtain

$$N(\eta_1 + \eta_2, \tau) = N_\epsilon(\eta_2, \tau) + e^{-\alpha t}\hat{Q}(\eta_1 + \eta_2, \tau) + \left[\int_0^1 N'(s\eta_1 + \eta_2, \tau)\, ds \right] \eta_1.$$

Since $N_\epsilon(\eta_2, \tau) = 0$ and $N'(s\eta_1 + \eta_2, \tau)$ is the linearization $\Delta + |\mathbf{A}|^2$ for the surface given as the normal graph of $s\eta_1 + \eta_2$ over $\Sigma(\tau)$, every term on the right here is in $H^s_{-\delta}$ so long as $\delta < \alpha$. Furthermore, every term is real analytic in (η_1, η_2, τ). Real analyticity of N for functions supported in the interior, away from the ends, follows directly from the real analyticity of the mean curvature equation.

We now apply the real analytic implicit function theorem to conclude that there is a set $\mathcal{U}_0 \subset H^{s+2}_{-\delta} \oplus \mathbb{W}$, consisting of those functions annihilated by N on Σ, such that $\mathcal{U}_0$ is a finite dimensional real analytic manifold. The dimension of $\mathcal{U}_0$ is simply the dimension of the nullspace L in (3.2). This is precisely the bounded nullspace $\mathcal{B}$, whose dimension is $3k$ as shown in Theorem 2.9. The open set $\mathcal{U} \subset \mathcal{M}_k$ is obtained by identifying those functions in $\mathcal{U}_0$ which arise from the (real analytic) space of isometries of $\mathbb{R}^3$. The dimension of $\mathcal{U}$ is the dimension of the space of solutions to the partial differential equation $N(\phi, \tau) = 0$ minus the dimension of $\mathrm{Isom}(\mathbb{R}^3)$ plus the dimension of $\mathrm{Iso}(\Sigma)$, the isotropy subgroup of Σ. When $k \geq 3$, $\mathrm{Iso}(\Sigma)$ is finite (and possibly trivial), and when $k = 2$ it consists of a subgroup of rotations (isomorphic to $\mathbb{S}^1$) and a subgroup (discrete unless $\epsilon = 1$ and Σ is the cylinder) of translations of $\Sigma = D_\epsilon$ along the Delaunay axis. Thus the dimension of $\mathcal{U}$ is $3k-6$ when $k \geq 3$, and $3k-5 \equiv 1$ when $k=2$. When $k \geq 3$ the existence of a finite isotropy subgroup of Σ implies that the action of Isom($\mathbb{R}^3$) on $\mathcal{U}_0$ is not free and thus $\mathcal{U}$ is a real analytic orbifold. This completes the proof of Theorem 3.1

Remark 3.3. Every element $\hat{\Sigma} \in \mathcal{U} \subset \mathcal{M}_k$, may be represented by some $(\eta, \tau) = (\eta_1, \eta_2, \tau) \in H^{s+2}_{-\delta} \oplus \mathbb{W}$. The function $\eta_2 \in \mathbb{W}_0$ consists of some combination of changes in the asymptotic Delaunay parameters, and asymptotic translations along and orthogonal to the axes of each end. τ corresponds to rotations of these axes. Determining precisely which of these $6k$ parameters occur in the deformations of a given surface Σ is an interesting geometric problem. The deformations which can occur as global solutions of the equation $N(\phi, \tau) = 0$ correspond, on the linear level, to the bounded nullspace $\mathcal{B}$. We cannot determine $\mathcal{B}$ explicitly, however we remark here that if $W \cong \mathbb{R}^{6k}$ is endowed with a natural symplectic structure, then $\mathcal{B} \subset W$ is a Lagrangian subspace. We refer to ([8], Lemma 7.5) for the proof of this fact.

IV. The General Case

In this section we establish the main result about the local structure of $\mathcal{M}_k$ in the general case, when the linearization L may have an L^2-nullspace.

Theorem 4.1. *The space $\mathcal{M}_k$ is locally a finite dimensional real analytic variety.*

Proof. Given $\Sigma \in \mathcal{M}_k$, we here consider the possibility that the Jacobi operator L is not injective when acting on the space $H^{s+2}_{-\delta}$ for any $0 < \delta < \delta_0$. (Note that the alternative "good case" has been covered in §3.) Proposition (2.4) implies that the bounded operator

$$L : H^{s+2}_{+\delta} \to H^s_{+\delta}$$

is Fredholm and, by duality, has a finite dimensional cokernel which we identify with $K_{(1)} \equiv K_{-\delta}$, the kernel of L restricted to $H^{s+2}_{-\delta}$. Consider the surjective Fredholm operator

$$\mathcal{L} : H^{s+2}_{+\delta} \oplus K_{(1)} \to H^s_{+\delta}$$

defined by $\mathcal{L}(\eta, \phi) = L(\eta) + \phi \in H^s_{+\delta}$.

Since $H^s_{-\delta} \subset H^s_{\delta}$, we may now apply the Linear Decomposition Lemma (2.7) to conclude that the operator

$$\mathcal{L} : H^{s+2}_{-\delta} \oplus W \oplus K_{(1)} \to H^s_{-\delta} \tag{4.2}$$

defined by $\mathcal{L}(\eta_1, \eta_2, \phi) = L(\eta_1 + \eta_2) + \phi \in H^s_{-\delta}$ is surjective. Recall that elements of our parameter space W are composed of functions which are the Jacobi fields corresponding to one parameters families of Delaunay surfaces (corresponding to the deformations of the asymptotic Delaunay surfaces of Σ.) This allowed us to define a corresponding $6k$

dimensional manifold, $\mathbb{W}$, annihilated by the model operators N_{ϵ_j} on D_{ϵ_j} such that $T_0\mathbb{W} = W$. Thus the nonlinear operator

$$\mathcal{N} : H^{s+2}_{-\delta} \oplus \mathbb{W} \oplus K_{(1)} \to H^{s}{}_{-\delta}$$

defined by $\mathcal{N}(\eta_1, \eta_2, \tau, \phi) = N(\eta_1 + \eta_2, \tau) + \phi$ is a real analytic map with $\mathcal{L}$ as its surjective linearization. We are interested in the space

$$\begin{aligned}\mathcal{S} &= \{\tilde{\phi} \in H^{s+2}_{-\delta} \oplus \mathbb{W} : N(\tilde{\phi}) = 0\} \\ &= \{(\tilde{\phi}, \phi_0) \in H^{s+2}_{-\delta} \oplus \mathbb{W} \oplus K_{(1)} : \mathcal{N}(\tilde{\phi}, \phi_0) = \phi_0\}.\end{aligned}$$

Since $K_{(1)} \subset H^{s}{}_{-\delta}$ is a linear subspace (in particular a real analytic submanifold), the implicit function theorem shows that

$$\mathcal{T} = \{(\tilde{\phi}, \phi_0) \in H^{s+2}_{-\delta} \oplus \mathbb{W} \oplus K_{(1)} : \mathcal{N}(\tilde{\phi}, \phi_0) \in K_{(1)}\}$$

is a finite dimensional real analytic manifold. In fact, $\mathcal{T} = \ker(\Pi^\perp \circ \mathcal{N})$, where $\Pi : H^{s}{}_{-\delta} \to K_{(1)}$ is the orthogonal projection and $\Pi^\perp = \mathrm{I} - \Pi$. The linearization of $\Pi^\perp \circ \mathcal{N}$ is $\Pi^\perp \circ \mathcal{L}$, and by (4.2) this is surjective. The nullspace of $\Pi^\perp \circ \mathcal{L}$ is the set of (η_1, η_2, ϕ) such that $\mathcal{L}(\eta_1, \eta_2, \phi) \subset K_{(1)}$, since $\Pi^\perp$ annihilates any $\phi \in K_{(1)}$. If $\mathcal{L}(\eta_1, \eta_2, \phi) \in K_{(1)}$ then $\phi \in K_{(1)}$ is arbitrary and $L(\eta_1 + \eta_2) = 0$, since the range of L is orthogonal to $K_{(1)}$ in $H^{s}{}_{-\delta}$. This implies that $\eta_1 + \eta_2 \in K_{(2)} \oplus B$, where $K_{(2)} \equiv K_{-\delta}$, is a second copy of the kernel of L acting on $H^{s+2}_{-\delta}$, and B is the bounded nullspace. $\mathcal{T}$ may thus be represented locally as a graph in $\mathbb{T} = H^{s+2}_{-\delta} \oplus \mathbb{W} \oplus K_{(1)}$, i.e. there exists a real analytic map

$$\Psi : K_{(2)} \oplus B \oplus K_{(1)} \to (H^{s+2}_{-\delta} \ominus K_{(2)}) \oplus (W \ominus B)$$

such that

$$\mathcal{T} = \{\phi + \Psi(\phi) \in \mathbb{T} : \phi \in K_{(2)} \oplus B \oplus K_{(1)}\}.$$

Clearly $\mathcal{S} \subset \mathcal{T}$, in fact writing $\phi = (\phi_1, \phi_2, \phi_0)$, we have

$$\mathcal{S} = \{\phi + \Psi(\phi) \in \mathbb{T} : \mathcal{N}(\phi) = \phi_0, \phi \in K_{(2)} \oplus B \oplus K_{(1)}\}.$$

Equivalently,

$$\begin{aligned}\mathcal{S} =& \{\phi + \Psi(\phi) \in \mathbb{T} : N(\phi_1 + \phi_2 + \Psi(\phi)) = 0, \phi \in K_{(2)} \oplus B \oplus K_{(1)}\} \\ =& \{\phi \in K_{(2)} \oplus B \oplus K_{(1)} : N \circ (\tilde{I} + \Psi)(\phi) = 0\},\end{aligned}$$

where $\tilde{I}(\phi) = (\phi_1, \phi_2)$, is the projection onto the first two factors. Finally, since for any $\phi \in K_{(2)} \oplus B \oplus K_{(1)}$, $\mathcal{N} \circ (I + \Psi)(\phi) = N \circ (\tilde{I} + \Psi)(\phi) + \phi_0 \in K_{(1)}$, we see that $N \circ (\tilde{I} + \Psi)(\phi) \in K_{(1)}$. Thus we have represented locally the space of solutions to our nonlinear equation $N(v, \tau) = 0$, with v and τ small, as the zero set of

$$N \circ (\tilde{I} + \Psi) : K_{(2)} \oplus B \oplus K_{(1)} \to K_{(1)}$$

which is a real analytic map between finite dimensional spaces. As before, the space $\mathcal{M}_k$ is obtained from this space by identifying those functions which arise from an isometry of $\mathbb{R}^3$. This completes the proof of Theorem 4.1.

Remark 4.3. The technique of the proof of Theorem 4.1 is adapted from an argument of L. Simon ([12], Theorem 3), where a similar result is established in the course of his extending the Łojasiewicz inequality to infinite dimensions. This simple technique for establishing real analyticity for the solution space of a real analytic elliptic operator is well known in the setting of compact manifolds, and is often referred to as the 'Liapunov-Schmidt reduction'.

V. CMC Hypersurfaces in Hyperbolic Space

In this section we sketch how the results above apply to a class of CMC hypersurfaces in $\mathbb{H}^{n+1}$. Generalizations of the Delaunay surfaces to hypersurfaces in $\mathbb{H}^{n+1}$ (and $\mathbb{S}^{n+1}$) were given by W.Y. Hsiang [3]. He showed that there exists a one parameter family of embedded, periodic CMC surfaces of revolution about a geodesic line $\gamma \in \mathbb{H}^{n+1}$ which interpolate between a cylinder and a string of geodesic spheres.

In [6], an analogue of Theorem 2.2 was established. In dimension $n+1=3$, any complete surface in $\mathbb{H}^3$ with constant mean curvature greater than $2=n$ (that of a horosphere) has $k \geq 2$ ends, each of which is cylindrically bounded. In particular any such surface Σ is conformally a compact Riemann surface with finitely many punctures (as is the case [7] for CMC surfaces in $\mathbb{R}^3$), and the limit set of $\Sigma \subset \mathbb{H}^3$ is a set of k points in the sphere at infinity S_∞. Any such surface is asymptotic to a generalized Delaunay surface in the sense of Theorem 2.2. When $n+1>3$ this is also true provided one makes the *assumption* that the ends of Σ are cylindrically bounded, i.e. remain within a bounded distance of some geodesic ray. (For CMC hypersurfaces $\Sigma \subset \mathbb{H}^{n+1}$, cylindrical boundedness of the ends implies that $\mathbf{H} > n$.)

We let $\Lambda = \{p_1, \cdots, p_k\} \subset S_\infty \cong \mathbb{S}^n$ denote the limit set of a hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ with constant mean curvature $\mathbf{H} > n$. The space $\mathcal{M}_\Lambda$ is defined to be the set of all such hypersurfaces having Λ as their limit set, where any hypersurfaces which differ by an isometry of $\mathbb{H}^{n+1}$ are identified.

The local structure of $\mathcal{M}_\Lambda$ is given by the following theorem.

Theorem 5.1. *$\mathcal{M}_\Lambda$ is locally a finite dimensional real analytic variety. Moreover, if $\Sigma \in \mathcal{M}_\Lambda$ satisfies hypothesis (1.4) then in a neighborhood of Σ, $\mathcal{M}_\Lambda$ is the quotient of a real analytic manifold of dimension k for $k \geq 3$ (and of dimension 1 for $k=2$) by the finite isotropy subgroup of Σ in the group of hyperbolic motions, i.e. $\mathcal{M}_\Lambda$ is a real analytic orbifold.*

The operator L in Theorem 5.1 is now the linearization of the quasilinear equation which specifies the mean curvature of a surface Σ_ϕ, written as a graph over $\Sigma \in \mathbb{H}^{n+1}$, to be identically $\mathbf{H}$. The results of [6] discussed above imply that this operator is an asymptotically periodic operator as in (2.3). In particular, analogues of Proposition 2.4, Corollary 2.6, Lemma 2.7 and Theorem 2.9 hold. The distinction is that the dimension of the bounded nullspace is k as opposed to $3k$. This is due to the fact that the Jacobi fields arising from the rotations and parabolic translations fixing either end of the asymptotic Delaunay axes about the n orthogonal axes have exponential growth or decay and hence do not contribute to the relative index. The proof of Theorem 5.1 is identical to the proof given above for Theorems 3.1 and 4.1.

One may note that the dimension k of the space $\mathcal{M}_\Lambda$ is the same as the dimension given in [8] for the solutions to the singular Yamabe problem on $\mathbb{S}^n\backslash\Lambda$. In the hyperbolic setting fixing the limit set is precisely analogous to fixing the singular set $\Lambda \subset \mathbb{S}^n$ in the singular Yamabe problem. It was suggested to us by Jonathan Poritz that one might try to recover this dimension count for the moduli space of CMC surfaces in $\mathbb{R}^3$ by specifying that the asymptotic axes remain fixed as rays in $\mathbb{R}^3$. In other words, we define a new space $\mathcal{M}_k^0 \subset \mathcal{M}_k$, whose elements are those surfaces in $\mathcal{M}_k$ which have the same asymptotic Delaunay axes as Σ. We conjecture that, in the case when Σ satisfies hypothesis (1.4), $\mathcal{M}_k^0$ is also a real analytic manifold of dimension k. The point is that those Jacobi fields arising from the rotations and translations of the asymptotic Delaunay axes about the orthogonal axes should be ruled out since they correspond to one parameter families of solutions about the asymptotic Delaunay surfaces which change the Delaunay axis. It is not clear how this may be done in the proof of Theorem 1.3.

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