Stochastic Ising models
and
anisotropic front propagation

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ABSTRACT

We study Ising models with general spin flip dynamics obeying the detailed balance law. After passing to suitable macroscopic limits, we obtain interfaces moving with normal velocity depending anisotropically on their principal curvatures and direction. In addition we deduce (direction-dependent) Kubo-Green-type formulae for the mobility and the hessian of the surface tension, thus obtaining an explicit description of anisotropy in terms of microscopic quantities. The choice of dynamics affects only the mobility, a scalar function of the direction.

Keywords. Ising model with general spin flip dynamics, interfaces, anisotropy, motion by curvature, Kubo-Green formulae for mobility and surface tension.

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0. Introduction.

In this paper we consider the mesoscopic and macroscopic behavior of stochastic Ising models with long range interactions and general spin flip dynamics. We derive a mean field equation as the interaction range tends to infinity (mesoscopic limit–grain coarsening), we study its asymptotic behavior and we show that it yields a front moving with normal velocity which is an anisotropic function of the principal curvatures. This function is actually described by a Kubo-Green-type formula which also specifies the relationship between the mobility and the surface tension of the moving interface. Finally we study macroscopic limits for the particle system. We show that, for a continuum of appropriate scalings, the particle system yields in the limit a front moving with the same normal velocity as the one governing the asymptotics of the mean field equation.

Our asymptotic results are stated and proved in this paper up to first time the underlying motion develops singularities. They can, however, be extended to hold globally in time, i.e., past the first time the evolving front develops singularities. This is done in a forthcoming paper by Barles and Souganidis(1). The results of our paper allow for the better understanding of the relationship between the phenomenological and microscopic theories of phase transition in the general setting where anisotropies are present. They may also be thought as providing a theoretical justification for the Monte-Carlo simulations performed by physicists to compute moving fronts.

The paper is organized as follows: In Section 1 we briefly discuss the phenomenological and microscopic theories to model phase transitions, we recall some recent results about them and set the ground for the results of this paper, which we present and discuss in Section 2. Section 3 is devoted to the proofs.

1. Phenomenological and Microscopic Theories of phase transitions.

Distinct thermodynamic phases in disequilibrium are in general separated by sharp transition regions (interfaces), where an order parameter changes rapidly from one phase to another. The modelling of phase transitions is mainly approached by either phenomenological or microscopic theories. Below we briefly describe these two types of modelling for non-conservative, isothermal, two-phase systems, in the presence of anisotropies.

In the phenomenological approach models are divided roughly in two categories. The first one is about macroscopic, sharp interface models, derived by continuum mechanics arguments (see Gurtin[2] and references therein), where interfaces are represented as $(N-1)$ dimensional hypersurfaces in $\mathbb{R}^N$ evolving with a prescribed normal velocity V given by

$$V = v(n, \kappa_1, \ldots, \kappa_{N-1}). \tag{1.1}$$

Here n is the normal vector and $\kappa_1, \ldots, \kappa_{N-1}$ are the principal curvatures of the evolving interface Γ_t. The function v in (1.1) is specified by a set of constitutive relations. An example arising in the isotropic case, which captures many important features of this class of hypersurface evolutions, is the motion by mean curvature, where the normal velocity V of Γ_t is proportional to its mean curvature, i.e.,

$$V = -\mu\sigma \sum_{i=1}^{N-1} \kappa_i, \tag{1.2}$$

the constants σ and μ is related to the interfacial energy and the mobility of the interface respectively.

The hypersurfaces $\{\Gamma_t\}_{t\geq 0}$ may develop singularities, change topological type and exhibit various other pathologies even when the initial set Γ_0 is smooth. A great deal of work has been done in order to interpret (1.1) past singularities. A rather general approach to provide a weak formulation for the motion past singularities, known as the level-set approach, was introduced for numerical computations by Osher and Sethian[3] and was developed rigorously by Evans and Spruck[4] for (1.2) and by Chen, Giga and Goto[5] for more general geometric evolutions including (1.1) – see also Barles, Soner, Souganidis[6], Goto[7] and Ishii and Souganidis[8].

In the level-set approach the evolving set Γ_t is represented as the zero-level set of an auxiliary function u, i.e., $\Gamma_t = \{x : u(x,t) = 0\}$, which solves the geometric pde

$$u_t = F(Du, D^2u) \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.3}$$

where, for $X \in \mathcal{S}^N$, the set of $N \times N$ symmetric matrices, and $p \in \mathbb{R}^N \backslash \{0\}$, F is related to v in (1.1) by

$$F(p, X) = -|p|^{-1} v(\overline{p}, X(I - \overline{p} \otimes \overline{p})),$$

with

$$\overline{p} = |p|^{-1}p.$$

In the special case of (1.2), the geometric pde has the form

$$u_t = \mu\sigma \ \mathrm{tr}((I - \overline{Du} \otimes \overline{Du})D^2u) \qquad \text{in } \mathbb{R}^N \times (0, \infty).$$

Nonlinear, singular, degenerate parabolic equations like (1.3) typically have only weak solutions, known as viscosity solutions. This nevertheless allows to define a unique weakly propagating interface Γ_t as the zero level set of the viscosity solution of (1.3), globally in time, past possible singularities.

Another way to define a weakly propagating front using the properties of the signed distance function was introduced by Soner(9) for motion governed by (1.1) and later extended by Barles, Soner and Souganidis(6).

Finally, recently Barles and Souganidis(1) put forward yet another equivalent way to describe the weak front propagation. This new approach, which is based on defining maximal and minimal evolutions using smooth surfaces evolving by approximately the same law as test surfaces (barriers) from inside and outside (see Ref. 1 for the details), is fundamental in understanding and justifying the appearance of moving interfaces globally in time in anisotropic regimes like in this paper.

A second class of phenomenological models is about the long time behavior of order parameters, which solve Ginzburg-Landau type equations and vary continuously between two distinct phases of the material. In such models there is a narrow transition region separating the two different phases instead of sharp interfaces. In this framework Allen and Cahn(10) proposed the asymptotic limit of the rescaled reaction-diffusion equation

$$v_t^\epsilon - \mu\sigma\Delta v^\epsilon + \epsilon^{-2} f(v^\epsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{1.4}$$

where $f(r) = 2\mu r(r^2 - r)$, as a model for the motion of antiphase boundaries in polycrystalline materials. Formal results (see for example, Ref. 10, 11) have indicated that these interfaces move with prescribed normal velocity proportional to their mean curvature. Evans, Soner and Souganidis(12) proved rigorously this conjecture by showing that, in the asymptotic limit $\epsilon \to 0$, the solutions of (1.4) develop interfaces moving by mean curvature in the viscosity sense with the result being valid globally in time, i.e., past singularities. (See also Ref. 1 and 6 for more general results and Souganidis[13],[14] for a general survey of the subject.)

Non-equilibrium statistical mechanics theories provide a microscocpic approach to the modelling of phase transitions using Interacting Particle Systems, IPS for short, which are Markov processes set on the lattice $\mathbb{Z}^N$. One distinguishes between stochastic Ginzburg-Landau models where the order parameter takes continuous values and Ising spin systems with either $(+)$ or $(-)$ spins at each lattice site. Here we only consider the latter type of models with general spin flip dynamics. Stochastic Ising systems, which describe phase transitions, $(+)$'s being converted to $(-)$'s and vice versa, starting from an initial state of disequilibrium, are jump Markov processes $\{\sigma_t\}_{t\geq 0}$ taking values in the configuration space $X = \{-1, 1\}^{\mathbb{Z}^N}$. A configuration $\sigma = \{\sigma(x) \in \{-1, 1\}, x \in \mathbb{Z}^N\}$ is updated by a sequence of spin flips, i.e., when a spin changes sign at a site x with a rate $c(x, \sigma)$ depending on an interaction potential J. (See the next Section for the detailed description of the model.)

For the stochastic Ising models there exists a mesoscopic space scaling (grain coarsening), giving rise, through the respective BBGKY hierarchies, to deterministic equations. Such mesoscopic (mean field) equations describe the limiting evolution of the average magnetization $E\sigma_t(x)$. In the case of Glauber dynamics with radially symmetric potentials J, De Masi, Orlandi, Presutti and Triolo[15] obtained, in the mean field limit as the interaction range tends to infinity, the fully nonlinear nonlocal equation

$$m_t + m - \tanh \beta(J * m) = 0 \ \text{ in } \ \mathbb{R}^N \times [0, \infty) \tag{1.5}$$

where $J * m$ denotes the usual convolution in $\mathbb{R}^N$.

The Allen-Cahn equation (1.4) may be viewed as a mesoscopic equation for a suitable IPS. Indeed, De Masi, Ferrari and Lebowitz[16] derived (1.4), with $\epsilon = 1$, from an IPS with Glauber-Kawasaki $(G + K)$ dynamics, i.e., a stochastic system evolving under the combined influence of slow spin flips (Glauber dynamics) and fast spin exchanges (Kawasaki dynamics).

Some aspects of the complex relations between the above micro-, meso- and macroscopic models for phase transitions were explored by the authors in Ref. 17 and 18, where they rigorously derived phenomenological pde's describing evolving phase boundaries, e.g. (1.3), from interacting particle systems. In Ref. 17 we studied an IPS with Glauber-Kawasaki dynamics proving that there is a continuum of suitable scaling of time and space, such that in the limit the sites of the spin system separate in clusters of $(+)$ and $(-)$, whose boundaries move towards equilibrium according to the mean curvature rule. In Ref. 18 we investigated the macroscopic limit of an appropriately rescaled stochastic Ising model with long range interactions evolving with Glauber dynamics as well as rescalings of the corresponding mesoscopic equation (1.5). In both scales we obtained an interface evolving with normal velocity $\mu\sigma\kappa$, where κ is the mean curvature and $\theta = \mu\sigma$ is a transport coefficient. The novelty of the results in Ref. 18, besides dealing with a fully nonlinear, nonlocal mesoscopic equation, is the identification of θ, through a homogenization technique, yielding an effective Green-Kubo type formula. The transport coefficient appears neither at the microscopic level, i.e., the particle system, nor at the level of the mesoscopic equation and it is actually the outcome of an averaging effect taking place during the limiting process. All the above results are again valid globally in time, the motion of the interface being interpreted in the viscosity sense after the onset of the geometric singularities. Moreover, the "propagation of chaos" property holds globally for both models. In the case of the Glauber-Kawasaki dynamics we obtained in addition that the resulting interfaces are varifolds evolving by their mean curvature in the Brakke sense, which eliminates some of the nuisance due to the possible interface fattening (see, for example, Ref. 6 and 17). Concluding this discussion, we would like to underline the critical role played by the mesoscopic equations (1.4) and (1.5), and their asymptotics, in the rigorous transition from the IPS to the macroscopic equations.

Our objective in this work is to study how anisotropy is manifested in the transition from microscopic to macroscopic models. To account for anisotropies in the Ising model we replace the assumption of the radial symmetry of the interaction potential by the requirement that J is even. The continuum theory (see Ref. 2 and references therein) suggests that, in the absence of faceting phenomena and for stable (strictly convex) interfacial energies H, the evolution of the phase boundaries $\{\Gamma_t\}_{t\geq 0}$ is governed by the geometric equation

$$u_t = \mu(\overline{Du})\mathrm{tr}[A(\overline{Du})D^2u(I - \overline{Du}\otimes\overline{Du})] \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.6}$$

with $\Gamma_t = \{r \in \mathbb{R}^N : u(r,t) = 0\}$, where $A = D^2H$. The direction dependent scalar μ is the mobility of the interface and H a positively homogeneous of degree one function. Notice that in the isotropic case where $H(e) = \sigma|e|$, (1.6) simply reduces to motion by mean curvature.

Our goal here is to derive rigorously such equations from Ising models with general spin flip dynamics and at the same time provide a Green-Kubo formula for the direction- and dynamics-dependent mobility $\mu(e)$ as well as the hessian of the interfacial energy $H(e)$.

We conclude this section noting that Spohn[(19)] has also derived (formally) Green-Kubo formulae for the mobility and the interfacial energy, using corresponding microscopic definitions, bypassing the issue of the macroscopic equation. Furthermore Butta[(20)] proved the validity of an Einstein relation for the transport coefficient of the isotropic mean curvature evolution. An approach similar to ours was taken in the physics literature by a number of authors – see, for example, Vredensky et. al.[(21)], Krug et. al.[(22)] and the references therein – where the macroscopic equation along with the Green-Kubo formulae are directly derived from the microsocpic dynamics. These works primarily refer to conservative dynamics (spin exchange dynamics) where, in addition, surface diffusion may enter in the macroscopic equations. Such questions have been addressed in a series of papers by Giacomin and Lebowitz[(23),(24),(25)], who studied phase segregation dynamics in particle systems with local mean field interactions and obtained, formally, interface evolution laws similar to the ones obtained in the analogous limit for the Cahn-Hilliard equations.

Finally we note that the results of this paper were already announced in Souganidis[(13),(14)].

2. The main results.

We begin with a description of general ferromagnetic Ising models, i.e., spin systems interacting by nonnegative symmetric (even) Kač potentials and evolving with general spin flip dynamics. For a much more detailed discussion, at least for Glauber dynamics, we refer, for example, to the papers by De Masi et. al.[15], Comets[26] and the references therein.

The energy H of the particle system, evaluated at a configuration σ, is given by

$$H(\sigma) = \sum_{x \neq y} J_\gamma(x,y)\sigma(x)\sigma(y) + h \sum \sigma(x),$$

where h is attributed to an external magnetization field and J_γ is the Kač potential defined by

$$J_\gamma(x,y) = \gamma^N J(\gamma(x-y)) \qquad (x, y \in \mathbb{Z}^N), \tag{2.1}$$

$\gamma^{-1} > 0$ being the interaction range. Here $J : \mathbb{R}^N \to \mathbb{R}$ is assumed to be such that

$$J \in C^1(\mathbb{R}^N),\ J(r) = J(-r) \geq 0 \quad \text{and} \quad J(r) = 0 \quad \text{for } |r| > R \text{ for some } R > 0. \tag{2.2}$$

The assumption that J has compact support is made only to simplify the arguments below and can be easily removed by specifying appropriate growth assumptions on J at infinity. We leave this task to the interested reader. The assumption that J is nonnegative is an important one from the physical point of view, since it implies that the Ising model is ferromagnetic.

The dynamics of the model consist of a sequence of flips. If σ is the configuration before a flip at x, then after the flip at x the configuration is

$$\sigma^x(y) = \begin{cases} -\sigma(x), & \text{if } y = x, \\ \sigma(y), & \text{if } y \neq x. \end{cases}$$

We assume that a flip occurs at x, when the configuration is σ, with a rate $c_\gamma(x,\sigma)$, given by

$$c_\gamma(x,\sigma) = \Psi(-\beta(H(\sigma^x) - H(\sigma))), \tag{2.3}$$

where $\beta > 0$ is identified with the inverse temperature, $H(\sigma^x) - H(\sigma)$ is the change in the energy due to a spin flip at x and $\Psi : \mathbb{R} \to (0,\infty)$ is a locally Lipschitz continuous function satisfying the detailed balance law (or reversibility condition)

$$\Psi(r) = \Psi(-r)e^{-r} \qquad (r \in \mathbb{R}). \tag{2.4}$$

Typical choices of Ψ's are $\Psi(r) = (1+e^r)^{-1}$ (Glauber dynamics), $\Psi(r) = e^{-r/2}$ (Arrhenius dynamics) or $\Psi(r) = e^{-r^+}$ (Metropolis dynamics). Dynamics obeying (2.4) leave the underlying Gibbs measures, which are associated with the Hamiltonian H and the inverse temperature β, invariant.

The underlying process is a jump process on $L^\infty(\Sigma;\mathbb{R})$ with generator given by

$$L_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^N} c_\gamma(x,\sigma)[f(\sigma^{(x)}) - f(\sigma)].$$

A very basic question in the theory of stochastic Ising models with Kač potentials is the behavior of the system as the interaction range tends to infinity, i.e., in the limit $\gamma \to 0$. The passage in the limit $\gamma \to 0$, which in the physics literature is identified with grain coarsening, of quantities like the thermodynamical pressure, total magnetization, etc., is known as the Lebowitz-Penrose limit (see for example, Ref. 27, 28, 29, etc.).

Along these lines we study the asymptotics, as $\gamma \to 0$, of the averaged magnetization

$$m_\gamma(x,t) = \mathbb{E}_{\mu^\gamma}\sigma_t(x) \qquad ((x,t)\in\mathbb{Z}^N\times[0,\infty)) \tag{2.5}$$

of the system, where $\mathbb{E}_{\mu^\gamma}$ denotes the expectation of the IPS starting from a measure μ^γ.

The relevant mesoscopic mean field equation is

$$m_t + \Phi(\beta(J*m+h))[m - \tanh\beta(J*m+h)] = 0 \quad \text{in } \mathbb{R}^N\times[0,\infty), \tag{2.6}$$

where

$$\Phi(r) = \Psi(-2r)(1+e^{-2r}). \tag{2.7}$$

Notice that for Glauber dynamics $\Phi(r) = 1$ and (2.6) reduces to the equation (1.5), studied, at least for radial potentials, in Ref. 15 and 18. In fact following the techniques of Ref. 15 we can prove the following Theorem:

Theorem 2.1: *Assume that the IPS defined earlier has as initial measure a product measure μ^γ such that, for $x\in\mathbb{Z}^N$, $\mathbb{E}_{\mu^\gamma}(\sigma(x)) = m_0(\gamma x)$, where m_0 is Lipschitz continuous, and that (2.2) holds. Then, for each $n\in\mathbb{Z}^+$,*

$$\lim_{\gamma\to 0}\sup_{\underline{x}\in\mathbb{Z}^N_n} |\mathbb{E}_{\mu^\gamma}\Big(\prod_{i=1}^n \sigma_t(x_i)\Big) - \prod_{i=1}^n m(\gamma x_i,t)| = 0,$$

where m is the unique solution of (2.6) with initial datum m_0.

In the above statement and henceforth, for each n,

$$\mathbb{Z}_n^N = \{\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^N : x_1 \neq \cdots \neq x_n\}.$$

Next we review some basic properties of (2.6). To this end, assume that

$$\beta \overline{J} > 1, \tag{2.8}$$

where

$$\overline{J} = \int J(r)dr.$$

In the sequel we refer to the value $\beta_{\text{cr}} = (\overline{J})^{-1}$ as the critical temperature.

It follows easily that there exists some $h_0 > 0$ such that, if $|h| < h_0$, then (2.6) has three steady solutions $m_\beta^{h,-} < m_\beta^{h,0} < m_\beta^{h,+}$, which are the solutions of the algebraic equation $x = \tanh(\beta(\overline{J}x + h))$. Note that the steady state solutions are independent of Φ and, when $h = 0$, $m_\beta^{h,\pm} = \pm m_\beta$ and $m_\beta^{h,0} = 0$. It also turns out that (2.6) admits a comparison principle stated in the following lemma, under an additional hypothesis, which is, however, satisfied by the Arrhenius, Glauber and Metropolis dynamics. Its proof is rather elementary and we will leave it as an exercise.

Lemma 2.2: *(i) Assume (2.2) and let m be a solution of (2.6) with initial datum m_0. Then, for all $t > 0$, $|m(\cdot, t)| \leq \|m_0\|$ on $\mathbb{R}^N$.*

(ii) Assume that Φ is locally Lipschitz continuous and that, for all $m \in [m_\beta^{h,-}, m_\beta^{h,+}]$ and $r \in [\beta\overline{J}m_\beta^{h,-}, \beta\overline{J}m_\beta^{h,+}]$,

$$r \mapsto \Phi(r + \beta h)(m - \tanh(r + \beta h)) \text{ is nonincreasing in } r. \tag{2.9}$$

If m_1, m_2 are solutions of (2.6) and $m_1(\cdot, 0) \leq m_2(\cdot, 0)$ on $\mathbb{R}^N$, then

$$m_1(\cdot, t) \leq m_2(\cdot, t) \quad \text{on } \mathbb{R}^N$$

$\square$

It also turns out – and this is crucial for our analysis below – that, for sufficiently small $|h|$, (2.6) admits, for each $e \in S^{N-1}$, the unit sphere in $\mathbb{R}^N$, special travelling wave solutions in the direction e connecting $m_\beta^{h,-}$ and $m_\beta^{h,+}$, with speed $c^h(e)$, i.e., solutions of the form

$$m(r,t) = q^h(r \cdot e + c^h(e)t, e),$$

where q^h solves the fully nonlinear integral – differential equation

$$c^h(e)\dot{q}^h(\xi,e) + \Phi(\beta(J * q^h + h))\Big[q^h(\xi,e) - \tanh\beta(J * q^h(\xi,e) + h)\Big] = 0. \tag{2.10}$$

Above and hencetoforth we write, for all
$(\xi,e) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}$,

$$J * q^h(\xi,e) = \int J(r')q^h(\xi + r'e, e)dr'.$$

In addition q^h satisfies, for appropriate positive constants $\lambda_\pm^h(e)$ and $a_\pm^h(e)$,

$$\begin{cases} q^h(\pm\infty,e) = m_\beta^{h,\pm}, \quad q^h(0,e) = m_\beta^{h,0} \quad \text{and} \quad \dot{q}^h(\xi,e) > 0 \\ \text{and} \\ \lim\limits_{\xi\to\pm\infty} \exp(\lambda_\pm^h(e)|\xi|)|q^h(\xi,e) - [m_\beta^{h,\pm} \pm a_\pm^h(e)\exp(-\lambda_\pm^h(e)|\xi|)]| = 0. \end{cases} \tag{2.11}$$

It follows that the domain of q^h can be extended from $\mathbb{R} \times S^{N-1}$ to $\mathbb{R} \times \mathbb{R}^N \backslash \{0\}$ by

$$q^h(\xi,e) = q^h(|e|^{-1}\xi, \bar{e}). \tag{2.12}$$

It also turns out, as we explain below, that

$$D_e q^h(\xi,e) \quad \text{is continuous in} \ \ \mathbb{R} \times \mathbb{R}^N \backslash \{0\}. \tag{2.13}$$

Finally,

$$\text{if} \ \ h = 0, \ \ \text{then} \ \ c^0(e) = 0 \ \ \text{and} \ \ q^0 \ \ \text{is odd in} \ \ \xi, \tag{2.14}$$

i.e., the travelling wave is a standing wave.

The existence and stability of such q^h's, when J is isotropic, i.e., $J(r) = J(|r|)$, was studied by De Masi, Orlandi, Presutti and Triolo[(30)], when $h = 0$, and Bates, Fife, Ren and Wang[(31)] in general. For a detailed study of travelling wave solutions of (2.6) in the presence of an external field but always in the isotropic case, we also refer to the papers by De Masi, Gobron and Presutti[(32)] and Orlandi and Triolo[(33)]. As one can see immediately, in the isotropic case the standing and travelling wave solutions are independent of the direction e.

The anisotropic case is, however, dramatically different. The standing wave solutions of (2.6) are expected to depend on the direction, as the next simple example indicates. Of course, this is the novelty here!

Assume that $h = 0$, $J = \frac{1}{4} 1_{[-\alpha^{-1}, \alpha^{-1}] \times [\alpha, \alpha]}$, for some $\alpha > 0$, where 1_A is the characteristic function of the set A. Substituting in (2.10) we immediately see that

$$q^0(\xi, (1,0)) = q(\alpha\xi) \ \text{ and } \ q^0(\xi, (0,1)) = q(\alpha^{-1}\xi),$$

where $q(\xi) = m_\beta \tanh(\beta m_\beta \xi)$ is the direction independent standing wave corresponding to $J = \frac{1}{2} 1_{[-1,1]}$. (See Ref. 30 for this last statement.)

It should also be noted that the dependence on the direction is of nonlocal nature and, hence, can not be removed a priori by some change of metric. This can be easily seen from the above example or by some elementary analysis of the behavior of the q^h's as $|\xi| \to \infty$. We would also like to point out that a similar phenomenon occurs, i.e., the existence of travelling waves which depend nontrivially on the direction, in the study of reaction-diffusion equations with oscillatory coefficients – see Xin[(34),(35),(36)] and Barles and Souganidis[(1)] – or quasilinear reaction-diffusion equations with nonlinearities depending on the direction of the gradient of the solutions.

The existence of q^h's satisfying (2.10), (2.11) and (2.13) has not been worked out explicitely anywhere but it can be obtained, as we sketch below for the convenience of the reader, by a more or less straightforward adaptation of the results of De Masi et al.[(30)], De Masi et al.[(32)] and Bates et. al.[(31)].

For simplicity, below we only discuss the case $h = 0$. When $h \neq 0$, one argues using the Implicit Function Theorem as in Theorem 3.1 of Ref. 32, with the appropriate modifications to deal with explicit dependence on the direction e. Finally to simplify the notation in what follows, we write q and λ instead of q^0 and λ^0 respectively.

To this end observe that we can apply the analysis of Ref. 30 and 31, for each fixed direction e, to the corresponding one dimensional potentials

$$\hat{J}(\rho, e) = \int_{N_e} J(\rho e + y) dy, \qquad (\rho \in \mathbb{R})$$

where $N_e = \{y \in \mathbb{R}^N : y \cdot e = 0\}$. We thus obtain a standing wave $q(\cdot, e)$ satisfying (2.10) and (2.11) with $\lambda(e)$ the unique positive solution of the algebraic equation

$$\beta[1 - m_\beta^2] \int J(r) \exp(-\lambda(e) e \cdot r) dr = 1.$$

To study the regularity of q in e asserted in (2.13), we need to consider, for each $e \in \mathbb{R}^N \backslash \{0\}$, the unbounded, self adjoint operator $\mathcal{L}(e) : L^2(\mathbb{R}) \cap C_0(\mathbb{R}) \to L^2(\mathbb{R}) \cap C_0(\mathbb{R})$, $C_0(\mathbb{R})$ being the space of bounded continuous functions on $\mathbb{R}$ vanishing at infinity, defined by

$$\mathcal{L}(e) p(\xi) = \beta \int J(r) p(\xi + r \cdot e) dr - [1 - q(\xi, e)^2]^{-1} p(\xi), \tag{2.15}$$

which is obtained by linearizing the standing wave equation (2.10) around $q(\cdot, e)$. It follows from Ref. 30 and 31 that, for each $e \in \mathbb{R}^N \backslash \{0\}$,

$$\ker \mathcal{L}^*(e) = \ker \mathcal{L}(e) = \dot{q}(\cdot, e) \mathbb{R}\,, \tag{2.16}$$

and that $\mathcal{L}(e)^{-1} : L^2(\mathbb{R}) \cap C_0(\mathbb{R}) \cap \ker \mathcal{L}(e)^{\perp} \to L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ is a bounded operator, the last claim being a consequence of Fredholm's alternative.

Next, for each $e \in \mathbb{R}^N \backslash \{0\}$ and each $i = 1, ..., N$, we consider the solution $p_i(\cdot, e) \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ of

$$\mathcal{L}(e) p_i(\xi, e) = -\beta \int J(r) \dot{q}(\xi + r \cdot e, e) r_i dr. \tag{2.17}$$

Since J is even and q is odd (recall (2.2) and (2.14)), the existence of p_i follows from the discussion above, since

$$\int \int J(r) \dot{q}(\xi + r \cdot e, e) \dot{q}(\xi, e) r_i dr d\xi = 0\,.$$

It also follows that p_i is continuous with respect to e. Indeed observe that, since (2.10) can be rewritten as

$$q = \tanh \beta(\hat{J}(\cdot, e) * q)\,,$$

if q_1 and q_2 are the solutions of (2.10) corresponding to e_1 and e_2, then

$$||q_1 - q_2||_\infty \le C|e_1 - e_2|\,,$$

with the constant C depending on $\pm m_\beta$ and the C^1-norm of J. An elementary integration by parts with respect to r of the right hand side of (2.17) together with the fact $\mathcal{L}(e)^{-1}$ is continuous with respect to e, following from the continuity of q in e, yield the above asserted regularity on p_i.

Fix now a unit vector $e_i \in \mathbb{R}^N$ and consider for $\rho \neq 0$ the finite difference

$$Q_i^\rho(\xi) = \rho^{-1}[q(\xi, e + \rho e_i) - q(\xi, e)]\,,$$

which solves, as an elementary computation reveals, for a suitable P_i^ρ, the equation

$$\mathcal{L}(e) Q_i^\rho = P_i^\rho\,.$$

It follows from the symmetry properties of J and q that $P_i^\rho \in L^2(\mathbb{R}) \cap C_0(\mathbb{R}) \cap \ker \mathcal{L}(e)^\perp$. Moreover, it is also immediate that, as $\rho \to 0$,

$$P_i^\rho \to -\beta \int J(r)\dot{q}(\cdot + r \cdot e, e) r_i dr \ \text{ in } \ L^2 \cap C_0\,.$$

The boundedness of $\mathcal{L}(e)^{-1}$ now gives in the limit $\rho \to 0$,

$$Q_i^\rho \to D_{e_i} q = p_i \ \text{ in } \ L^2 \cap C_0\,.$$

The above yield that

$$\mathcal{L}(e) D_{e_i} q(\xi, e) = -\beta \int J(r)\dot{q}(\xi + r \cdot e, e) r_i dr\,,$$

as well as the regularity of q in e asserted in (2.13).

As mentioned earlier the existence of q^h's satisfying (2.10), (2.11) and (2.13) is a very important technical tool for our analysis. It is by no means, however, what creates the curvature effects in the asymptotic limits, although the latter are expressed quantitatively in terms of expressions which depend on q^h. Moreover, it is worth remarking that it is only (2.10), (2.11) and (2.13) that play a role in our analysis and not the stability properties of q, which require in addition to (2.16), a spectral estimate on $\mathcal{L}(e)$. We refer the reader to the related analysis for reaction-diffusion equations – see, for example, Ref. 1, 6, 12 – and for (1.5) – see Ref. 18. Although spectral estimates played a crucial role in the analysis performed for short times by a number of authors, it turns out that they play no role whatsoever in the approach we are using here. We refer the reader to Ref. 1, 6, 12, 17, 18 etc. for further discussion of this point.

We continue now with the presentation of our main results which are about the long time asymptotics of (2.6) and the IPS. For the former it is convenient to rescale (2.6) using the parabolic scaling $(r,t) \to (\epsilon^{-1}r, \epsilon^{-2}t)$. The effect of scaling space and time is, of course, to reproduce in bounded space regions and for finite times the long time behavior of (2.6).

For any $\alpha \in \mathbb{R}$, let m_ϵ be the solution of (2.6) with $h = \alpha\epsilon$ and define, for $(r,t) \in \mathbb{R}^N \times (0,\infty)$,

$$m^\epsilon(r,t) = m_\epsilon(\epsilon^{-1}r, \epsilon^{-2}t).$$

It follows that m^ϵ solves the rescaled equation

$$m_t^\epsilon + \epsilon^{-2}\Phi(\beta(J^\epsilon * m^\epsilon + \alpha\epsilon))[m^\epsilon - \tanh\beta(J^\epsilon * m^\epsilon + \alpha\epsilon)] = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{2.18}$$

where

$$J^\epsilon(r) = \epsilon^{-N}J(\epsilon^{-1}r) \qquad (r \in \mathbb{R}^N).$$

To state the results we also need to introduce the scalar $\mu : S^{N-1} \to \mathbb{R}$ identified with the mobility of the interface and the matrix $A(e) : S^{N-1} \to \mathcal{S}^N$ related to the surface tension given by

$$\mu(e) = \beta\Big[\int \frac{(\dot{q}(\xi,e))^2}{\Phi(\beta J * q(\xi,e)dr)(1-q^2(\xi,e))}\,d\xi\Big]^{-1} \tag{2.19}$$

and

$$A(e) = \tfrac{1}{2} \iint J(r)\dot{q}(\xi, e)[\dot{q}(\xi + r\cdot e, e)(r \otimes r) + D_e q(\xi + r\cdot e, e) \otimes r \tag{2.20}$$

$$+ r \otimes D_e q(\xi + r\cdot e, e)] dr d\xi.$$

Notice that if J is radially symmetric then $A(e)$ reduces to θI, with

$$\theta = \tfrac{1}{2} \int\int J(r)\dot{q}(\xi)\dot{q}(\xi + r\cdot e)(r \otimes r) dr d\xi.$$

Next define the matrix $\tilde{A} : \mathbb{R}^N \backslash \{0\} \times \mathcal{S}^N \to \mathcal{S}^N$ by

$$\tilde{A}(e, X) = A(\bar{e})X(I - \bar{e} \otimes \bar{e}), \tag{2.21}$$

and consider the function $F : \mathbb{R}^N \backslash \{0\} \times \mathcal{S}^N \to \mathbb{R}$ given by

$$F(e, X) = \mu(\bar{e})\Big[\mathrm{tr}\ \tilde{A}(\bar{e}, X) + 2\alpha m_\beta |e|\Big].$$

It follows from the general theory developed in Barles and Souganidis[1] that F is degenerate elliptic, i.e., for all $e \in \mathbb{R}^N \backslash \{0\}$ and $X, Y \in \mathcal{S}^N$,

$$\text{if } X \le Y \ \text{ then } F(e, X) \le F(e, Y). \tag{2.22}$$

This last fact is crucial for the analysis below. It is worth remarking that in principle one should be able to check (2.22) by a direct computation without using Ref. 1, as it is the case for a number of other examples. This, however, requires a more detailed knowledge of properties of the standing wave, which may not be easily obtained if not at all. The theory of Ref. 1 circumvents this problem.

Consider now the initial value problem

$$\begin{cases} m_t^\epsilon + \epsilon^{-2}\Phi(\beta(J^\epsilon * m^\epsilon + \alpha\epsilon))[m^\epsilon - \tanh\beta(J^\epsilon * m^\epsilon + \alpha\epsilon)] = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ m^\epsilon = m_0^\epsilon & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{2.23}$$

and assume that there exists an open set $\Omega_0 \subset \mathbb{R}^N$ and a closed set $\Gamma_0 \subset \mathbb{R}^N$ such that $\mathbb{R}^N = \Omega_0 \cup \overline{\Omega}_0^c \cup \Gamma_0$ and

$$\Omega_0 = \{r \in \mathbb{R}^N : m_0^\epsilon > 0\} \ \text{ and } \ \Gamma_0 = \{r \in \mathbb{R}^N : m_0^\epsilon = 0\}; \tag{2.24}$$

notice that this last assumption on m_0^ϵ can be easily generalized – see, for example, Ref. 18.

Finally consider the geometric pde

$$
\begin{cases} u_t = F(Du, D^2u) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{2.25}
$$

where u_0 is a bounded uniformly continuous function such that

$$
\Gamma_0 = \{x : u_0(x) = 0\},\ \Omega_0 = \{x : u_0(x) > 0\} \quad \text{and} \quad \overline{\Omega}_0^c = \{x : u_0(x) < 0\}. \tag{2.26}
$$

As discussed in Section 1, the set $\Gamma_t = \{x \in \mathbb{R}^N : u(x,t) = 0\}$ is by the definition the weak front propagation of Γ_0 with normal velocity

$$
V = -\mu(n)\mathrm{tr}[A(n)Dn + 2\alpha m_\beta]. \tag{2.27}
$$

The first main result is

Theorem 2.3: *Assume (2.8), (2.9), (2.24) and let m^ϵ be the solution of (2.23). Then, as $\epsilon \to 0^+$, $m^\epsilon \to m_\beta$ in $\{u > 0\}$ and $m^\epsilon \to -m_\beta$ in $\{u < 0\}$, with both limits local uniform, where u is the unique solution of (2.25) with u_0 satisfying (2.26).*

As mentioned in the Introduction here we only prove Theorem 2.3 under the assumption that the weak evolution of Γ_0 with normal velocity (2.27) is smooth. Theorem 2.3 is proved, for the weak evolution, in Ref. 1.

To state our result for the IPS, if u is the solution of (2.25), for $t > 0$, we define the sets

$$
\begin{cases} P_t^\gamma = \{x \in \mathbb{Z}^N : u(\gamma\epsilon(\gamma)x, t) > 0\}, \quad N_t^\gamma = \{x \in \mathbb{Z}^N : u(\gamma\epsilon(\gamma)x, t) < 0\} \\ \text{and} \\ M_{\gamma,t}^n = \{\underline{x} \in \mathbb{Z}_n^N : x_i \in P_t^\gamma \cup N_t^\gamma\}. \end{cases}
$$

The result is

Theorem 2.4: *Assume (2.8), (2.9) and (2.24). Under the assumptions of Theorem 2.1 on the initial measure, there exists a $\rho^* > 0$ such that for any $\epsilon(\gamma)$ such that $\gamma^{-\rho^*}\epsilon(\gamma) \to +\infty$,* *as $\gamma \to 0$, and, for all $t > 0$ with the limit local uniform in t,*

$$\lim_{\gamma \to 0} \sup_{\underline{x} \in M^n_{\gamma,t}} |E_{\mu^\gamma} \prod_{i=1}^{n} \sigma_{t\epsilon(\gamma)^{-2}}(x_i) - m^n_\beta \prod_{i \in N^\gamma_t} (-1)| = 0.$$

Theorem 2.4 follows from Theorem 2.3 the same way as the analogous theorem in Ref. 18, we therefore do not present its proof here.

We conclude this section with a discussion about the history of this problem as well as the meaning of our results.

To our knowledge, Theorem 2.3 and 2.4 are the first rigorous results in a non-equilibrium setting where an anisotropic macroscopic equation (1.6) as well as a Green-Kubo formula for the direction-dependent transport matrix (2.20) and mobility (2.19) are derived from mesoscopic and microscopic dynamics, namely (2.6) and the underlying stochastic Ising model.

As already mentioned earlier a result analogous to Theorem 2.3 was obtained for the isotropic case, i.e., when $J(r) = J(|r|)$, first under the assumption that the evolving front remains smooth in Ref. 15 and later extended past all possible singularities by the authors in Ref. 18. In this case it turns out that the limiting motion is governed by (2.27), where $V = -\mu\theta \ \mathrm{tr} Dn$, where the constants θ and μ are given by

$$\theta = \iint J(|r|)\dot{q}(\xi + e \cdot r)\dot{q}(\xi)(\hat{e} \cdot r)^2 dr d\xi \quad \text{and} \quad \mu = \beta \int (1 - q^2(\xi))^{-1} \dot{q}^2(\xi) d\xi,$$

where e, $\hat{e}$ are any two orthogonal vectors in S^{N-1}. Note that due to the symmetry of J, both θ and μ are independent of the particular choice of e and $\hat{e}$. In addition q is the direction-independent travelling wave corresponding to the symmetric J.

One may simplify (2.6) by substituting $J_2(\Delta m - m)$ for the convolution term $J * m$ (see, for example, Penrose[37]), where $\overline{J}_2 = \int J(|r|)|r|^2 dr$ or even additionally linearize the hyperbolic tangent, thus obtaining a Ginzburg-Landau equation (1.1). It is known (see Jerrard[38], Evans, Soner and Souganidis[12] and Barles, Soner and Souganidis[6]) that in the isotropic case, both simplified models have the same qualitative asymptotic behavior as (2.6) with different though transport coefficients. In the anisotropic case, however, this picture is not true anymore. The second order approximations described earlier still give, in the limit $\epsilon \to 0$, isotropic motion by mean curvature with a constant transport coefficient, while (2.6), according to our analysis should yield the anisotropic equation (2.23) with the Green-Kubo formulae (2.19) and (2.20). It appears that anisotropy is a higher order effect which cannot be accounted for, only with second order approximating equations. This phenomenon is also pointed out by Caginalp and Fife[39], where depending on the type of anisotropy expected, they "correct" (1.4) by suitably adding higher order derivatives.

Fronts moving with normal velocity given by (2.27) can also be obtained at the scaled limit of monotone threshold dynamics – see Ishii, Pires and Souganidis[40], which can be thought as deterministic analogues to Ising models.

3. The proof of Theorem 2.3.

As mentioned earlier here we prove Theorem 2.3 under the additional hypothesis that

$$\Gamma_0 \text{ is smooth,} \tag{3.1}$$

which yields, by classical arguments, that there exists $T > 0$ such that

$$\text{the evolution } \Gamma_t \text{ of } \Gamma_0 \text{ according to (2.27) is smooth for } t \in [0, T]. \tag{3.2}$$

Here we only present the argument for $\alpha = 0$, the general case follows by replacing in the proof below, a by $a + \alpha$.

Let u be the solution of (2.25) with u_0 satisfying (2.26) and define the signed distance d to Γ_t by

$$d(r,t) = \begin{cases} d(r, \Gamma_t) & \text{if } r \in \{r' \in \mathbb{R}^N : u(r',t) > 0\}, \\ -d(r, \Gamma_t) & \text{if } x \in \{r' \in \mathbb{R}^N : u(r',t) < 0\}, \end{cases} \tag{3.3}$$

where $d(x, B)$ is the usual distance from x to the set B. Then (3.2) is quantified by saying that, for some fixed $\epsilon > 0$ and $\delta_0 > 0$,

$$d_t, Dd, Dd_t, D^2 d \in C_b^1(\mathbb{R}^N \times (0, T + \epsilon)) \cap \{(r,t) : |d(r,t)| < \delta_0\}. \tag{3.4}$$

Finally, throughout this section we will assume that the system starts at a local equilibrium, i.e., that

$$m^\epsilon = q(\epsilon^{-1} d_0, Dd_0) \qquad \text{on } \mathbb{R}^N \times \{0\}, \tag{3.5}$$

d_0 being the signed distance to Γ_0. This additional assumption can also be removed. We refer to Ref. 1, 6, 17 and 30 for such arguments. Recall that for simplicity we write q instead of q^0.

The proof of Theorem 2.3 relies on the construction of suitable super- and sub-solutions of (2.23), which, as $\epsilon \to 0^+$, drive the solution of (2.23) to $\pm m_\beta$ in the appropriate regions of the (r, t) space. A similar approach was taken in Ref. 1, 6, 12 and 42 for the study of the asymptotics of reaction-diffusion equations.

A crucial part in the construction of super- and sub-solutions of (2.23) is played by a lower order corrector terms the existence of which leads to the identification of the matrix $A(e)$ and the coefficient $\mu(e)$. Notice that the transport matrix $\mu(e)A(e)$ does not appear in (2.23). It arises as a result of an averaging effect, in the limit $\epsilon \to 0^+$, due to the highly nonlinear form of the equation as well as its nonlocal character.

More precisely, but still heuristically, our super- and sub-solutions will be of the form

$$q(\epsilon^{-1} d(r,t), Dd(r,t)) + \epsilon Q(\epsilon^{-1} d(r,t), Dd(r,t)) + O(\epsilon^2),$$

Q being the corrector, which is identified by solving an appropriate "cell" problem. As usual, it is the condition guaranteeing the solvability of the cell problem that yields the result.

To this end, for $a \in \mathbb{R}$, $e \in \mathbb{R}^N \backslash \{0\}$ and $\epsilon > 0$, let $q^{a\epsilon} = q^{a\epsilon}(\xi, e)$ be the travelling wave corresponding to

$$m_t + \Phi(\beta(J * m + a\epsilon))[m - \tanh[\beta(J * m + a\epsilon)]] = 0 \qquad \text{in } \mathbb{R}^N \times (0, \infty), \tag{3.6}$$

with speed $c^{a\epsilon}(e)$ satisfying, see for example Ref. 30,

$$c(a, e) := \lim_{\epsilon \to 0^+} \epsilon^{-1} c^{a\epsilon}(e) = 2a m_\beta \mu(e), \tag{3.7}$$

with $\mu(e)$ given by (2.19).

Next fix $\mathcal{B} \in S^N$ and $\mathcal{A} \in \mathbb{R}$. A corrector $Q^\epsilon = Q^\epsilon(\xi, e) : \mathbb{R} \times \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ is the unique solution of

$$
\begin{aligned}
\mathcal{L}^{a\epsilon}(e)Q^\epsilon =& \mathcal{A}\dot{q}^{a\epsilon} - \Phi(\beta(J * q^{a\epsilon} + a\epsilon)) \\
& \times \Big[1 - \Big(q^{a\epsilon} + c^{a\epsilon}(e)(\Phi(\beta(J * q^{a\epsilon} + a\epsilon)))^{-1}\dot{q}^{a\epsilon}\Big)^2\Big] \\
& \times \frac{\beta}{2}\int J(r')[\dot{q}^{a\epsilon}(\xi + r'\cdot e, e)(\mathrm{tr}(r'\otimes r')\mathcal{B}) \\
& + \mathrm{tr}([(D_e q^{a\epsilon}(\xi + r'\cdot e, e)\otimes r' + r'\otimes D_e q^{a\epsilon}(\xi + r'\cdot e, e))]\mathcal{B})]dr',
\end{aligned} \tag{3.8}
$$

which is such that

$$
\begin{cases}
Q^\epsilon(0,e) = 0, \ \ |Q^\epsilon(\xi,e)| \le Ce^{-\lambda|\xi|}, \ |\dot{Q}^\epsilon(\xi,e)| \le Ce^{-\lambda|\xi|}, \\
\text{for some positive constants } C \text{ and } \lambda, \\
\text{and} \\
D_e Q^\epsilon \ \text{ is continuous.}
\end{cases} \tag{3.9}
$$

In the above equation $\mathcal{L}^{a\epsilon}(e)$, which is the linearization around $q^{a\epsilon}$ of the equation satisfied by $q^{a\epsilon}$, is given by

$$
\begin{aligned}
\mathcal{L}^{a\epsilon}(e)Q =& c^{a\epsilon}(e)\Big[\Phi(\beta(J * q^{a\epsilon} + a\epsilon))\dot{Q} - \frac{\Phi'(\beta(J^\epsilon * q^{a\epsilon} + a\epsilon))}{\Phi(\beta(J^\epsilon * q^{a\epsilon} + a\epsilon))} J^\epsilon * \dot{q}^{a\epsilon}Q\Big] \\
& + \Phi(\beta(J * q^{a\epsilon} + a\epsilon))\{Q - [1 - (q^{a\epsilon} + c^{a\epsilon}(e)(\Phi(\beta(J * q^{a\epsilon} + a\epsilon)))^{-1}\dot{q}^{a\epsilon})^2] \\
& \times \beta\int J(r')Q(\xi + r'e, e)dr'.
\end{aligned} \tag{3.10}
$$

It follows, see for example the discussion in Section 2, that

$$
\ker(\mathcal{L}^{a\epsilon}(e))^* = \ker\,\mathcal{L}^{a\epsilon}(e) = \dot{q}^{a\epsilon}(\cdot,e)\mathbb{R}.
$$

Hence the existence of such a Q^ϵ follows from Fredholm's alternative, provided the right hand side of (3.10) is orthogonal to the kernel of the operator $\mathcal{L}^{a\epsilon}(e)$. This leads to the compatibility condition

$$
\mathcal{A} = \mathrm{tr}\ \mu^\epsilon(e)A^\epsilon(e)\mathcal{B}\,,
$$

where

$$A^\epsilon(e) = \tfrac{1}{2}\iint J(r)\dot{q}^{a\epsilon}(\xi,e)[\dot{q}^{a\epsilon}(\xi+e\cdot r,e)r\otimes r$$

$$+ D_e q^{a\epsilon}(\xi+e\cdot r,e)\otimes r + r\otimes D_e q^{a\epsilon}(\xi+e\cdot r,e)]drd\xi$$

and

$$\mu^\epsilon(e) = \beta\Big[\int \frac{(\dot{q}^{a\epsilon}(\xi,e))^2}{\Phi(\beta(J*q^{a\epsilon}+a\epsilon))[1-(q^{a\epsilon}+c^{a\epsilon}(e)(\Phi(\beta(J*q^{a\epsilon}+a\epsilon)))^{-1}\dot{q}^{a\epsilon}(\xi,e))^2]}d\xi\Big]^{-1}$$

Notice that, as $\epsilon\to 0$,

$$A^\epsilon(e)\to A(e) \text{ and } \mu^\epsilon(e)\to\mu(e) \text{ in } \mathbb{R}^N\backslash\{0\}. \tag{3.11}$$

Furthermore, as we will see later, $\mathcal{B}$ will be chosen to be $D^2d(r,t)$, hence Q^ϵ will depend on (r,t). Since d satisfies (3.4), it follows that there exists a positive constant C such that

$$|Q^\epsilon_t| + |D_r Q^\epsilon| \le C. \tag{3.12}$$

We now introduce super- and sub-solutions for (2.23) refining ideas of Ref. 6, 12, 17, 18, etc., with the use of the appropriate correctors defined earlier. We begin with some preliminary constructions.

For fixed δ and a, let $u^{\delta,a}$ be the solution

$$\begin{cases} u^{\delta,a}_t - F(Du^{\delta,a}, D^2u^{\delta,a}) - c(a,\overline{Du^{\delta,a}})|Du^{\delta,a}| = 0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ u^{\delta,a}(r,0) = d_0(r)+\delta & \text{on } \mathbb{R}^N\times\{0\}, \end{cases} \tag{3.13}$$

set $\Gamma^{\delta,a}_t = \{r : u^{\delta,a}(r,t)=0\}$ and let $d^{\delta,a}(r,t)$ be the signed distance from $\Gamma^{\delta,a}_t$.

Since d satisfies (3.4) in $[0,T]$, there is $a_0>0$ such that for all $a\in(-a_0,a_0)$, $d^{\delta,a}$ satisfies (3.4) in $[0,T+\epsilon)$. Furthermore we have

$$d^{\delta,a}_t - \mu(Dd^{\delta,a})\mathrm{tr}\{A(Dd^{\delta,a})D^2d^{\delta,a}\} - c(a,Dd^{\delta,a}) = 0 \quad \text{on } \Gamma^{\delta,a}_t, \tag{3.14}$$

and

$$d^{\delta,a}_t - \mu(Dd^{\delta,a})\mathrm{tr}\{A(Dd^{\delta,a})D^2d^{\delta,a}\} - c(a,Dd^{\delta,a}) = O(|d^{\delta,a}|) \quad \text{on } \{|d^{\delta,a}|<\delta_0\}. \tag{3.15}$$

We now define our candidate $U = U(r,t)$ for the super- and sub-solution of (2.23). If $|d^{\delta,a}| \le \delta \le \delta_0/2$, set

$$U(r,t) = q^{a\epsilon}(\epsilon^{-1}d^{\delta,a}(r,t), Dd^{\delta,a}(r,t)) + \epsilon Q^{\epsilon}(\epsilon^{-1}d^{\delta,a}(r,t), Dd^{\delta,a}(r,t)). \tag{3.16}$$

If $d^{\delta,a} > \delta$ we extend U so that it is uniformly continuous in (r,t), continuously differentiable in t and satisfies, uniformly in ϵ,

$$\begin{cases} |U(r,t) - m_\beta^{a\epsilon,+}| \le a_{\max} e^{-\lambda_{\min}\frac{\delta}{\epsilon}} + o_\delta(1) \quad \text{in } \{d^{\delta,a} > \delta\}, \\ \text{and} \\ |U_t| \le C. \end{cases} \tag{3.17}$$

Here

$$\lambda_{\min} = \min_{|e|=1} \lambda(e) \quad \text{and} \quad a_{\max} = \max_{|e|=1} a(e),$$

where a and λ are defined in (2.11) for $h = 0$. Similarly we extend U when $d^{\delta,a} < -\delta$ by requiring that

$$|U(r,t) - m_\beta^{a\epsilon,-}| \le a_{\max} e^{-\lambda_{\min}\frac{\delta}{\epsilon}} + o_\delta(1). \tag{3.18}$$

We can now state and prove the key lemma leading to the proof of Theorem 2.3.

Lemma 3.1: *The function U defined in (3.16) is a super-solution (respectively sub-) of (2.23), (3.5) if a is positive (respectively negative).*

Proof. 1. We only argue for $a > 0$. 2. If $r \in \{d^{\delta,a} > \delta\}$, then, for ϵ uniformly small,

$$U(r,0) \ge m_\beta^{a\epsilon,+} - a_{\max} e^{-\lambda_{\min}\frac{\delta}{\epsilon}} > m_\beta > m(r,0).$$

Similarly, if $r \in \{d^{\delta,a} < -\delta\}$, $U(r,0) \ge m(r,0)$. If $r \in \{|d^{\delta,a}| \le \delta\}$, using the properties of q and $q^{a\epsilon}$ we obtain, for ϵ sufficiently small,

$$U(r,0) = q^{a\epsilon}(\epsilon^{-1}(d_0(r) + \delta), Dd_0(r)) \ge q(\epsilon^{-1}(d_0(r)), Dd_0(r)).$$

Hence

$$U(\cdot,0) \ge m(\cdot,0) = q(d_0, Dd_0).$$

3. Next we show U is supersolution of (2.23) in $\{d^{\delta,a} > \delta\} \cup \{d^{\delta,a} < -\delta\}$. Using the fact J has compact support, we obtain, for ϵ uniformly small, that

$$U_t + \epsilon^{-2}\Phi(\beta J^\epsilon * U)\Big[U - \tanh\Big(\beta \int J(r')U(r+\epsilon r', t)dr'\Big)\Big]$$

$$\geq -c + \epsilon^{-2}\Phi(\beta J^\epsilon * U)[m_\beta^{a\epsilon,+} + O(e^{-\lambda_{\min}\frac{\delta}{\epsilon}}) - \tanh(\beta \overline{J} m_\beta^{a\epsilon,+} + O(e^{-\lambda_{\min}\frac{\delta}{\epsilon}}))]$$

$$= -c + \epsilon^{-2}\Phi(\beta J^\epsilon * U)[\tanh(\beta \overline{J} m_\beta^{a\epsilon,+} + a\epsilon) - \tanh(\beta J m_\beta^{a\epsilon,+}) + O(e^{-\lambda_{\min}\frac{\delta}{\epsilon}})]$$

$$\geq -c + \epsilon^{-2}\Phi(\beta J^\epsilon * U)[\tanh'(\beta \overline{J} m_\beta^{a\epsilon,+})a\epsilon + O(\epsilon^2) + O(e^{-\lambda_{\min}\frac{\delta}{\epsilon}})] > 0.$$

4. If $|d^{\delta,a}(r,t)| < \delta$, then, since $q^{a\epsilon}$ is a travelling wave solution of (3.6) with speed $c^\epsilon(a,e)$,

$$U_t + \epsilon^{-2}\Phi(\beta J^\epsilon * U)[U - \tanh J^\epsilon * U] = \epsilon^{-1}\dot{q}^{a\epsilon} d_t^{\delta,a} + \epsilon^{-1}\Phi(\beta J^\epsilon * U)Q^\epsilon +$$

$$+ D_e q^{a\epsilon} D d_t^{\delta,a} + \dot{Q}^\epsilon d_t^{\delta,a} + \epsilon D_e Q^\epsilon D d_t^{\delta,a} + \epsilon Q_t^\epsilon - \epsilon^{-2} c^{a\epsilon}(e)\frac{\Phi(\beta J^\epsilon * U)}{\Phi(\beta(J^\epsilon * q^{a\epsilon} + a\epsilon)}\dot{q}^{a\epsilon} +$$

$$+ \epsilon^{-2}\Phi(\beta J^\epsilon * U)\Big\{\tanh\beta\Big[\int J(r')q^{a\epsilon}(\epsilon^{-1}d^{\delta,a} + er', e)dr' + a\epsilon\Big] -$$

$$- \tanh\beta\Big[\int J(r')(q^{a\epsilon}(\epsilon^{-1}d^{\delta,a}(r+\epsilon r', t), Dd^{\delta,a}(r+\epsilon r', t)) +$$

$$+ \epsilon Q^\epsilon(\epsilon^{-1}d^{\delta,a}(r+\epsilon r', t), Dd^{\delta,a}(r+\epsilon r', t), r+\epsilon r', t))dr'\Big]\Big\}$$

where we denote by e the gradient $Dd^{\delta,a}(r,t)$ and whenever we evaluate a function at (r,t) we omit the arguments.

5. Call C^ϵ the term in the curly bracket in the equation of Step 3. Expanding tanh to second order we obtain

$$C^\epsilon = \tanh'\Big[\beta\Big(\int J(r')q^{a\epsilon}(\epsilon^{-1}d^{\delta,a} + er', e)dr' + a\epsilon\Big)\Big] \cdot D^\epsilon + \tanh''(\zeta) \cdot (D^\epsilon)^2$$

where

$$D^\epsilon = \beta \int J(r')[q^{a\epsilon}(\epsilon^{-1} d^{\delta,a} + er', e) - q^{a\epsilon}(\epsilon^{-1} d^{\delta,a}(r + \epsilon r', t), Dd^{\delta,a}(r + \epsilon r', t))]dr$$

$$+ \beta a\epsilon - \epsilon\beta \int J(r') Q^\epsilon(\epsilon^{-1} d^{\delta,a}(r + \epsilon r', t), Dd^{\delta,a}(r + \epsilon r', t), r + \epsilon r', t)dr'$$

$$= B^\epsilon - \epsilon E^\epsilon.$$

6. Using (3.4) and the properties of the corrector Q^ϵ we find that

$$E^\epsilon = \beta \int J(r') Q^\epsilon(\epsilon^{-1} d^{\delta,a} + er' + O(\epsilon), e + O(\epsilon), r + \epsilon r', t)dr'$$

(3.19)

$$= \beta \int J(r') Q^\epsilon(\epsilon^{-1} d^{\delta,a} + er', e, r, t)dr' + O(\epsilon).$$

and

$$B^\epsilon = \beta \int J(r')[q^{a\epsilon}(\epsilon^{-1} d^{\delta,a} + er', e) - q^{a\epsilon}(\epsilon^{-1} d^{\delta,a} + er' + \frac{\epsilon}{2}(D^2 d^{\delta,a} r', r') + O(\epsilon^2),$$

(3.20)

$$e + \epsilon D^2 d^{\delta,a} r' + O(\epsilon^2))\Big] dr' + \beta a\epsilon,$$

where the $O(\epsilon)$ and $O(\epsilon^2)$ depend only on ϵ and the constants in (3.4).

Expanding q to second order we obtain

$$B^\epsilon = \beta \int J(r')\Big\{\dot{q}^{a\epsilon}(\epsilon^{-1} d^{\delta,a} + e \cdot r', e)\Big[\frac{\epsilon}{2}(D^2 d^{\delta,a} r', r') + O(\epsilon^2)\Big] +$$

(3.21)

$$+ D_e q^{a\epsilon}(\epsilon^{-1} d^{\delta,a} + er', e)(\epsilon D^2 d^{\delta,a} r')\Big\} dr' + \beta a\epsilon + O(\epsilon^2).$$

Combining (3.19) and (3.21) yields that

$$D^\epsilon = O(\epsilon).$$

7. Using this last fact as well as (3.19) and (3.20) and

$$\tanh'(J * q) = 1 - \tanh^2(J * q)$$

we obtain

$$C^\epsilon = [1 - (q^{a\epsilon} + c^{a\epsilon}(e)(\Phi(\beta J * q^{q\epsilon}))^{-1}\dot{q}^{a\epsilon})^2]$$

$$\times \Big\{ \frac{\beta\epsilon}{2} \int J(r')\dot{q}^{a\epsilon}(\epsilon^{-1}d^{\delta,a} + e\cdot r', e)(D^2 d^{\delta,a} r', r')dr'$$

$$+ \beta a\epsilon + \beta\epsilon \int J(r') D_e q^{a\epsilon}(\epsilon^{-1}d^{\delta,a} + e\cdot r', e) D^2 d^{\delta,a} r' dr'$$

$$- \beta\epsilon \int J(r') Q^\epsilon(\epsilon^{-1}d^{\delta,a} + e\cdot r', e, r, t\Big) dr' + O(\epsilon^2)\Big\} + O(\epsilon^2).$$

Using that Φ is Lipschitz and subsequently expanding $J^\epsilon * U$ as in (3.20), (3.21), we get that

$$\Phi(\beta J^\epsilon * U) = \Phi(\beta J^\epsilon * q^{a\epsilon}) + O(\epsilon).$$

Going now all the way back to the equation of Step 4 we obtain

$$U_t + \epsilon^{-2}\Phi(\beta J^\epsilon * U)[\Phi - \tanh \beta J^\epsilon * U] =$$

$$\epsilon^{-1}\Big\{\dot{q}^{a\epsilon} d_t^{\delta,a} - \Phi(\beta J * q^{a\epsilon})[1 - (q^{a\epsilon} + c^{a\epsilon}(e)(\Phi(\beta J * q^{a\epsilon}))^{-1}\dot{q}^{a\epsilon})^2]\Big[\frac{\beta}{2}\int J(r')$$

$$[\dot{q}^{a\epsilon}(\epsilon^{-1}d^{\delta,a} + e\cdot r', e)(D^2 d^{\delta,a} r', r') + 2D_e q^{a\epsilon}(\epsilon^{-1}d^{\delta,a} + e\cdot r', r') D^2 d^{\delta,a} r']dr'$$

$$- \beta \int J(r') Q^\epsilon(\epsilon^{-1}d^{\delta,a} + e\cdot r', e, r, t)dr' + \beta a + O(\epsilon)\Big] + \Phi(\beta J * q^{a\epsilon})Q^\epsilon -$$

$$- \epsilon^{-1} c^{a\epsilon}(\epsilon)\frac{\Phi(\beta J^\epsilon * U)}{\Phi(\beta(J^\epsilon * q^{a\epsilon} + a\epsilon))}\dot{q}^{a\epsilon}\Big\} + [D_e q^{a\epsilon} D d_t^{\delta,a} + \dot{Q}^\epsilon d_t^{\delta,a} + \epsilon D_e Q^\epsilon D d_t^{\delta,a} + \epsilon Q_t^\epsilon] + O(1).$$

8. Recall the definition of Q^ϵ through the cell problem (3.7), where $\mathcal{B} = D^2 d(r,t)$ and

$\mathcal{A} = \mathrm{tr}\{\mu^\epsilon(Dd^{\delta,a})A^\epsilon(Dd^{\delta,a})D^2 d^{\delta,a}\}$. Then

$$U_t + \epsilon^{-2}\Phi(\beta J * m)[U - \tanh J^\epsilon * U] = \epsilon^{-1}\Big\{\dot{q}^{a\epsilon}\Big[d_t^{\delta,a} - \mathrm{tr}\mu^\epsilon(Dd^{\delta,a})A^\epsilon(Dd^{\delta,a})D^2 d^{\delta,a}$$

$$- c(a,e) + (c(a,e) - \epsilon^{-1}c^{a\epsilon}(e))\Big] + \beta a + O(\epsilon) - c^{a\epsilon}(e)[\Phi(\beta J * q^{a\epsilon})\dot{Q}^\epsilon$$

$$- \frac{\Phi'}{\Phi} J * q^{a\epsilon} Q] + O(\epsilon)\Big\} + O(1).$$

Since, as $\epsilon \to 0$,

$$\epsilon^{-1}c^{a\epsilon}(e) \to c(a,e) \quad \text{and} \quad A^\epsilon \to A \text{ and } \mu^\epsilon \to \mu,$$

and

$$|d_t^{\delta,a} - \mu(Dd^{\delta,a})\mathrm{tr}\{A(Dd^{\delta,a})D^2 d^{\delta,a}\} - c(a, Dd^{\delta,a})| = O(|d^{\delta,a}|) \leq O(\delta),$$

for ϵ, δ small, the right hand side of the last equality is positive, thus U a supersolution of (2.23) in $\{|d^{\delta,a}| < \delta\}$. □

We conclude with the

Proof of Theorem 2.3: 1. Pick $(r_0, t_0) \in \mathbb{R}^N \times [0, T)$ such that $u(r_0, t_0) = -\gamma < 0$, where u solves (2.23). The stability of solutions for pde's of the type (2.23) yields that $u^{\delta,a} \to u$ locally uniformly in $\mathbb{R}^N \times [0, T)$ as $\delta, a \to 0$. Therefore, for sufficiently small δ and a, we have

$$u^{\delta,a}(r_0, t_0) < -\frac{\epsilon}{2} < 0 \quad \text{and} \quad d^{\delta,a}(r_0, t_0) < 0. \tag{3.22}$$

2. Lemma 2.2 and Lemma 3.1 yield

$$U \geq m^\epsilon \geq -m_\beta \quad \text{in } \mathbb{R}^N \times [0, T),$$

which combined with (3.22) yields

$$\lim_{\epsilon \to 0^+} m^\epsilon(r_0, t_0) = \lim_{\epsilon \to 0^+} \Phi(r_0, t_0) = -m_\beta.$$

3. Using a subsolution constructed as in Lemma 3.1 we see that

$$\lim_{\epsilon \to 0^+} m^\epsilon = m_\beta \quad \text{in } \{u > 0\}.$$ □

Remark. Notice that the above proof does not quite work for the Metropolis dynamics, since in (3.10) we used the differentiability of Φ. However we may mollify the singularity of the Metropolis dynamics at 0 by introducing a new small parameter ζ. Then we may proceed in the proof of Theorem 2.3, using the stability of equations (2.21) and (2.23) and letting first $\zeta \to 0$ and then $\epsilon \to 0$.

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