Dressing Orbits of Harmonic Maps

F.E. Burstall and F. Pedit

Introduction

At the heart of the modern theory of harmonic maps from a Riemann surface to a Riemannian symmetric space is the observation that, in this setting, the harmonic map equations have a zero curvature representation [19, 24, 28] and so correspond to loops of flat connections. This fact was first exploited in the mathematical literature by Uhlenbeck in her study [24] of harmonic maps $\mathbb{R}^2 \to G$ into a compact Lie group G. Uhlenbeck discovered that harmonic maps correspond to certain holomorphic maps, the *extended solutions*, into the based loop group ΩG and used this to define an action of a certain loop group on the space of harmonic maps. However, the main focus of [24] was on harmonic maps of a 2-sphere and for these maps the action reduces to an action of a finite-dimensional quotient group (see also [1, 9]).

In another direction, the zero curvature representation has been central to recent progress in the understanding of the harmonic map equations as soliton equations, i.e. as completely integrable Hamiltonian PDE . By solving certain Lax flows on loop algebras a rather complete description of all harmonic tori in symmetric spaces and Lie groups has been obtained. Of particular importance in this approach are the harmonic maps of finite type: these arise from Lax flows on finite dimensional subspaces of loop algebras and correspond to linear flows on Jacobians of certain algebraic curves [2, 3, 4, 8, 10, 18]. Among these harmonic maps, we further distinguish those of *semisimple* finite type (see Section 1.3 below) which are characterised by a semisimplicity condition on their derivative. Semisimple finite type harmonic maps account for all non-conformal harmonic tori in rank one symmetric spaces of compact type [4], all non-isotropic harmonic tori in spheres and complex projective spaces [3], and all doubly periodic solutions to the abelian affine Toda field equations for simple Lie groups [2].

The purpose of this paper is to describe some interactions between these two approaches. Our starting point is the fact that underlying all of the above results is the existence of Iwasawa type decompositions of the loop groups and algebras concerned. On the one hand, the Lax equations mentioned above arise from an Iwasawa decomposition of certain twisted loop algebras via the Adler–Kostant–Symes scheme [5]. On the other hand, the loop group action of Uhlenbeck is essentially the *dressing* action arising from the Iwasawa decompositions of the corresponding loop groups [9]. Moreover, a bridge beween these constructions is provided by Symes's formula for the solution of the Lax equations: in this construction, first applied by Symes [21] to solve the open

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Toda lattice, projection of certain complex geodesics on a factor in the Iwasawa decomposition yields (extended framings of) harmonic maps. This provides a map from a certain subspace of the loop algebra to the space of harmonic maps which intertwines the dressing and adjoint actions.

This set-up is very familiar in soliton theory where the dressing method was first developed. Here the idea is to use the dressing action to construct new solutions from old and in many cases the dressing orbits through trivial or *vacuum* solutions account for all the solutions one is interested in. It is this theme that we develop in the present paper.

We treat (primitive) harmonic maps $\mathbb{R}^2 \to G/K$ where G/K is a (k-)symmetric space. In this context, our vacuum solutions are the maps $f^A : \mathbb{R}^2 \to G/K$ defined by

$$f^A(z) = \exp(zA + \overline{zA})K,$$

where A is an element of the Lie algebra of G satisfying $[A, \overline{A}] = 0$. These maps (the geometry of which has recently been studied by Jensen–Liao [11] in the case $G/K = \mathbb{C}P^n$) are equivariant with respect to actions of the abelian group generated by A and $\overline{A}$.

Our main results concern the orbit $\mathcal{O}_A$ of f^A under the dressing action. This orbit is infinite-dimensional in contrast to those studied by Uhlenbeck and we show that every harmonic map of semisimple finite type lies in some $\mathcal{O}_A$. As a special case, we deduce that every non-isotropic harmonic torus in a sphere or complex projective space is dressing equivalent to a vacuum solution.

A distinctive feature of the orbits $\mathcal{O}_A$ is that they admit a hierarchy of commuting flows (conservation laws). We show that this hierarchy can be used to characterise the harmonic maps of finite type: a harmonic map in $\mathcal{O}_A$ is of finite type if and only if its orbit under the hierarchy is finite-dimensional.

In all these results, just as in those of [3, 4], essential use is made of the semisimplicity assumption on the derivative of the harmonic maps. In an appendix, we examine the situation when this assumption is dropped and discover intriguing relationships between this case and harmonic maps of finite uniton number in the sense of Uhlenbeck.

Special cases and partial versions of some of our results already exist in the literature: non-conformal harmonic tori in S^2—the Gauss maps of contant mean curvature tori—were studied by Dorfmeister–Wu [7] and a similar analysis of non-superminimal minimal tori in S^4 was performed by Wu [27]. However, even in these cases, our methods are different and we believe them to be more transparent.

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Notation Throughout this work, when a Lie group is denoted by an upper case letter, its Lie algebra will be denoted by the corresponding lower case gothic letter. Thus G is a Lie group with Lie algebra $\mathfrak{g}$.

1 Primitive harmonic maps and their extended framings

We are going to study primitive harmonic maps of a Riemann surface into a k-symmetric space. Such maps (or, rather, their framings) have a zero-curvature representation and so give rise to maps into a loop group. This construction is fundamental for everything that follows so we begin by reviewing this circle of ideas to establish notation and to give a context for our results.

1.1 Primitive harmonic maps

Let G be a compact semisimple Lie group. A (regular) k-symmetric G-space [14] is a coset space $N = G/K$ where $(G^\tau)_0 \subset K \subset G^\tau$ for some automorphism $\tau : G \to G$ of finite order $k \geq 2$.

Example A 2-symmetric space is just a Riemannian symmetric space of compact type.

In general, the k-symmetric spaces form a large class of reductive homogeneous spaces which include the generalised flag manifolds (that is, G/K where K is the centraliser of a torus).

The automorphism τ induces a $\mathbb{Z}_k$-grading of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{\ell \in \mathbb{Z}_k} \mathfrak{g}_\ell, \tag{1.1}$$

where, setting $\omega = e^{2\pi i/k}$, $\mathfrak{g}_\ell$ is the ω^ℓ-eigenspace of (the derivative of) τ. We have $\mathfrak{g}_0 = \mathfrak{k}^{\mathbb{C}}$, $\overline{\mathfrak{g}_\ell} = \mathfrak{g}_{-\ell}$ and

$$[\mathfrak{g}_j, \mathfrak{g}_\ell] = \mathfrak{g}_{j+\ell},$$

where all arithmetic is modulo k. In particular, defining $\mathfrak{m} \subset \mathfrak{g}$ by $\mathfrak{m}^{\mathbb{C}} = \sum_{\ell \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_\ell$, we have a reductive decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}. \tag{1.2}$$

Example When $k = 2$, $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \mathfrak{m}^{\mathbb{C}}$ so that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and (1.2) is the familiar symmetric decomposition.

The decomposition (1.1) induces a G-invariant decomposition of the tangent bundle of N which is non-trivial when $k > 2$. Indeed, set $o = eK \in N$ and let $p = g \cdot o \in N$. Then the map $\mathfrak{g} \to T_pN$ given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot p$$

is a surjection with kernel $\operatorname{Ad} g\, \mathfrak{k}$ and so gives an isomorphism between $\operatorname{Ad} g\, \mathfrak{m} \subset \mathfrak{g}$ and T_pN.

Notation If $\mathfrak{l} \subset \mathfrak{g}$ is an $\operatorname{Ad} K$-invariant subspace, we denote by $[\mathfrak{l}]$ the sub-bundle of $N \times \mathfrak{g}$ with fibres given by

$$[\mathfrak{l}]_{g \cdot o} = \operatorname{Ad} g\, \mathfrak{l}.$$

With this notation, we identify $[\mathfrak{m}]$ with TN and have a decomposition

$$TN^{\mathbb{C}} = \sum_{\ell \in \mathbb{Z}_k \setminus \{0\}} [\mathfrak{g}_\ell].$$

Definition [3] A map $f : M \to N$ of a Riemann surface into a k-symmetric space is *primitive* if $\mathrm{d}f(T^{1,0}M) \subset [\mathfrak{g}_{-1}]$.

Note that when $k = 2$, $[\mathfrak{g}_{-1}] = TN^{\mathbb{C}}$ and the primitivity condition is vacuous.

The study of primitive maps can be motivated by the following considerations: firstly, they arise naturally as twistor lifts (prolongations) of certain harmonic maps into Riemannian symmetric spaces [3, 5]. Secondly, there is a close relationship between primitive maps and solutions of the affine Toda field equations, both abelian and non-abelian. In particular, there is an essentially bijective correspondence between certain primitive maps into the full flag manifold G/T (modulo the left action of G) and solutions to the abelian affine Toda field equations [2, 5] (in this case, τ is the Coxeter–Killing automorphism).

In [5], it is shown that primitive maps are harmonic with respect to suitable invariant metrics on G/K. Moreover, it is shown that the structure equations for such maps have the same form as those for harmonic maps into 2-symmetric spaces. This motivates the following definition:

Definition [5] A map $f : M \to N$ of a Riemann surface into a k-symmetric space is *primitive harmonic* if $k = 2$ and f is harmonic or $k > 2$ and f is primitive.

1.2 Extended framings

Henceforth, we assume that the Riemann surface M is contractible (in our applications, we shall take $M = \mathbb{R}^2$). As a consequence, all maps $M \to G/K$ have global framings $g : M \to G$. We will study primitive harmonic maps via their framings.

Let $\pi : G \to G/K$ be the coset projection and $f : M \to G/K$ a primitive harmonic map with framing $g : M \to G$, thus $f = \pi \circ g$. Let $\alpha = g^{-1}\mathrm{d}g$ be the pull-back by g of the Maurer–Cartan form of G. We have a decomposition of α according to the eigenspace decomposition (1.1) of $\mathfrak{g}^{\mathbb{C}}$:

$$\alpha = \sum_{\ell \in \mathbb{Z}_k} \alpha_\ell$$

which, for a primitive harmonic map, reduces to

$$\alpha = \alpha'_{-1} + \alpha_0 + \alpha''_1,$$

where α'_{-1} is a $\mathfrak{g}_{-1}$-valued $(1,0)$-form and $\alpha''_1 = \overline{\alpha'_{-1}}$ [5]. Moreover, the condition that f be primitive harmonic amounts to demanding that the Maurer–Cartan equation for α

$$\mathrm{d}\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0 \tag{1.3}$$

decouples into three equations:

$$\mathrm{d}\alpha'_{-1} + [\alpha_0 \wedge \alpha'_{-1}] = 0 \tag{$1.4'_{\mathfrak{m}}$}$$

$$\mathrm{d}\alpha_0 + \tfrac{1}{2}[\alpha_0 \wedge \alpha_0] + [\alpha'_{-1} \wedge \alpha''_1] = 0 \tag{$1.4_{\mathfrak{k}}$}$$

$$\mathrm{d}\alpha''_1 + [\alpha_0 \wedge \alpha''_1] = 0. \tag{$1.4''_{\mathfrak{m}}$}$$

(For $k = 2$, the equations $(1.4'_{\mathfrak{m}})$ and $(1.4''_{\mathfrak{m}})$ are the harmonic map equations while, for $k > 2$, they are the projections of (1.3) onto $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$.)

For $\lambda \in \mathbb{C}^*$, define a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form by

$$\alpha_\lambda = \lambda^{-1}\alpha'_{-1} + \alpha_0 + \lambda\alpha''_1.$$

By construction, α_λ enjoys the following properties:

1. $\alpha_{\lambda=1} = \alpha$.
2. For all $\lambda \in \mathbb{C}^*$, $\tau\alpha_\lambda = \alpha_{\omega\lambda}$.
3. For all $\lambda \in \mathbb{C}^*$, $\overline{\alpha_\lambda} = \alpha_{1/\bar{\lambda}}$ so that α_λ is $\mathfrak{g}$-valued when $\lambda \in S^1$.

The crucial observation is that α satisfies the equations (1.4) if and only if

$$\mathrm{d}\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0,$$

for all $\lambda \in \mathbb{C}^*$. Thus, for each λ, we can integrate the Maurer–Cartan equations to obtain a map $F_\lambda : M \to G^{\mathbb{C}}$, unique up to left translation by a constant, satisfying $F_\lambda^{-1}\mathrm{d}F_\lambda = \alpha_\lambda$. Moreover, in view of the properties of α_λ listed above, we may choose the constants of integration to ensure

1. $F_1 = g$.
2. For all $\lambda \in \mathbb{C}^*$, $\tau F_\lambda = F_{\omega\lambda}$.
3. For all $\lambda \in \mathbb{C}^*$, $\overline{F_\lambda} = F_{1/\bar{\lambda}}$ where conjugation is the Cartan involution of $G^{\mathbb{C}}$ fixing G. In particular, $F_\lambda : M \to G$ when $\lambda \in S^1$.
4. For each $p \in M$, $\lambda \mapsto F_\lambda(p)$ is holomorphic on $\mathbb{C}^*$.

Otherwise said, we have defined a map F from M into the group $\Lambda_{\mathrm{hol}}G_\tau$ given by

$$\Lambda_{\mathrm{hol}}G_\tau = \{g : \mathbb{C}^* \to G^{\mathbb{C}} : g \text{ is holomorphic and } \overline{g(\lambda)} = g(1/\bar{\lambda}),\ g(\omega\lambda) = \tau g(\lambda)\}.$$

This prompts the following definition:

Definition A map $F : M \to \Lambda_{\mathrm{hol}}G_\tau$ is an *extended framing* if

$$F^{-1}\mathrm{d}F = \lambda^{-1}\alpha'_{-1} + \alpha_0 + \lambda\alpha''_1,$$

with α'_{-1} a $(1,0)$-form on M, or equivalently, if $\lambda F^{-1}\partial F$ is holomorphic at $\lambda = 0$ (here $F^{-1}\partial F$ is the $(1,0)$-part of $F^{-1}\mathrm{d}F$).

We have seen that any primitive harmonic map f admits an extended framing F such that F_1 is a framing of f. Conversely, it is clear that when F is an extended framing then F_1 frames a primitive harmonic map. (In fact, F_λ frames a primitive harmonic map for each $\lambda \in S^1$).

This correspondence between primitive harmonic maps and extended framings can be made bijective modulo gauge transformations by imposing base-point conditions. Fix $p_0 \in M$ and let

$$\mathcal{H} = \{f : M \to G/K : f \text{ is primitive harmonic with } f(p_0) = o\}$$

be the space of based primitive harmonic maps. Similarly, let

$$\mathcal{E} = \{F : M \to \Lambda_{\mathrm{hol}} G_\tau : F \text{ is an extended framing with } F(p_0) \in K\}$$

be the space of based extended framings (here we identify K with the constant elements of $\Lambda_{\mathrm{hol}} G_\tau$). It is then straightforward to see that we have a bijective correspondence

$$\mathcal{H} \cong \mathcal{E}/\mathcal{K}$$

where the gauge group $\mathcal{K} = \mathrm{C}^\infty(M, K)$ acts by point-wise multiplication on the right.

With all this in place, our constructions involving primitive harmonic maps will be made at the level of the corresponding extended framings. Before turning to this, however, we pause to briefly describe a class of primitive harmonic maps which will be important in the sequel.

1.3 Primitive harmonic maps of finite type

In [5], building on earlier work of several authors [2, 3, 4, 8, 18], we described a method for constructing primitive harmonic maps $\mathbb{R}^2 \to G/K$ from commuting Hamiltonian flows on loop algebras.

The loop algebra in question is $\Lambda\mathfrak{g}_\tau$ given by

$$\Lambda\mathfrak{g}_\tau = \{\xi : S^1 \to \mathfrak{g} : \quad \tau\xi(\lambda) = \xi(\omega\lambda)\}.$$

Any $\xi \in \Lambda\mathfrak{g}_\tau$ has a Fourier decomposition

$$\xi = \sum_{n\in\mathbb{Z}} \lambda^n \xi_n$$

and we distinguish the finite-dimensional subspaces $\Lambda_d \subset \Lambda\mathfrak{g}_\tau$ given by

$$\Lambda_d = \{\xi \in \Lambda\mathfrak{g}_\tau : \xi_n = 0 \text{ for } |n| > d\}.$$

Now fix $d \equiv 1 \mod k$. A *polynomial Killing field* is a map $\xi : \mathbb{R}^2 \to \Lambda_d$ satisfying the Lax equation

$$\mathrm{d}\xi = [\xi, (\lambda^{-1}\xi_{-d} + r(\xi_{1-d}))\,\mathrm{d}z + (\lambda\xi_d + \overline{r(\xi_{1-d})})\,\mathrm{d}\bar{z}]. \tag{1.4}$$

Here z is the usual holomorphic co-ordinate on $\mathbb{R}^2$ and $r : \mathfrak{g}_0 \to \mathfrak{g}_0$ is a certain linear map constructed from an Iwasawa decomposition of $\mathfrak{g}_0$ (see [5] for more details).

Polynomial Killing fields simultaneously integrate a pair of commuting Hamiltonian vector fields on Λ_d and so there is a unique such having any prescribed value at $z = 0$. A polynomial Killing field gives rise to a primitive harmonic map because α_λ defined by

$$\alpha_\lambda = (\lambda^{-1}\xi_{-d} + r(\xi_{1-d}))\,\mathrm{d}z + (\lambda\xi_d + \overline{r(\xi_{1-d})})\,\mathrm{d}\bar{z} \tag{1.5}$$

satisfies the Maurer–Cartan equations. We may therefore integrate to get an extended framing F with $F_\lambda^{-1}\mathrm{d}F_\lambda = \alpha_\lambda$ and so a primitive harmonic map. The primitive harmonic maps that arise in this way are said to be of *finite type*.

There is a necessary condition for a primitive harmonic map f to be of finite type: the Lax equation (1.4) implies that $\xi_{-d} : \mathbb{R}^2 \to \mathfrak{g}_{-1}$ must take values in a single $\mathrm{Ad}\, K^{\mathbb{C}}$-orbit so that, from (1.5), we see that, for some, and hence every, framing of f, $\alpha'_{-1}(\partial/\partial z)$ must take values in a single orbit also. For doubly periodic primitive harmonic maps (that is, those which cover a map of a torus), this condition is almost sufficient:

Theorem 1.1 [5] *A doubly periodic primitive harmonic map $\mathbb{R}^2 \to G/K$ is of finite type if, for any framing, $\alpha'_{-1}(\partial/\partial z)$ takes values in an $\mathrm{Ad}\, K^{\mathbb{C}}$-orbit of* semisimple *elements of $\mathfrak{g}_{-1}$.*

In view of this, we make the following definition:

Definition A primitive harmonic map $\mathbb{R}^2 \to G/K$ is of semisimple finite type if it admits a polynomial Killing field ξ with ξ_{-d} semisimple.

The primitive harmonic maps of semisimple type include:

(*i*) All doubly periodic non-conformal harmonic maps of $\mathbb{R}^2$ into a rank one symmetric space [4].

(*ii*) Twistor lifts of any doubly periodic non-isotropic harmonic map of $\mathbb{R}^2$ into a sphere or complex projective space [3] (see also [2, 8]). Here G/K is a flag manifold in most cases.

(*iii*) Twistor lifts of certain doubly periodic harmonic maps of $\mathbb{R}^2$ into low dimensional complex Grassmannians and quaternionic projective spaces [22, 23].

2 Dressing actions of loop groups

We have seen that primitive harmonic maps correspond to extended framings $M \to \Lambda_{\mathrm{hol}} G_\tau$. Various loop groups have non-trivial actions on $\Lambda_{\mathrm{hol}} G_\tau$ which induce actions on extended framings. In this section we define these groups and collect some elementary facts concerning their generalised Birkhoff factorisations. Using these facts, we will be able to define these actions on extended framings as well as provide a simple construction of extended framings via an analogue of the Symes formula for solutions of the Toda lattice [21].

2.1 Decompositions of loop groups

A k-symmetric space is defined by the following ingredients, which we fix once and for all:

1. A compact semisimple group G.

2. An automorphism $\tau : G \to G$ of finite order $k \geq 2$ with fixed set K.

3. The primitive k-th root of unity $\omega = e^{2\pi i/k}$.

Moreover, we fix an Iwasawa decomposition of the reductive group $K^{\mathbb{C}}$:

$$K^{\mathbb{C}} = KB$$

where B is a solvable subgroup of $K^{\mathbb{C}}$. Thus any element $k \in K^{\mathbb{C}}$ can be uniquely written as a product

$$k = k_K k_B$$

with $k_K \in K$, $k_B \in B$.

The loop groups of interest to us are spaces of smooth maps from a pair of circles in $\mathbb{C}$ to $G^{\mathbb{C}}$ which are equivariant with respect to τ and satisfy a reality condition. To define them, we fix $0 < \epsilon < 1$ and partition the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ as follows: let C_ϵ and $C_{1/\epsilon}$ denote the circles of radius ϵ and $1/\epsilon$ about $0 \in \mathbb{C}$ and define open sets by

$$I_\epsilon = \{\lambda \in \mathbb{P}^1 \colon |\lambda| < \epsilon\}, \quad I_{1/\epsilon} = \{\lambda \in \mathbb{P}^1 \colon |\lambda| > 1/\epsilon\}, \quad E^{(\epsilon)} = \{\lambda \in \mathbb{P}^1 \colon \epsilon < |\lambda| < 1/\epsilon\}.$$

Now put $I^{(\epsilon)} = I_\epsilon \cup I_{1/\epsilon}$ and $C^{(\epsilon)} = C_\epsilon \cup C_{1/\epsilon}$ so that $\mathbb{P}^1 = I^{(\epsilon)} \cup C^{(\epsilon)} \cup E^{(\epsilon)}$.

We define the group of smooth maps $\Lambda^\epsilon G_\tau$ by

$$\Lambda^\epsilon G_\tau = \{g : C^{(\epsilon)} \to G^{\mathbb{C}} : g(\omega\lambda) = \tau g(\lambda), \overline{g(\lambda)} = g(1/\bar{\lambda}), \text{ for all } \lambda \in C^{(\epsilon)}\}.$$

Here, again, the conjugation is the Cartan involution of $G^{\mathbb{C}}$ which fixes G.

Remark Observe that the reality condition $\overline{g(\lambda)} = g(1/\bar{\lambda})$ implies that $g \in \Lambda^\epsilon G_\tau$ is completely determined by its values on C_ϵ so that we have an isomorphism between $\Lambda^\epsilon G_\tau$ and the group of τ-equivariant maps $C_\epsilon \to G^{\mathbb{C}}$. In particular, $\Lambda^\epsilon G_\tau$ becomes in this way a *complex* Lie group. In what follows, we shall use this isomorphism to identify elements of $\Lambda^\epsilon G_\tau$ with their restrictions to C_ϵ.

We now define some subgroups of G:

$$\begin{aligned}\Lambda^\epsilon_E G_\tau &= \{g \in \Lambda^\epsilon G_\tau \colon g \text{ extends holomorphically to } g : E^{(\epsilon)} \to G^{\mathbb{C}}\},\\ \Lambda^\epsilon_I G_\tau &= \{g \in \Lambda^\epsilon G_\tau \colon g \text{ extends holomorphically to } g : I^{(\epsilon)} \to G^{\mathbb{C}}\}.\end{aligned}$$

By unique continuation, any element g of these subgroups satisfies the reality and equivariance conditions

$$\overline{g(\lambda)} = g(1/\bar{\lambda}), \qquad g(\omega\lambda) = \tau g(\lambda),$$

for all λ in its domain of definition. In particular, $g \in \Lambda^\epsilon_I G_\tau$ has $g(0) \in K^{\mathbb{C}}$ and we distinguish the subgroup

$$\Lambda^\epsilon_{I,B} G_\tau = \{g \in \Lambda^\epsilon_I G_\tau \colon g(0) \in B\}.$$

The main tool in all our constructions is the following Iwasawa type decomposition for the complex loop groups $\Lambda^\epsilon G_\tau$ which is due to McIntosh [16].

Theorem 2.1 *Multiplication $\Lambda_E^\epsilon G_\tau \times \Lambda_{I,B}^\epsilon G_\tau \to \Lambda^\epsilon G_\tau$ is a diffeomorphism onto.*

Remark The important fact here is the surjectivity of the multiplication which amounts to the assertion that a certain Riemann–Hilbert problem is always solvable. That this is indeed the case is a consequence of the reality conditions we have imposed on our loops.

Remark The limiting case of Theorem 2.1 as $\epsilon \to 1$ is the more familiar assertion that a loop $S^1 \to G^{\mathbb{C}}$ can be factorised as a product of a loop $S^1 \to G$ and a loop which has a holomorphic extension to $\{|\lambda| < 1\}$. This result is due to Pressley–Segal [20] and was extended to the twisted setting by Dorfmeister–Pedit–Wu [6].

As a consequence of Theorem 2.1, we have a diffeomorphism

$$\Lambda_E^\epsilon G_\tau \cong \Lambda^\epsilon G_\tau / \Lambda_{I,B}^\epsilon G_\tau$$

so that $\Lambda^\epsilon G_\tau$ and, particularly, $\Lambda_I^\epsilon G_\tau$ acts on $\Lambda_E^\epsilon G_\tau$. To describe this action explicitly, note that any $g \in \Lambda^\epsilon G_\tau$ has a unique factorisation

$$g = g_E g_I$$

with $g_E \in \Lambda_E^\epsilon G_\tau$ and $g_I \in \Lambda_{I,B}^\epsilon G_\tau$. The action of $\Lambda_I^\epsilon G_\tau$ on $\Lambda_E^\epsilon G_\tau$ is now given by

$$g \#_\epsilon h = (gh)_E, \tag{2.1}$$

for $g \in \Lambda_I^\epsilon G_\tau$, $h \in \Lambda_E^\epsilon G_\tau$. We call this action the *dressing action* on $\Lambda_E^\epsilon G_\tau$.

Remark These ideas fit into the general framework of dressing actions on Poisson–Lie groups described by Lu–Weinstein [15]: there is a Poisson–Lie structure on $\Lambda_E^\epsilon G_\tau$ for which $\Lambda^\epsilon G_\tau$ is the double group and $\Lambda_{I,B}^\epsilon G_\tau$ is the dual group. In this context, our dressing action is precisely the right dressing action of the dual group in the sense of Lu–Weinstein made into a left action in the usual way. We shall return to these matters elsewhere.

In our applications to harmonic maps, we need to factor out the constant loops K in $\Lambda_E^\epsilon G_\tau$. This is compatible with the dressing action:

Lemma 2.2 *The dressing action* (2.1) *descends to an action on $\Lambda_E^\epsilon G_\tau / K$.*

Proof First observe that K is stable under the dressing action: if $g \in \Lambda_I^\epsilon G_\tau$ with $g_0 = g(0)$ and $k \in K$, we use the Iwasawa decomposition of $K^{\mathbb{C}}$ to write

$$gk = (g_0 k)_K \left((g_0 k)_B (k^{-1} g_0^{-1} g k)\right)$$

so that

$$g \#_\epsilon k = (g_0 k)_K \in K.$$

Now, for $h \in \Lambda_E^\epsilon G_\tau$, we have

$$g \#_\epsilon (hk) = (ghk)_E = \left((gh)_E (gh)_I k\right)_E = (g \#_\epsilon h)\left((gh)_I \#_\epsilon k\right)$$

and $(gh)_I \#_\epsilon k = \left((gh)_I(0) k\right)_K \in K$. □

To see how these constructions vary with ϵ, note that, for $0<\epsilon<\epsilon'<1$, restriction of the holomorphic extensions provides injections

$$\Lambda_I^{\epsilon'}G_\tau^{\mathbb{C}} \subset \Lambda_I^{\epsilon}G_\tau, \qquad \Lambda_E^{\epsilon}G_\tau \subset \Lambda_E^{\epsilon'}G_\tau^{\mathbb{C}}$$

and similarly, for $0<\epsilon<1$, we have

$$\Lambda_{\mathrm{hol}}G_\tau \subset \Lambda_E^{\epsilon}G_\tau.$$

Indeed, it is easy to see that

$$\Lambda_{\mathrm{hol}}G_\tau = \bigcap_{0<\epsilon<1} \Lambda_E^{\epsilon}G_\tau. \tag{2.2}$$

The dressing actions are compatible with these inclusions because the following generalisation of a result of Guest–Ohnita [9]:

Proposition 2.3 *For $0<\epsilon<\epsilon'<1$, $g\in\Lambda_I^{\epsilon'}G_\tau^{\mathbb{C}}\subset\Lambda_I^{\epsilon}G_\tau$ and $h\in\Lambda_E^{\epsilon}G_\tau\subset\Lambda_E^{\epsilon'}G_\tau^{\mathbb{C}}$, we have*

$$g\#_{\epsilon'}h = g\#_{\epsilon}h \in \Lambda_E^{\epsilon}G_\tau.$$

Proof On $C^{(\epsilon')}$, we have

$$g\#_{\epsilon'}h = gh(gh)_{I^{(\epsilon')}}^{-1}$$

where the left hand side has a holomorphic extension to $E^{(\epsilon')}$ while, since $h\in\Lambda_E^{\epsilon}G_\tau$, the right hand side has a holomorphic extension to $I^{(\epsilon')}\cap E^{(\epsilon)}$. It now follows from a theorem of Painlevé that $g\#_{\epsilon'}h$ has a holomorphic extension to $E^{(\epsilon')}\cup C^{(\epsilon')}\cup(I^{(\epsilon')}\cap E^{(\epsilon)})=E^{(\epsilon)}$. Thus $g\#_{\epsilon'}h\in\Lambda_E^{\epsilon}G_\tau$ while $(gh)_{I^{(\epsilon')}}\in\Lambda_{I,B}^{\epsilon}G_\tau$ and the proposition follows from the uniqueness of the factorisation of $\Lambda^{\epsilon}G_\tau$. □

In particular, the action of $\Lambda_I^{\epsilon'}G_\tau^{\mathbb{C}}$ preserves each $\Lambda_E^{\epsilon}G_\tau$ for $0<\epsilon<\epsilon'<1$ and so, taking (2.2) into account, we conclude

Corollary 2.4 *The action of each $\Lambda_E^{\epsilon}G_\tau$ preserves $\Lambda_{\mathrm{hol}}G_\tau$ and, for $0<\epsilon<\epsilon'<1$, $g\in\Lambda_I^{\epsilon'}G_\tau^{\mathbb{C}}\in\Lambda_I^{\epsilon}G_\tau$ and $h\in\Lambda_{\mathrm{hol}}G_\tau$, we have*

$$g\#_{\epsilon'}h = g\#_{\epsilon}h.$$

Notation Henceforth, we simply write $g\#h$ for the action on $\Lambda_{\mathrm{hol}}G_\tau$.

Remark It follows from Corollary 2.4 that we can take a direct limit as $\epsilon\to0$ and so obtain an action on $\Lambda_{\mathrm{hol}}G_\tau$ of the group of germs at zero of τ-equivariant maps $\mathbb{C}\to G^{\mathbb{C}}$. This action is very similar to the one discussed by Uhlenbeck [24] although she considers only the subgroup of rational maps $\mathbb{P}^1\to G^{\mathbb{C}}$ which satisfy the reality conditions and are holomorphic at zero.

Finally, we note that, by virtue of Lemma 2.2, we have

Corollary 2.5 *The dressing action of each $\Lambda_I^{\epsilon}G_\tau$ descends to an action on $\Lambda_{\mathrm{hol}}G_\tau/K$.*

2.2 Symes formula: point-wise version

In [21], Symes gave a formula for solutions of the Toda lattice in terms of the projection of a one-parameter subgroup onto one of the factors in an Iwasawa decomposition. We shall see that a similar formula holds for extended framings. We begin by describing the space of generators for these one-parameter subgroups.

The Lie algebras of the loop groups of the previous section are simply the corresponding algebras of τ-equivariant maps $C^{(\epsilon)} \to \mathfrak{g}^{\mathbb{C}}$ and are denoted $\Lambda^{\epsilon}\mathfrak{g}_\tau$, $\Lambda^{\epsilon}_{E}\mathfrak{g}_\tau$ and so on. Throughout this section, we use the reality condition to identify elements of $\Lambda^{\epsilon}\mathfrak{g}_\tau$ and its subalgebras with maps $C_\epsilon \to \mathfrak{g}^{\mathbb{C}}$.

Define the subspace $\Lambda^{\epsilon}_{-1,\infty} \subset \Lambda^{\epsilon}\mathfrak{g}_\tau$ by

$$\Lambda^{\epsilon}_{-1,\infty} = \{\xi \in \Lambda^{\epsilon}\mathfrak{g}_\tau \colon \lambda\xi \text{ has a holomorphic extension to } I_\epsilon\}.$$

Thus $\Lambda^{\epsilon}_{-1,\infty}$ consists of those elements of $\Lambda^{\epsilon}\mathfrak{g}_\tau$ which extend meromorphically to $I^{(\epsilon)}$ with at most simple poles at 0 and ∞.

Observe that $\Lambda^{\epsilon}_{-1,\infty}$ is stable under the adjoint action of $\Lambda^{\epsilon}_{I}G_\tau$ on $\Lambda^{\epsilon}\mathfrak{g}_\tau$: if $g \in \Lambda^{\epsilon}_{I}G_\tau$ and $\xi \in \Lambda^{\epsilon}_{-1,\infty}$ then $\lambda\xi$ extends holomorphically to I_ϵ whence

$$\lambda \operatorname{Ad} g(\xi) = \operatorname{Ad} g(\lambda\xi)$$

does also.

Now define $\Phi_\epsilon : \Lambda^{\epsilon}_{-1,\infty} \to \Lambda^{\epsilon}_{E}G_\tau$ by

$$\Phi_\epsilon(\xi) = (\exp\xi)_E,$$

where the exponential map is defined point-wise: $(\exp\xi)(\lambda) = \exp_{G^{\mathbb{C}}} \xi(\lambda)$.

Arguing as in Proposition 2.3, one proves

Proposition 2.6 *For $0 < \epsilon < \epsilon' < 1$ and $\xi \in \Lambda^{\epsilon'}_{-1,\infty} \subset \Lambda^{\epsilon}_{-1,\infty}$, we have*

$$\Phi_{\epsilon'}(\xi) = \Phi_\epsilon(\xi) \in \Lambda^{\epsilon}_{E}G_\tau.$$

In particular, $\Phi_{\epsilon'}$ has image in $\bigcap_{\epsilon<\epsilon'} \Lambda^{\epsilon}_{E}G_\tau = \Lambda_{\mathrm{hol}}G_\tau$.

Notation In view of this, we shall henceforth simply write Φ for Φ_ϵ.

A useful property of Φ is that it essentially intertwines the adjoint and dressing actions of $\Lambda^{\epsilon}_{I}G_\tau$:

Proposition 2.7 *Let $g \in \Lambda^{\epsilon}_{I}G_\tau$ and $\xi \in \Lambda^{\epsilon}_{-1,\infty}$. Then*

$$\Phi(\operatorname{Ad} g\,\xi) = g\#(\Phi(\xi)k),$$

where $k = \big((\exp\xi)_I(0)g(0)^{-1}\big)_K \in K$.

Note that when $g(0) \in B$, $k = 1$ so that Φ truly intertwines the actions of $\Lambda^{\epsilon}_{I,B}G_\tau$.

Proof Using $\exp \operatorname{Ad} g\, \xi = g \exp(\xi) g^{-1}$, we have

$$\begin{aligned}\Phi(\operatorname{Ad} g\, \xi) &= (\exp \operatorname{Ad} g\, \xi)_E = (g \exp(\xi) g^{-1})_E \\ &= \left(g\Phi(\xi)(\exp \xi)_I g^{-1}\right)_E = (g\Phi(\xi)k\widetilde{g})_E \\ &= g\#(\Phi(\xi)k)\end{aligned}$$

where $\widetilde{g} \in \Lambda^\epsilon_{I,B}G_\tau$ and $k = ((\exp \xi)_I(0)g(0)^{-1})_K$ so that $\exp(\xi)_I g^{-1} = k\widetilde{g}$. □

In particular, denoting by $\overline{\Phi} : \Lambda^\epsilon_{-1,\infty} \to \Lambda_{\mathrm{hol}}G_\tau/K$ the composition of Φ with the coset projection, we conclude from Proposition 2.7 together with Corollary 2.5:

Corollary 2.8 *$\overline{\Phi}$ intertwines the adjoint and (descended) dressing actions of $\Lambda^\epsilon_I G_\tau$:*

$$\overline{\Phi}(\operatorname{Ad} g\, \xi) = g\#\overline{\Phi}(\xi),$$

for $\xi \in \Lambda^\epsilon_{-1,\infty}$, $g \in \Lambda^\epsilon_I G_\tau$.

2.3 Dressing action on extended framings

The relevance of the results of Section 2.1 to primitive harmonic maps is that the point-wise dressing action of $\Lambda^\epsilon_I G_\tau$ on $\Lambda_{\mathrm{hol}}G_\tau$ induces an action on extended framings that preserves gauge orbits and the base-point condition. We therefore arrive at an action of $\Lambda^\epsilon_I G_\tau$ on $\mathcal{H}$:

Proposition 2.9 *Let $g \in \Lambda^\epsilon_I G_\tau$ and $F : M \to \Lambda_{\mathrm{hol}}G_\tau$ be an extended framing. Define $g\#F : M \to \Lambda_{\mathrm{hol}}G_\tau$ by*

$$(g\#F)(p) = g\#(F(p)),$$

for $p \in M$. Then

- *(i) $g\#F$ is also an extended solution.*
- *(ii) If F is based (that is, $F \in \mathcal{E}$) then so is $g\#F$.*
- *(iii) If $k \in \mathcal{K}$ then*

$$g\#(Fk) = (g\#F)\widetilde{k} \tag{2.3}$$

with $\widetilde{k} \in \mathcal{K}$.

Thus $\Lambda^\epsilon_I G_\tau$ acts on $\mathcal{H} = \mathcal{E}/\mathcal{K}$.

Proof To see that $g\#F$ is an extended frame, write

$$gF = ab$$

where $a = g\#F$ and $b : M \to \Lambda^\epsilon_{I,B}G_\tau$. Then

$$a^{-1}\mathrm{d}a = \operatorname{Ad} b(F^{-1}\mathrm{d}F - b^{-1}\mathrm{d}b)$$

so that

$$\lambda a^{-1}\partial a = \operatorname{Ad} b(\lambda F^{-1}\partial F - \lambda b^{-1}\partial b).$$

Now all ingredients on the right hand side are holomorphic in λ on I_ϵ so that $\lambda a^{-1}\partial a$ is also whence a is an extended framing.

Now suppose that $F \in \mathcal{E}$ so that $F(p_0) \in K$. In the proof of Lemma 2.2 we saw that the action of $\Lambda_I^\epsilon G_\tau$ preserves K so that $(g\#F)(p_0) = g\#(F(p_0)) \in K$ as required.

Finally, (2.3) is an immediate consequence of Lemma 2.2. □

Remark Proposition 2.9 is a simple generalisation to our setting of results of Uhlenbeck [24] and Bergveldt–Guest [1] for harmonic maps into Lie groups.

2.4 Symes formula for extended framings

We now use the map Φ of Section 2.2 to construct based extended solutions $\mathbb{R}^2 \to \Lambda_{\mathrm{hol}} G_\tau$ from complex lines in $\Lambda^\epsilon_{-1,\infty}$. (In all that follows, we base maps of $\mathbb{R}^2$ at 0.)

For $\eta \in \Lambda^\epsilon_{-1,\infty}$, define $F^\eta : \mathbb{R}^2 \to \Lambda_{\mathrm{hol}} G_\tau$ by

$$F^\eta(z) = \Phi(z\eta) = (\exp z\eta)_E.$$

Proposition 2.10 *F^η is a based extended framing.*

Proof We have

$$\exp(z\eta) = F^\eta b,$$

with $b : \mathbb{R}^2 \to \Lambda^\epsilon_{I,B} G_\tau$ so that

$$\lambda(F^\eta)^{-1}\partial F^\eta = \operatorname{Ad} b(\lambda\eta\, \mathrm{d}z - \lambda b^{-1}\partial b). \tag{2.4}$$

Again all ingredients on the right hand side are holomorphic near $\lambda = 0$ (since $\eta \in \Lambda^\epsilon_{-1,\infty}$) so that $\lambda(F^\eta)^{-1}\partial F^\eta$ is also whence F^η is an extended framing. Finally, $F^\eta(0) = 1 \in K$ so that $F^\eta \in \mathcal{E}$. □

Observe that primitive harmonic maps that arise from Proposition 2.10 in this way have the same restriction on their framings as the maps of finite type: indeed, let $\eta_{-1} = (\lambda\eta)(0)$ and put $b_0 = b|_{\lambda=0} : \mathbb{R}^2 \to B$. Now set $\lambda = 0$ in (2.4) to conclude that

$$(F^\eta)^{-1}\mathrm{d}F^\eta = \lambda^{-1}\alpha'_{-1} + \alpha_0 + \lambda\alpha''_1$$

with

$$\alpha'_{-1} = \operatorname{Ad} b_0(\eta_{-1})\, \mathrm{d}z. \tag{2.5}$$

Otherwise said, $\alpha'_{-1}(\partial/\partial z)$ takes values in a single B-orbit.

In fact, all primitive harmonic maps of finite type arise from this construction: let $f : \mathbb{R}^2 \to G/K$ be a based primitive harmonic map of finite type with polynomial Killing field $\xi : \mathbb{R}^2 \to \Lambda_d$ and let $\eta = \lambda^{d-1}\xi(0) \in \Lambda^\epsilon_{-1,\infty}$. The arguments of [5, Section 4.2] carry over directly to our setting to prove:

1. F^η is an extended framing for f.
2. $\xi = \mathrm{Ad}(F^\eta)^{-1}\xi(0)$.

To summarise:

Proposition 2.11 *$f : \mathbb{R}^2 \to G/K$ is of finite type if and only if it admits an extended framing of the form F^η where, for some $d \equiv 1 \mod k$, $\eta \in \lambda^{d-1}\Lambda_d \subset \Lambda^\epsilon_{-1,\infty}$*

Finally, we note the following useful fact: viewing the Symes formula as a map

$$\Lambda^\epsilon_{-1,\infty} \to \mathcal{H},$$

we see that this map intertwines the adjoint and dressing actions of $\Lambda^\epsilon_I G_\tau$. Indeed, denoting by $[F^\eta]$ the gauge equivalence class of F^η, Corollary 2.8 gives

Lemma 2.12 *For $g \in \Lambda^\epsilon_I G_\tau$ and $\eta \in \Lambda^\epsilon_{-1,\infty}$,*

$$[F^{\mathrm{Ad}\, g\, \eta}] = g\#[F^\eta].$$

3 Dressing orbits of vacuum solutions

The main philosophy of the dressing construction [26, 28] is to construct non-trivial solutions of partial differential equations by applying a dressing action of a suitable loop group to a trivial or *vacuum* solution. In this section, we define vacuum solutions for our problem and show that any primitive harmonic map of semisimple finite type lies in the dressing orbit of such a solution.

Some care must be exercised when developing the notion of a vacuum solution of the primitive harmonic map equations: the simplest harmonic maps are the constant maps but we learn from Lemma 2.2 that the constant element of $\mathcal{H}$ is preserved by the action of $\Lambda^\epsilon_I G_\tau$ so that dressing will give us nothing new. (Similar remarks apply to the harmonic maps of finite uniton number in the sense of Uhlenbeck [24], see the appendix.) However, if one is guided by the relationship between primitive harmonic maps and Toda fields, one is led to the following class of solutions which correspond to constant solutions of the appropriate Toda field equations.

Definition A *vacuum solution* is an extended framing of the form F^{η_A} where $\eta_A = \lambda^{-1}A$ with $A \in \mathfrak{g}_{-1}$ and $[A, \overline{A}] = 0$.

Notice that it follows from the vanishing of $[A, \overline{A}]$ that A is semisimple.

In this simple case, we can perform the Iwasawa decomposition explicitly:

$$\exp(z\eta_A) = \exp(z\lambda^{-1}A + \bar{z}\lambda\overline{A}) \exp(-\bar{z}\lambda\overline{A}),$$

on C_ϵ so that

$$F^{\eta_A}(z) = \exp(z\lambda^{-1}A + \bar{z}\lambda\overline{A})$$

and the corresponding primitive harmonic map $f:\mathbb{R}^2 \to G/K$

$$f(z)=\exp(z\lambda^{-1}A+\bar{z}\lambda\overline{A})o$$

is equivariant for actions of the abelian group generated by A and $\overline{A}$.

Denote by $\mathcal{O}_A \subset \mathcal{H}$ the $\Lambda^{\epsilon}_{I,B}G_\tau$-orbit of $[F^{\eta_A}]$ so that

$$\mathcal{O}_A \cong \Lambda^{\epsilon}_{I,B}G_\tau/\Gamma^{\epsilon}_{I,B}$$

where $\Gamma^{\epsilon}_{I,B}$ is the stabiliser of $[F^{\eta_A}]$:

$$\Gamma^{\epsilon}_{I,B}=\{g\in\Lambda^{\epsilon}_{I,B}G_\tau : \quad g\#[F^{\eta_A}]=[F^{\eta_A}]\}.$$

Remark Let us pause to justify our decision to study $\mathcal{O}_A$ rather than the slightly larger $\Lambda^{\epsilon}_{I}G_\tau$-orbit of $[F^{\eta_A}]$. We shall see in Section 4 that $\mathcal{O}_A$ admits a hierarchy of commuting flows which can be used to characterise the primitive harmonic maps of finite type. These flows do not extend to the $\Lambda^{\epsilon}_{I}G_\tau$-orbit. Moreover, since $\Lambda^{\epsilon}_{I}G_\tau = K\Lambda^{\epsilon}_{I,B}G_\tau$ and the dressing action on $\mathcal{H}$ of K is merely that induced by the action of K on G/K, we see that we lose nothing essential by adopting this approach.

We will study $\mathcal{O}_A$ by using the equivariance of the map $\eta \to [F^\eta] : \Lambda^{\epsilon}_{-1,\infty} \to \mathcal{H}$ to replace the dressing action of $\Lambda^{\epsilon}_{I,B}G_\tau$ on $\mathcal{H}$ by the easier adjoint action on $\Lambda^{\epsilon}_{-1,\infty}$. To accomplish this, we must describe the fibres of this map. We begin with the following observation:

Lemma 3.1 *Let ζ, $\eta\in\Lambda^{\epsilon}_{-1,\infty}$. Then $[F^\zeta]=[F^\eta]$ if and only if $(\lambda\zeta)(0)=(\lambda\eta)(0)$ and*

$$(\operatorname{ad}\eta)^n\zeta\in\Lambda^{\epsilon}_{I}\mathfrak{g}_\tau, \tag{3.1}$$

for all $n\geq 1$.

Proof $[F^\zeta]=[F^\eta]$ if and only if $F^\zeta=F^\eta k$, for some $k\in\mathcal{K}$, and, using the definitions of F^ζ and F^η, it is straight-forward to see that this is the case precisely when

$$e(z):=\exp(-z\zeta)\exp(z\eta)\in\Lambda^{\epsilon}_{I}G_\tau,$$

for $z\in\mathbb{R}^2$. This, in turn, is the same as demanding that

$$e^{-1}\mathrm{d}e=(-\operatorname{Ad}\exp(-z\eta)\,\zeta+\eta)\,\mathrm{d}z$$

be $\Lambda^{\epsilon}_{I}\mathfrak{g}_\tau$-valued, that is,

$$e^{-\operatorname{ad} z\eta}\zeta-\eta\in\Lambda^{\epsilon}_{I}\mathfrak{g}_\tau,$$

for all $z\in\mathbb{R}^2$. Expanding this last in powers of z and comparing coefficients proves the lemma. $\square$

Applying this to the case where $\eta=\eta_A$ gives:

Proposition 3.2 $[F^\zeta]=[F^{\eta_A}]$ *if and only if* $(\lambda\zeta)(0)=A$ *and* $[\zeta,A]=0$.

Proof Write $\zeta = \sum_{n \geq -1} \lambda^n \zeta_n$ on C_ϵ. Comparing coefficients of λ in (3.1) gives

$$(\operatorname{ad} A)^n \zeta_{n-1} = 0,$$

for all $n \geq 1$. However, since A is semisimple, $\ker(\operatorname{ad} A)^n = \ker \operatorname{ad} A$ whence $[\zeta, A] = 0$ as required. $\square$

Taking Lemma 2.12 into account, we obtain the following characterisation of the fibres of $\Lambda^\epsilon_{-1,\infty} \to \mathcal{H}$ over $\mathcal{O}_A$:

Proposition 3.3 *For $\zeta \in \Lambda^\epsilon_{-1,\infty}$, $g \in \Lambda^\epsilon_{I,B} G_\tau$, $[F^\zeta] = g\#[F^{\eta_A}] \in \mathcal{O}_A$ if and only if $(\lambda\zeta)(0) = \operatorname{Ad} g(0)\, A$ and*

$$[\zeta, \operatorname{Ad} g\, A] = 0.$$

Proof By Lemma 2.12, $[F^\zeta] = g\#[F^{\eta_A}]$ if and only if $[F^{\operatorname{Ad} g^{-1}\zeta}] = [F^{\eta_A}]$. From Proposition 3.2, we see that this is the case precisely when

$$(\lambda \operatorname{Ad} g^{-1} \zeta)(0) = A$$

and

$$[\operatorname{Ad} g^{-1}\zeta, A] = 0,$$

whence the result. $\square$

As an immediate corollary to this result and Proposition 2.11, we characterise the maps of finite type in $\mathcal{O}_A$:

Corollary 3.4 *$g\#[F^{\eta_A}]$ is of finite type if and only if, for some $d \equiv 1 \mod k$, there is $\xi \in \Lambda_d$ such that $\xi_{-d} = \operatorname{Ad} g(0)\, A$ and*

$$[\xi, \operatorname{Ad} g\, A] = 0.$$

For example, the vacuum solutions are themselves of finite type: take $\xi = \lambda^{-1} A + \lambda \overline{A} \in \Lambda_1$. These solutions have *constant* polynomial Killing fields and so correspond to fixed points of the Hamiltonian flows on Λ_1.

On the other hand, it is a surprisingly simple matter to construct solutions in $\mathcal{O}_A$ which are not of finite type as the following example shows:

Example Consider the Riemann sphere as a 2-symmetric space so that $G = \mathrm{SU}(2)$, $K = S^1$ and B consists of diagonal matrices in $\mathrm{SL}(2, \mathbb{C})$ of the form

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix},$$

where $r > 0$. In this case, $\mathfrak{g}_0$ consists of trace zero diagonal matrices and $\mathfrak{g}_1 = \mathfrak{g}_{-1}$ of off-diagonal matrices. We take

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{g}_1$$

so that $A = \overline{A}$ and consider the dressing action of $B \subset \Lambda^\epsilon_{I,B} G_\tau$ on the vacuum solution $[F^{\eta_A}]$. For $b \in B$, we know from Corollary 3.4 that $b\#[F^{\eta_A}]$ is of finite type if and only if we can find $d \equiv 1 \mod k$ and $\xi \in \Lambda_d$ such that $\xi_{-d} = \operatorname{Ad} b\, A$ and

$$[\xi, \operatorname{Ad} b\, A] = 0.$$

Writing $\xi = \sum_{|n| \leq d} \lambda^n \xi_n$, this would force each $[\xi_n, \operatorname{Ad} b\, A]$ to vanish and, in particular, since

$$\overline{\xi_{-d}} = \xi_d,$$

we must have

$$[\overline{\operatorname{Ad} b\, A}, \operatorname{Ad} b\, A] = 0.$$

However, when

$$b = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$

we have

$$[\overline{\operatorname{Ad} b\, A}, \operatorname{Ad} b\, A] = \begin{pmatrix} r^4 - r^{-4} & 0 \\ 0 & r^{-4} - r^4 \end{pmatrix} \neq 0,$$

unless $r = 1$. Thus no $b\#[F^{\eta_A}]$ is of finite type unless $b = 1$. This illustrates the complexity of even the simplest case of the dressing action.

We are now in a position to identify the stabiliser group $\Gamma^\epsilon_{I,B}$:

Theorem 3.5 *$\Gamma^\epsilon_{I,B}$ is the centraliser in $\Lambda^\epsilon_{I,B} G_\tau$ of $\lambda^{-1} A \in \Lambda^\epsilon_{-1,\infty}$:*

$$\Gamma^\epsilon_{I,B} = \{g \in \Lambda^\epsilon_{I,B} G_\tau : \quad \operatorname{Ad} g\, A = A\}.$$

Proof By Proposition 3.3, $g \in \Gamma^\epsilon_{I,B}$ if and only if $\operatorname{Ad} g(0)\, A = A$ and $[\operatorname{Ad} g\, A, A]$ vanishes. Thus $\operatorname{Ad} g\, A$ takes values in $\operatorname{Ad} G^{\mathbb{C}}\, A \cap \ker \operatorname{ad} A$. On the other hand, since A is semisimple,

$$\mathfrak{g}^{\mathbb{C}} = \ker \operatorname{ad} A \oplus [A, \mathfrak{g}^{\mathbb{C}}]$$

from which it follows that $\operatorname{Ad} G^{\mathbb{C}}\, A$ intersects $\ker \operatorname{ad} A$ transversely at A so that A is an isolated point of $\operatorname{Ad} G^{\mathbb{C}}\, A \cap \ker \operatorname{ad} A$. The continuity of $\lambda \mapsto \operatorname{Ad} g(\lambda)\, A$ on the connected set I_ϵ now guarantees that $\operatorname{Ad} g(\lambda)\, A = A$ for all $\lambda \in I_\epsilon$. $\square$

We conclude from this theorem that $\mathcal{O}_A$ is diffeomorphic to the adjoint orbit of $\lambda^{-1} A$ in $\Lambda^\epsilon_{-1,\infty}$. In Section 4 we shall make use of this fact to define a hierarchy of commuting flows on $\mathcal{O}_A$.

Let us now turn to the main theorem of this section and prove that all primitive harmonic maps of semisimple finite type lie in some $\mathcal{O}_A$. We begin with a lemma which was proved in the unpublished thesis of I. McIntosh. For completeness of exposition, we shall give a proof here.

Lemma 3.6 *Let $X \in \mathfrak{g}_{-1}$ be semisimple. Then there is $A \in \operatorname{Ad} B\, X$ with $[A, \overline{A}] = 0$.*

Proof We begin by finding an element in $\operatorname{Ad} K^{\mathbb{C}} X$ with the desired property. For this, let $\|\,.\,\|$ denote the norm on $\mathfrak{g}_{-1}$ induced by the Killing inner product $(\,.\,)$ on $\mathfrak{g}$. Since X is semisimple, $\operatorname{Ad} K^{\mathbb{C}} X$ is closed in $\mathfrak{g}_{-1}$ [25] so that the restriction of $\|\,.\,\|^2$ to $\operatorname{Ad} K^{\mathbb{C}} X$ attains a minimum at some $Y \in \operatorname{Ad} K^{\mathbb{C}} X$. Now, for $\chi \in \sqrt{-1}\mathfrak{k}$, we have

$$0 = \left.\frac{d}{dt}\right|_{t=0} \| \operatorname{Ad} \exp t\chi\, Y\|^2 = ([\chi, Y], \overline{Y}) - (Y, [\chi, \overline{Y}]) = 2(\chi, [Y, \overline{Y}]).$$

However, $[Y, \overline{Y}] \in \sqrt{-1}\mathfrak{k}$ and the Killing form is definite there so that $[Y, \overline{Y}]$ vanishes as required. Thus, for some $k \in K^{\mathbb{C}}$, we have $\operatorname{Ad} k\, X = Y$. Now put $A = \operatorname{Ad} k_K^{-1} Y$ so that $A = \operatorname{Ad} k_B\, X$ and $[A, \overline{A}] = \operatorname{Ad} k_k^{-1}\, [Y, \overline{Y}] = 0$. $\square$

With this in hand, we prove:

Theorem 3.7 *Let $\eta = \sum_{n \geq -1} \lambda^n \eta_n \in \Lambda^{\epsilon'}_{-1,\infty}$ with η_{-1} semisimple. Then there is $\epsilon \leq \epsilon'$, $g \in \Lambda^{\epsilon}_{I,B} G_\tau$ and a vacuum solution F^{η_A} such that*

$$g\#[F^\eta] = [F^{\eta_A}].$$

In particular, since $\lambda^{d-1}\Lambda_d \subset \Lambda^{\epsilon'}_{-1,\infty}$ for all $0 < \epsilon' < 1$, we have

Corollary 3.8 *Any primitive harmonic map of semisimple finite type is in the $\Lambda^{\epsilon}_{I,B} G_\tau$-orbit of a vacuum solution for some $0 < \epsilon < 1$.*

Let us turn to the proof of Theorem 3.7: first, in view of Lemma 3.6, we can find $A \in \operatorname{Ad} B\, \eta_{-1}$ such that $[A, \overline{A}] = 0$ and after dressing by an element of B, we may assume that $\eta_{-1} = A$. By Proposition 3.3, it now suffices to find $g \in \Lambda^{\epsilon}_{I,B} G_\tau$, some $0 < \epsilon \leq \epsilon'$, such that

$$\operatorname{Ad} g(0)\, A = A, \quad [A, \operatorname{Ad} g\, \eta] = 0.$$

We shall construct g via the Inverse Function Theorem: since A is semisimple,

$$\mathfrak{g}^{\mathbb{C}} = \ker \operatorname{ad} A \oplus [A, \mathfrak{g}^{\mathbb{C}}]$$

and we define $\phi : \ker \operatorname{ad} A \oplus [A, \mathfrak{g}^{\mathbb{C}}] \to \mathfrak{g}^{\mathbb{C}}$ by

$$\phi(x, y) = \operatorname{Ad} \exp(y)\, x.$$

Observe that ϕ is equivariant in the following sense:

$$\omega\tau\phi(x, y) = \phi(\omega\tau x, \tau y), \tag{3.2}$$

for all $(x, y) \in \ker \operatorname{ad} A \oplus [A, \mathfrak{g}^{\mathbb{C}}]$.

Differentiating ϕ at $(A, 0)$ gives

$$\mathrm{d}_{(A,0)}\phi(v, w) = v + [w, A],$$

for $(v, w) \in \ker \operatorname{ad} A \oplus [A, \mathfrak{g}^{\mathbb{C}}]$, so that $\mathrm{d}_{(A,0)}\phi$ is an isomorphism. By the holomorphic Inverse Function Theorem, there are open neighbourhoods Ω_1 of $(A, 0)$ and Ω_2 of A such that $\phi : \Omega_1 \to$

Ω_2 is a biholomorphism. Moreover, since $(A, 0)$ is fixed by the order k linear automorphism $T : (x, y) \to (\omega\tau x, \tau y)$, we may assume, shrinking Ω_1 if necessary, that Ω_1 is T-stable.

Let $(\psi_1, \psi_2) = \phi^{-1} : \Omega_2 \to \Omega_1$ so that, for $\chi \in \Omega_2$,

$$\chi = \mathrm{Ad}(\exp \psi_2(\chi))\psi_1(\chi),$$

or, equivalently,

$$\mathrm{Ad}\exp(-\psi_2(\chi))\,\chi = \psi_1(\chi) \in \ker \mathrm{ad}\, A. \tag{3.3}$$

From (3.2) and the T-stability of Ω_1 we deduce that ψ_2 has the following equivariance property:

$$\psi_2(\omega\tau\chi) = \tau\psi_2(\chi), \tag{3.4}$$

for all $\chi \in \Omega_2$.

Since $\eta \in \Lambda^{\epsilon'}_{-1,\infty}$, $\lambda\eta$ is holomorphic on $I_{\epsilon'}$ with $(\lambda\eta)(0) = A$ so we can find $0 < \epsilon \leq \epsilon'$ such that $C_\epsilon \cup I_\epsilon \subset (\lambda\eta)^{-1}(\Omega_2)$. We may therefore define $g : C_\epsilon \cup I_\epsilon \to G^{\mathbb{C}}$ by

$$g(\lambda) = \exp(-\psi_2(\lambda\eta(\lambda))).$$

By construction g is holomorphic on I_ϵ and $g(0) = \exp(-\psi_2(A)) = 1 \in B$ so that

$$\mathrm{Ad}\, g(0)\, A = A.$$

Moreover, from (3.3), for $\lambda \in C_\epsilon$, we have

$$\mathrm{Ad}\, g(\lambda)\, \eta(\lambda) = \lambda^{-1}\, \mathrm{Ad}\exp(-\psi_2(\lambda\eta(\lambda)))\, \lambda\eta(\lambda) = \lambda^{-1}\psi_1(\lambda\eta(\lambda)) \in \ker \mathrm{ad}\, A$$

so that

$$[A, \mathrm{Ad}\, g\, \eta] = 0$$

on C_ϵ. Thus g will define our desired element of $\Lambda^{\epsilon}_{I,B} G_\tau$ so long as it satisfies the equivariance condition $g(\omega\lambda) = \tau g(\lambda)$. For this, recall that $\eta(\omega\lambda) = \tau\eta(\lambda)$ so that, using (3.4),

$$\begin{aligned} g(\omega\lambda) &= \exp(-\psi_2(\omega\lambda\eta(\omega\lambda))) = \exp(-\psi_2(\omega\tau\lambda\eta(\lambda))) \\ &= \exp(-\tau\psi_2(\lambda\eta(\lambda))) = \tau g(\lambda) \end{aligned}$$

as required. This completes the proof of Theorem 3.7.

Remark Corollary 3.8 is an extension of results of Dorfmeister–Wu [7] and Wu [27]. In those papers, a similar result was proved by different methods for $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(5)$, respectively, with k-symmetric structure given by the Coxeter–Killing automorphism. Geometrically, these cases correspond to non-conformal harmonic maps into S^2 (i.e. Gauss maps of constant mean curvature surfaces in $\mathbb{R}^3$) and (twistor lifts of) minimal non-superminimal maps into S^4. In view of [3, 4], the present result accounts for all harmonic non-isotropic 2-tori in any sphere or complex projective space.

To summarise: our results give a rather complete picture of the dressing orbits through primitive harmonic maps $[F^\eta]$ with $(\lambda\eta)(0)$ semisimple. Any non-constant map of this type lies in some $\mathcal{O}_A$ and each such orbit is infinite-dimensional. Moreover, if A and A' both generate vacuum solutions and lie in the same $\operatorname{Ad} B$-orbit, we clearly have

$$\mathcal{O}_A = \mathcal{O}_{A'}.$$

Finally, for $w \in \mathbb{C}^*$, we can scale the parameter z on $\mathbb{R}^2$ to identify $\mathcal{O}_A$ and $\mathcal{O}_{wA}$. We therefore conclude that the set of essentially different dressing orbits of vacuum solutions is parametrised by the projective $\operatorname{Ad} B$-orbits of semisimple elements of $\mathfrak{g}_1$. This is a finite-dimensional family.

We conclude this section with an application of Theorem 3.7 which provides a simplified version of Lemma 3.1. Recall that a semisimple $A \in \mathfrak{g}^{\mathbb{C}}$ is *regular* if its centraliser $\ker \operatorname{ad} A$ is abelian.

Proposition 3.9 *Let ζ, $\eta \in \Lambda^\epsilon_{-1,\infty}$ with $(\lambda\zeta)(0) = (\lambda\eta)(0)$ regular semisimple. Then $[F^\zeta] = [F^\eta]$ if and only if $[\zeta, \eta] = 0$.*

Proof In view of Lemma 3.1, only the forward implication requires proof. So suppose that $[F^\zeta] = [F^\eta]$. By Theorem 3.7, there is a vacuum solution $[F^{\eta_A}]$ and $g \in \Lambda^\delta_I G^C_\tau$, for $0 < \delta \leq \epsilon$, such that

$$[F^\zeta] = [F^\eta] = g\#[F^{\eta_A}].$$

Thus, by Proposition 3.3, $(\lambda\zeta)(0) = \operatorname{Ad} g(0)\, A$, whence A is regular semisimple, and

$$[\zeta, \operatorname{Ad} g\, A] = [\eta, \operatorname{Ad} g\, A] = 0.$$

Thus $\operatorname{Ad} g^{-1}\, \zeta$ and $\operatorname{Ad} g^{-1}\, \eta$ take values in $\ker \operatorname{ad} A$ and so commute. Thus $[\zeta, \eta] = 0$ as required. □

Example When τ is the Coxeter–Killing automorphism of a simple group G, then any semisimple element in $\mathfrak{g}_{-1}$ is regular [12].

4 Higher flows

It is characteristic of "completely integrable" systems of partial differential equations that the solution set admits a hierarchy of commuting flows. In this section, we shall define such a hierarchy on each dressing orbit $\mathcal{O}_A$ and characterise the primitive harmonic maps of finite type in $\mathcal{O}_A$ as precisely those whose orbit under these flows is finite-dimensional.

Notation We denote by $[g]$ the coset of g in $\Lambda^\epsilon_{I,B} G_\tau / \Gamma^\epsilon_{I,B}$.

Recall from Section 3 that we have a diffeomorphism $\Lambda^\epsilon_{I,B} G_\tau / \Gamma^\epsilon_{I,B} \cong \mathcal{O}_A$ given by

$$[g] \mapsto g\#[F^{\eta_A}].$$

In view of the identification $\Lambda^\epsilon_{I,B} G_\tau \cong \Lambda^\epsilon_E G_\tau \backslash \Lambda^\epsilon G_\tau$, we have a right action of $\Lambda^\epsilon G_\tau$ on $\Lambda^\epsilon_{I,B} G_\tau$ given by

$$(g, h) \mapsto (gh)_I.$$

We now identify an abelian subgroup of $\Lambda^\epsilon G_\tau$ for which this action descends to one on $\mathcal{O}_A$. For this, let $\mathfrak{z}_A \subset \mathfrak{g}^{\mathbb{C}}$ denote the centre of the centraliser of A and define $\mathcal{Z} \subset \Lambda^\epsilon \mathfrak{g}_\tau$ by

$$\mathcal{Z} = \{\zeta \in \Lambda^\epsilon \mathfrak{g}_\tau \colon \text{for some } N \in \mathbb{N},\ \zeta(\lambda) = \sum_{n \geq -N} \lambda^n \zeta_n, \text{ for } \lambda \in C_\epsilon, \text{ with } \zeta_n \in \mathfrak{z}_A \text{ for all } n.\}$$

Moreover, let $Z \subset \Lambda^\epsilon G_\tau$ be the abelian subgroup obtained by exponentiating $\mathcal{Z}$.

Since A is semisimple, the centraliser of A in $\operatorname{Ad} G^{\mathbb{C}}$ is connected [13, Lemma 5] from which it follows that Z commutes with $\Gamma^\epsilon_{I,B}$. Thus, for $\exp \zeta \in Z$ and $[g] = [g'] \in \mathcal{O}_A$, we have $g = g'\gamma$, some $\gamma \in \Gamma^\epsilon_{I,B}$, so that

$$(g \exp \zeta)_I = (g'\gamma \exp \zeta)_I = (g' \exp(\zeta)\gamma)_I = (g' \exp \zeta)_I \gamma$$

whence

$$[(g \exp \zeta)_I] = [(g' \exp \zeta)_I]$$

and we have proved:

Proposition 4.1 *Z acts on $\mathcal{O}_A \cong \Lambda^\epsilon_{I,B} G_\tau / \Gamma^\epsilon_{I,B}$ by*

$$\exp \zeta \cdot [g] = [(g \exp \zeta)_I].$$

Remark It is easy to see that $Z \cap \Gamma^\epsilon_{I,B} = Z \cap \Lambda^\epsilon_{I,B} G_\tau$ acts trivially on $\mathcal{O}_A$. With a little more work, one can also show that any $z \in Z \setminus \Lambda^\epsilon_I G_\tau$ acts non-trivially on $\mathcal{O}_A$. In particular, when $\mathfrak{z}_A \cap \mathfrak{g}_0 = \{0\}$ (which is the case, for instance, for abelian Toda fields), we deduce that there is an effective action of $Z / Z \cap \Lambda^\epsilon_{I,B} G_\tau$ on $\mathcal{O}_A$.

Our main result concerning the action of Z is the following characterisation of the primitive harmonic maps of finite type in $\mathcal{O}_A$:

Theorem 4.2 *$[F] \in \mathcal{O}_A$ is of finite type if and only if the Z-orbit of $[F]$ is finite-dimensional.*

Remark In their study of the sinh-Gordon equation, Dorfmeister–Wu prove this result by different methods for the case $G = \mathrm{SU}(2)$ equipped with the Coxeter–Killing automorphism.

We break up our proof of Theorem 4.2 into a sequence of steps. We begin with a simple technical lemma:

Lemma 4.3 *Let $d \geq 1 \in \mathbb{N}$ and suppose that $\zeta \in \Lambda^\epsilon \mathfrak{g}_\tau$ with $\lambda^d \zeta$ holomorphic on I_ϵ. Then $\zeta_E \in \Lambda_d$ and*

$$(\zeta_E)_{-d} = (\lambda^d \zeta)(0).$$

Proof On C_ϵ, we can write $\zeta = \sum_{n \geq -d} \lambda^n \zeta_n$ and it is easy to see that

$$\zeta_E = \sum_{-d \leq n \leq -1} \lambda^n \zeta_n + (\zeta_0)_{\mathfrak{k}} + \sum_{1 \leq n \leq d} \lambda^n \overline{\zeta_{-n}}$$

from which the lemma follows immediately. $\square$

We can now prove one part of the theorem:

Proposition 4.4 *If $[F] = g\#[F^{\eta_A}] \in \mathcal{O}_A$ has finite-dimensional Z-orbit then $[F]$ is of finite type.*

Proof According to Corollary 3.4, it suffices to find $d \equiv 1 \mod k$ and $\xi \in \Lambda_d$ such that $\xi_{-d} = \operatorname{Ad} g(0)\, A$ and

$$[\xi, \operatorname{Ad} g\, A] = 0.$$

By hypothesis, the map $\mathcal{Z} \to T_{[F]}\mathcal{O}_A$ given by

$$\zeta \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp t\zeta \cdot [g]$$

has finite-dimensional image. We use left translation by g to identify $T_{[F]}\mathcal{O}_A$ with $T_{[F^{\eta_A}]}\mathcal{O}_A \cong \Lambda^{\epsilon}_{I,}\mathfrak{g}_{\tau}/\mathcal{G}_{I,B}$ (here $\mathcal{G}_{I,B}$ is the Lie algebra of $\Gamma^{\epsilon}_{I,B}$) and then an easy calculation shows that the resulting map $\mathcal{Z} \to \Lambda^{\epsilon}_{I,}\mathfrak{g}_{\tau}/\mathcal{G}_{I,B}$ is given by

$$\zeta \mapsto \operatorname{Ad} g^{-1} (\operatorname{Ad} g\, \zeta)_I \mod \mathcal{G}_{I,B}. \tag{4.1}$$

Let $\mathcal{N} \subset \mathcal{Z}$ denote the kernel of the linear map (4.1). Since $\mathcal{N}$ has finite codimension, there is a monic polynomial p such that $\zeta = p(\lambda^{-k})\lambda^{-1}A \in \mathcal{Z}$ lies in $\mathcal{N}$ so that

$$\operatorname{Ad} g^{-1} (\operatorname{Ad} g\, \zeta)_I \in \mathcal{G}_{I,B}$$

or, equivalently,

$$[\operatorname{Ad} g^{-1} (\operatorname{Ad} g\, \zeta)_I, A] = 0.$$

We therefore see that $[(\operatorname{Ad} g\, \zeta)_I, \operatorname{Ad} g\, A] = 0$. On the other hand, since $\zeta \in \mathcal{Z}$, we have $[\zeta, A] = 0$ and we can conclude that

$$[(\operatorname{Ad} g\, \zeta)_E, \operatorname{Ad} g\, A] = 0. \tag{4.2}$$

Now set $d = k \deg p + 1 \equiv 1 \mod k$ and observe that $\lambda^d\zeta$ is holomorphic (indeed polynomial) on I_ϵ with $(\lambda^d\zeta)(0) = A$. It follows that $\lambda^d \operatorname{Ad} g\, \zeta$ is holomorphic on I_ϵ with $(\lambda^d \operatorname{Ad} g\, \zeta)(0) = \operatorname{Ad} g(0)\, A$ so that, from Lemma 4.3, we see that $\xi = (\operatorname{Ad} g\, \zeta)_E \in \Lambda_d$ with $\xi_{-d} = \operatorname{Ad} g(0)\, A$. This, taken together with (4.2), establishes the proposition. $\square$

Remark In case that $G = \mathrm{SO}(5)$ equipped with the Coxeter–Killing automorphism, this proposition was proved by Wu [27] using different methods.

It remains to prove the converse of Proposition 4.4. For this, we use an argument inspired by ideas of McIntosh [17]. Fix $d \equiv 1 \mod k$ and let $\mathcal{O}_A^{(d)}$ consist of those maps in $\mathcal{O}_A$ which admit a polynomial Killing field $\mathbb{R}^2 \to \Lambda_d$. $\mathcal{O}_A^{(d)}$ is the image under $\Lambda^{\epsilon}_{-1,\infty} \to \mathcal{H}$ of a subvariety of $\lambda^{d-1}\Lambda_d$ and so is finite-dimensional. With this in mind, the proof of Theorem 4.2 is completed by the following

Proposition 4.5 *$\mathcal{O}_A^{(d)}$ is Z-stable.*

Proof Let $[g] \in \mathcal{O}_A^{(d)}$ so that there is $\xi \in \Lambda_d$ such that $(\lambda^d \xi)(0) = \operatorname{Ad} g(0)\, A$ and

$$[\xi, \operatorname{Ad} g\, A] = 0.$$

Let $\zeta \in \mathcal{Z}$ and set

$$\hat{\xi} = \operatorname{Ad}(g \exp \zeta)_I \operatorname{Ad} g^{-1} \xi = \operatorname{Ad}(\exp \operatorname{Ad} g\, \zeta)_I\, \xi.$$

Clearly we have

$$[\hat{\xi}, \operatorname{Ad}(g \exp \zeta)_I\, A] = [\operatorname{Ad} g^{-1} \xi, A] = 0.$$

It therefore suffices to show that $\hat{\xi} \in \Lambda_d$ and that $\hat{\xi}_{-d} = \operatorname{Ad}(g \exp \zeta)_I(0)\, A$ to conclude that $\exp \zeta \cdot [g] \in \mathcal{O}_A^{(d)}$.

For this, first note that since $\operatorname{Ad} g^{-1} \xi$ centralises A,

$[\zeta, \operatorname{Ad} g^{-1} \xi] = 0$ whence

$$\operatorname{Ad}(g \exp \zeta)\ \operatorname{Ad} g^{-1} \xi = \operatorname{Ad} g\ \operatorname{Ad} g^{-1} \xi = \xi$$

so that

$$\operatorname{Ad}(\exp \operatorname{Ad} g\, \zeta)_E\ \operatorname{Ad}(\exp \operatorname{Ad} g\, \zeta)_I\, \xi = \xi$$

from which we conclude that

$$\hat{\xi} = \operatorname{Ad}(\exp \operatorname{Ad} g\, \zeta)_I\, \xi = \operatorname{Ad}(\exp \operatorname{Ad} g\, \zeta)_E^{-1}\, \xi \in \Lambda_E^\epsilon \mathfrak{g}_\tau.$$

On the other hand,

$$\lambda^d \hat{\xi} = \operatorname{Ad}(g \exp \zeta)_I\ \operatorname{Ad} g^{-1}\, \lambda^d \xi$$

which is holomorphic on I_ϵ and

$$(\lambda^d \hat{\xi})(0) = \operatorname{Ad}(g \exp \zeta)_I(0)\ \operatorname{Ad} g^{-1}(0)\, \xi_{-d} = \operatorname{Ad}(g \exp \zeta)_I(0)\, A.$$

It now follows from Lemma 4.3 that $\hat{\xi}$ has the required properties and the proof is complete. $\square$

Appendix: nilpotency of η_{-1} and finite uniton number

In Section 3, we saw that the primitive harmonic maps $[F^\eta]$ with $(\lambda\eta)(0)$ semisimple comprise a finite-dimensional family of infinite-dimensional dressing orbits. By contrast, we now consider the case where $(\lambda\eta)(0)$ is nilpotent (which is necessarily the case if the harmonic map $[F^\eta]$ has finite energy). Here our results are less complete and somewhat confused but examples and partial results indicate that the picture is completely different.

Example Let $\eta = \lambda^{-1}\eta_{-1} \in \Lambda^\epsilon_{-1,\infty}$ with η_{-1} nilpotent so that $(\operatorname{ad} \eta_{-1})^\ell = 0$, say. Comparing coefficients in (3.1), we conclude from Lemma 3.1 that $[F^\zeta] = [F^\eta]$ if and only if $\zeta_{-1} = \eta_{-1}$ and

$$(\operatorname{ad} \eta_{-1})^n \zeta_{n-1} = 0,$$

for $n < \ell$. From this it is easy to see that $g\#[F^\eta] = [F^\eta]$ for all $g \in \Lambda^\epsilon_{I,B} G_\tau$ whose ℓ-jet at zero coincides with that of 1. Such g comprise a normal subgroup of $\Lambda^\epsilon_{I,B} G_\tau$ of finite codimension so that the action of $\Lambda^\epsilon_{I,B} G_\tau$ on $[F^\eta]$ reduces to that of a finite-dimensional quotient group.

This is reminiscent of the behaviour of harmonic maps with finite uniton number in the sense of Uhlenbeck [24]. A primitive harmonic map has finite uniton number if it has an extended framing F which can be written as a Laurent polynomial of fixed degree in λ:

$$F(z) = \sum_{|n| \leq d} \lambda^n F_n(z),$$

where we fix a faithful representation of $G^{\mathbb{C}}$ to make sense of the Laurent expansion. We call the minimal such d the *uniton number* of $[F]$.

Concerning these maps, Uhlenbeck [24] and Bergveldt–Guest [1] prove

Proposition A.1 *The action of $\Lambda_I^\epsilon G_\tau$ preserves uniton number and the action on the space of maps of fixed uniton number reduces to that of a finite dimensional quotient group.*

Since the orbits $\mathcal{O}_A$ are infinite-dimensional, we deduce the following result announced in [5]:

Theorem A.2 *No $[F^\eta]$ with $(\lambda\eta)(0)$ non-zero semisimple and, in particular, no non-constant primitive harmonic map of semisimple finite type, has finite uniton number.*

By contrast, we have the following model result which we state for $G = \mathrm{SU}(2)$ for simplicity of exposition.

Proposition A.3 *Let $[F^\eta] : \mathbb{R}^2 \to S^2$ be harmonic with η_{-1} nilpotent. Then $[F^\eta]$ has finite uniton number.*

Proof Set $F = F^\eta$. In this setting the reality condition reads

$$F^*_{1/\bar{\lambda}} = F_\lambda^{-1}$$

from which it is easy to see that F is a Laurent polynomial precisely when, for some d, both $\lambda^d F_\lambda$ and $\lambda^d F_\lambda^{-1}$ are holomorphic (as matrix valued functions) at $\lambda = 0$. In view of the definition of F, this amounts to demanding that $\lambda^d \exp(\pm z\eta)$ be holomorphic at zero.

To prove this, we first note that after dressing with a constant element of $K^{\mathbb{C}}$, we may assume that

$$\eta_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For definiteness, we assume

$$\eta_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and write

$$\eta(\lambda) = \begin{pmatrix} a(\lambda) & \lambda^{-1} + b(\lambda) \\ c(\lambda) & -a(\lambda) \end{pmatrix}$$

where a, b, c are holomorphic on I_ϵ. The equivariance condition on η means that a is an even function while b and c are odd. In particular, $c(0) = 0$.

Now set

$$\gamma(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

and observe that

$$\operatorname{Ad}\gamma(\lambda)\,\eta(\lambda) = \begin{pmatrix} a(\lambda) & 1 + \lambda b(\lambda) \\ \lambda^{-1}c(\lambda) & -a(\lambda) \end{pmatrix}$$

is holomorphic at $\lambda = 0$ since $\lambda^{-1}c(\lambda)$ is. Thus

$$\gamma(\lambda)\exp(\pm z\eta(\lambda))\gamma(\lambda)^{-1}$$

is holomorphic at zero whence $\lambda\exp(\pm z\eta(\lambda))$ is also. □

Remark The trick of conjugating with γ amounts to passing between the principal and standard realisations of $\Lambda\mathrm{SL}(2,\mathbb{C})$ in the sense of Wilson [26].

It is not difficult to extend the above argument to numerous other cases which include that where G/K is a full flag manifold with Coxeter–Killing automorphism and η_{-1} is principal nilpotent. In view of this, one might feel tempted (as we were!) to conjecture that any primitive harmonic map $[F^\eta]$ with η_{-1} nilpotent has finite uniton number. However, this is not the case as the following analysis of harmonic maps $\mathbb{R}^2 \to \mathrm{SU}(2)$ shows.

Set $K = \mathrm{SU}(2)$ and put $G = K \times K$. We view K as the Riemannian symmetric space G/K with involution $\tau : G \to G$ given by $(k_1, k_2) \mapsto (k_2, k_1)$ and coset projection $G \to K$ given by $(k_1, k_2) \mapsto k_1 k_2^{-1}$.

Define $\Lambda_{\mathrm{hol}}K$ by

$$\Lambda_{\mathrm{hol}}K = \{k : \mathbb{C}^* \to K^{\mathbb{C}} : k \text{ is holomorphic and } \overline{k(\lambda)} = k(1/\bar{\lambda})\}.$$

It is easy to see that $\Lambda_{\mathrm{hol}}G_\tau$ is given by

$$\Lambda_{\mathrm{hol}}G_\tau = \{(k(-\lambda), k(\lambda)) : \quad k \in \Lambda_{\mathrm{hol}}K\}$$

so that projection onto the second factor gives an isomorphism $\Lambda_{\mathrm{hol}}G_\tau \cong \Lambda_{\mathrm{hol}}K$. Similarly, we can define subgroups $\Lambda_E^\epsilon K$ and so on, just by dropping the equivariance conditions on the loops $C^{(\epsilon)} \to K^{\mathbb{C}}$ and thus obtain isomorphisms $\Lambda_E^\epsilon G_\tau \cong \Lambda_E^\epsilon K$ and so on. Under these isomorphisms, the harmonic map $\phi : M \to K$ corresponding to the extended framing $F : M \to \Lambda_{\mathrm{hol}}K$ is given by

$$\phi = F_{-1}F_1^{-1}$$

(and, indeed, the *extended solution* corresponding to ϕ in the sense of Uhlenbeck is the map into the based loop group ΩK given by $F_\lambda F_1^{-1} : M \to \Omega K$).

Having adopted this view-point, we may think of $\eta \in \Lambda^\epsilon_{-1,\infty}$ as a holomorphic map $\eta : I_\epsilon \setminus \{0\} \to \mathfrak{k}^{\mathbb{C}}$ with at most a simple pole at zero. We now have

Proposition A.4 *$[F^\eta]$ has finite uniton number if and only if $\det\eta$ is holomorphic on I_ϵ.*

Proof Arguing as in Proposition A.3, we see that $[F^\eta]$ has finite uniton number if and only if, for some $d \in \mathbb{N}$, $\lambda^d \exp(z\eta)$ is holomorphic on I_ϵ, for all $z \in \mathbb{C}$.

For $A \in \mathfrak{sl}(2, \mathbb{C})$, one has

$$A^2 = (-\det A)\mathrm{I}$$

from which we deduce, setting $D = \sqrt{-\det A}$, that

$$\exp A = \cosh(D)\mathrm{I} + \frac{\sinh(D)}{D} A.$$

Thus

$$\exp(z\eta) = \cosh(\sqrt{-\det z\eta})\mathrm{I} + \frac{\sinh(\sqrt{-\det z\eta})}{\sqrt{-\det z\eta}} z\eta. \tag{A.1}$$

If $\det \eta$ is holomorphic on I_ϵ, we see that $\lambda \exp(z\eta)$ is also since, by hypothesis, $\lambda\eta$ is holomorphic on I_ϵ. Thus, in this case, $[F^\eta]$ has finite uniton number.

Conversely, if $\det \eta$ has a pole at zero, it is easy to see from (A.1) that at least one matrix entry of $\exp(z\eta)$ has an essential singularity at zero so that $[F^\eta]$ has infinite uniton number. □

With this in hand, it is easy to find $[F^\eta]$ with η_{-1} nilpotent and infinite uniton number:

Example Define $\eta \in \Lambda^\epsilon_{-1,\infty}$ by

$$\eta(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}$$

so that

$$\eta_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent while $\det \eta = -\lambda^{-1}$ whence $[F^\eta]$ has infinite uniton number by Proposition A.3.

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(F.E. Burstall) School of Mathematical Sciences, University of Bath, Bath BA2 7AY, England

E-mail address: f.e.burstall@maths.bath.ac.uk

(F. Pedit) Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA

E-mail address: franz@gang.umass.edu