BONNET PAIRS AND ISOTHERMIC SURFACES

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1. Introduction

A classical question in surface theory is which data are sufficient to describe a surface in space up to rigid motions. Bonnet suggested that mean curvature and metric should suffice to determine the surface generically. The local theory (without umbilic points) was developed by Bonnet [6], Cartan [4] and Chern [5] who showed the existence of various 1-parameter families of Bonnet surfaces, i.e., surfaces with the same induced metric and mean curvature. A comprehensive study of this problem and its relationship to the Painleve equations has recently been completed by Bobenko and Eitner [2]. On the other hand, Lawson and Tribuzy [9] have shown that for embedded compact surfaces there are at most two surfaces to a given metric and mean curvature. Moreover, uniqueness can be established under various global assumptions [8]. Up to date it is unknown whether such compact Bonnet pairs exist and if, how to construct them.

In this note we describe *all* Bonnet pairs on a simply connected domain, including surfaces with umbilic points. Our main intent was to apply what we call a *quaternionic function theory* to a concrete problem in differential geometry. In the first section we develop the necessary formulas and explain how usual Riemann surface theory can be viewed as a special case of our extended function theory. The ideas are simple: conformal immersions into quaternions or imaginary quaternions take the place of chart maps for a Riemann surface. Starting from a reference immersion we construct all conformal immersions of a given Riemann surface (up to translational periods) by *spin transformations* (Definition 2.1). With this viewpoint in mind we discuss in the second section how to construct all Bonnet pairs on a simply connected domain from isothermic surfaces and vice versa. Isothermic surfaces are solutions to a certain soliton equation [1] thus a simple dimension count tells us that we obtain Bonnet pairs which are not part of any of the families. The correspondence between Bonnet pairs and isothermic surfaces is explicit and to each isothermic surface we obtain a 4-parameter family of Bonnet pairs. An indication that such an explicit relationship might exist was first given by Cartan who observed that the general (local) Bonnet pair depends on 4 arbitrary functions of one variable which is the same generality as the general (local) isothermic surface.

2. Surface theory revisited

Let M be a Riemann surface and $f : M \to \mathbb{R}^3$ an immersion. We always regard $\mathbb{R}^4 = \mathbb{H}$ as quaternions and $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ as purely imaginary quaternions. Note that for $a, b \in \mathrm{Im}\mathbb{H}$ we have

$$ab = -<a, b> + a \times b\,.$$

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There is a natural wedge product over $\mathbb{H}$-valued 1-forms given by

$$\alpha \wedge \beta(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X) \tag{2.1}$$

which satisfies the obvious identities

$$\overline{\alpha \wedge \beta} = -\bar{\beta} \wedge \bar{\alpha}$$
$$\alpha \wedge h\beta = \alpha h \wedge \beta$$
$$\mathrm{d}(h\alpha) = dh \wedge \mathrm{d}\alpha + h\mathrm{d}\alpha$$
$$\mathrm{d}(\alpha h) = \mathrm{d}\alpha h - \alpha \wedge dh$$

for an $\mathbb{H}$-valued function $h : M \to \mathbb{H}$. Usually we will identify 2-forms on M with their quadratic forms via

$$\omega(X) = \omega(X, JX)$$

where $\omega \in \Omega^2(M,\mathbb{H})$ and $J : TM \to TM$ is the complex structure of M. Thus the only objects appearing in our theory will be $\mathbb{H}$-valued functions on M and TM. A key formula is

$$\alpha \wedge \beta = \alpha(*\beta) - (*\alpha)\beta \tag{2.2}$$

where $*\alpha = \alpha \circ J$ is (minus) the Hodge star operator. With this in place let us develop the necessary equations of surface theory as needed for our purposes.

Lemma 2.1. *$f : M \to \mathbb{R}^3$ is a conformal immersion if and only if there exists $N : M \to \mathbb{H}$ such that*

$$*df = N df\,. \tag{2.3}$$

If (2.3) holds then $N : M \to S^2 \subset \mathrm{Im}\mathbb{H}$ is the (oriented) unit normal field (Gauss map) to f.

Proof. Since $*^2 = -1$ and df is pointwise injective we have $N^2 = -1$. This implies $|N|^2 = 1$ and thus $\bar{N} = -N$. Conjugating (2.3) gives $Ndf = -dfN$ which says that $< N, df >= 0$. Thus $N : M \to S^2 \subset \mathrm{Im}\mathbb{H}$ is a unit normal field to f which defines the orientation since $*df = df \circ J$. □

Remark 1. Equation (2.3) is a generalization of the Cauchy-Riemann equation: to $h : M \to \mathbb{C}$ let $f = hj : M \to \mathrm{Im}\mathbb{H}$ with $N = i$. Then (2.3) becomes

$$*dh = idh$$

which is the Cauchy-Riemann equation for h. In this sense surface theory in $\mathrm{Im}\mathbb{H}$ can be viewed as a generalization of complex function theory to a quaternionic function theory. Holomorphic chart maps are special cases of conformal immersions.

Any $\mathbb{H}$-valued 1-form α on M can be decomposed into its conformal and anti-conformal parts:

$$\alpha = \alpha_+ + \alpha_-\,, \qquad *\alpha_\pm = \pm N\alpha_\pm \tag{2.4a}$$

where

$$\alpha_\pm = \frac{1}{2}(\alpha \mp N * \alpha)\,. \tag{2.4b}$$

Thus, using (2.3),

$$d * df = dN \wedge df = dN * df - *dNdf = (dNN - *dN)df = 2(dN)_+ Ndf$$

which is $\mathrm{Im}\mathbb{H}$-valued. This means

$$< (dN)_+, Ndf >= 0$$

or, since dN_+ is also tangential,

$$dN_+ = Hdf \tag{2.5}$$

for some real valued function $H : M \to \mathbb{R}$. Clearly, H is (minus) the mean curvature of f w.r.t. the induced metric $|df|^2$. Inserting (2.5) we obtain

$$\mathrm{d}N = Hdf + \omega\,, \qquad \omega = (dN)_-\,, \qquad *\omega = -N\omega \tag{2.6}$$

and thus

$$d * df = 2HN|df|^2\,. \tag{2.7}$$

Differentiating (2.6) we get the Codazzi equation

$$\mathrm{d}\omega = (*dH - dHN)df\,. \tag{2.8}$$

Definition 2.1. Two conformal immersions $f, \tilde{f} : M \to \mathbb{R}^3$ are called *spin-equivalent* if there exists $\lambda : M \to \mathbb{H}_*$ so that

$$d\tilde{f} = \bar{\lambda} df \lambda\,. \tag{2.9}$$

We call $\tilde{f}$ a *spin-transform* of f.

Remark 2.
1. If $\pi_1(M) = 0$ then any two conformal immersions are spin-equivalent: since $d\tilde{f}$ and df are conformal 1-forms pointwise they can be mapped into each other by a rotation and scaling. But this is exactly what (2.9) means. Notice that λ is unique up to sign.
2. If $\pi_1(M) \neq 0$ then f and $\tilde{f}$ are spin-equivalent if and only if they belong to the same regular homotopy class [10].
3. λ is constant if and only if $\tilde{f}$ is obtained from f by an Euclidean motion and scaling. If $|\lambda| = 1$ then $\tilde{f}$ and f are congruent immersions.
4. Spin-equivalence clearly is an equivalence relation.

To obtain new conformal immersions (locally) from a given *reference* immersion $f : M \to \mathbb{R}^3$ via spin-transformations we only have to solve

$$0 = d(\bar{\lambda} df \lambda) = d\bar{\lambda} \wedge df \lambda - \bar{\lambda} df \wedge d\lambda = -2\mathrm{Im}(\bar{\lambda} df \wedge d\lambda)\,.$$

But this says that $\bar{\lambda} df \wedge d\lambda$ is real valued and hence there exists a unique real valued function $\rho : M \to \mathbb{R}$ with

$$df \wedge d\lambda = -\rho |df|^2 \lambda = \rho df^2 \lambda\,.$$

Using (2.2), (2.3) and dividing by df we obtain the integrability equation

$$*d\lambda + N d\lambda = \rho df \lambda\,. \tag{2.10}$$

This is worth formulating as a

Lemma 2.2. *If $f, \tilde{f} : M \to \mathbb{R}^3$ are spin-equivalent via $d\tilde{f} = \bar{\lambda} df \lambda$ then $\lambda : M \to \mathbb{H}_*$ satisfies* (2.10)*.*

Conversely, if $\pi_1(M) = 0$ then to a given conformal immersion $f : M \to \mathbb{R}^3$ nowhere vanishing solutions to (2.10) *yield all conformal immersions $\tilde{f} : M \to \mathbb{R}^3$ via spin-transformations $d\tilde{f} = \bar{\lambda} df \lambda$.*

We now relate elementary geometric data of spin-equivalent immersions.

Lemma 2.3. *Let $f, \tilde{f} : M \to \mathbb{R}^3$ be spin-equivalent via $d\tilde{f} = \bar{\lambda} df \lambda$. Then*

1. *$\tilde{N} = \lambda^{-1} N \lambda$ is the oriented normal to $\tilde{f}$,*

2. $|d\tilde{f}|^2 = |\lambda|^4 |df|^2$,
3. $\tilde{H} = \frac{H+\rho}{|\lambda|^2}$ *where* $\rho : M \to \mathbb{R}$ *is given by* (2.10).

Proof. (i) and (ii) are obvious. To see (iii) we use (2.6):

$$\tilde{H} d\tilde{f} + \tilde{\omega} = d\tilde{N} = \frac{H}{|\lambda|^2} d\tilde{f} + \lambda^{-1}(\omega + N d\lambda\lambda^{-1} - d\lambda\lambda^{-1}N)\lambda\,,$$

hence, computing the conformal part (2.4) of the latter and inserting (2.10), we get

$$(\lambda^{-1}(\omega + N d\lambda\lambda^{-1} - d\lambda\lambda^{-1}N)\lambda)_+ = \lambda^{-1}(\omega + N d\lambda\lambda^{-1} - d\lambda\lambda^{-1}N)_+\lambda =$$

$$\lambda^{-1}\rho df\lambda = \frac{\rho d\tilde{f}}{|\lambda|^2}$$

and thus

$$\tilde{H} = \frac{H+\rho}{|\lambda|^2}\,.$$

□

Corollary 2.1. *Let* $f, \tilde{f} : M \to \mathbb{R}^3$ *be spin-equivalent via* $d\tilde{f} = \bar{\lambda} df \lambda$. *Then the following are equivalent:*

1. $\tilde{H}|d\tilde{f}| = H|df|$
2. $df \wedge \mathrm{d}\lambda = 0$ *which is the same as* $*d\lambda + N d\lambda = 0$.

3. Isothermic surfaces and Bonnet pairs

We now give the local classification of all Bonnett pairs and explain how they are obtained from isothermic surfaces.

Definition 3.1. A conformal immersion $f : M \to \mathbb{R}^3$ is *isothermic* if there exists an non-zero, $\mathrm{Im}\mathbb{H}$-valued, closed and anti-conformal 1-form $\tau \in \Omega^1(M, \mathrm{Im}\mathbb{H})$, i.e.,

$$\mathrm{d}\tau = 0\,, \qquad \tau \neq 0\,, \qquad *\tau + N\tau = 0 \text{ which is equivalent to } df \wedge \tau = 0\,.$$

Note that this is a Moebius invariant condition.

Remark 3.
1. The classical notion of an isothermic surface is that $f : M \to \mathbb{R}^3$ admits conformal curvature line parameters (away from umbilic points). If (x, y) are such parameters then one easily sees that the 1-form
$$\tau = \frac{1}{|f_x|^2}(f_x dx - f_y dy) = -f_x^{-1} dx + f_y^{-1} dy$$
is closed and anti-conformal. Thus, classically isothermic surfaces are isothermic in our notion.
2. Since $\mathrm{d}\tau = 0$ locally $\tau = df^*$ and with $N^* = -N$ one sees that f^* is a conformal immersion (a dual isothermic surface) away from zeros of τ.
3. Examples of isothermic surfaces include quadrics, surfaces of revolution and surfaces of constant mean curvature (and their Moebius transforms). The latter case follows at once from (2.8) and (2.6): $\mathrm{d}\omega = 0$ and $*\omega + N\omega = 0$ so that we may put $\tau = \omega$. For minimal surfaces ($H = 0$) we get $dN = \omega$ so that the Gauss map N is a dual surface.
4. Recently it has been shown [1, 3, 7] that isothermic surfaces can be obtained from an infinite dimensional integrable system (soliton equation). This provides an infinte dimesnional space of examples of local isothermic surfaces.

Definition 3.2. Two conformal immersions $f, \tilde{f} : M \to \mathbb{R}^3$ form a *Bonnet pair* if they induce the same metric $|d\tilde{f}|^2 = |df|^2$ and have the same mean curvature $\tilde{H} = H$, but are not congruent.

Finally we arrive at the local classification of *all* Bonnet pairs. We assume $\pi_1(M) = 0$.

Theorem 3.1. *Let $f : M \to \mathbb{R}^3$ be isothermic with dual surface $f^* : M \to \mathbb{R}^3$. Choose $\epsilon \in \mathbb{R}_*$, $a \in \mathrm{Im}\mathbb{H}$ and let $\lambda_\pm = \pm\epsilon + f^* + a$. Then the spin transforms $f_\pm : M \to \mathbb{R}^3$ given by $df_\pm = \overline{\lambda_\pm} df \lambda_\pm$ form a Bonnet pair.*

Conversely, every Bonnet pair arises from a 3-parameter family (determined up to scalings) of isothermic surfaces where the three parameters account for Euclidean rotations of the partners in the Bonnet pair w.r.t. each other.

Proof. From Definition 3.1 we have $df \wedge df^* = 0$ and thus also $df \wedge d\lambda_\pm = 0$, i.e., $*d\lambda_\pm + Nd\lambda_\pm = 0$, which is (2.10) with $\rho_\pm = 0$. Moreover, $|\lambda_+| = |\lambda_-|$ so that $|df_+|^2 = |df_-|^2$ and $H_+ = H_-$. Congruence of f_+ and f_- means that $\lambda_+^{-1}\lambda_-$ is constant which would imply that f^* is constant. Hence $f_\pm$ form a Bonnet pair.

To show the converse assume that we have a Bonnet pair $f_\pm$. Then

$$df_+ = \bar{\lambda} df_- \lambda \tag{3.1}$$

for some $\lambda : M \to \mathbb{H}$. But f_+ and f_- induce the same metric and same mean curvature so that $|\lambda| = 1$ and by Corollary 2.1 $df_- \wedge \mathrm{d}\lambda = 0$. We now seek an isothermic surface $f : M \to \mathbb{R}^3$ for which $df_\pm = \overline{\lambda_\pm} df \lambda_\pm$ with $\lambda_\pm$ as given in the statement of the Theorem. Comparing with (3.1) this gives $\lambda = \lambda_-^{-1}\lambda_+$. Solving for λ_- we obtain

$$\lambda_- = 2\epsilon(\lambda - 1)^{-1}$$

and

$$f^* = \epsilon - a + 2\epsilon(\lambda - 1)^{-1}.$$

To insure that $\lambda - 1$ vanishes nowhere we can multiply λ on the right by a unit quaternion (which induces a rotation of the partners of the Bonnet pair w.r.t. each other). From $|\lambda| = 1$ we conclude that f^* is immaginary. Thus (up to scale)

$$df = \overline{(\lambda - 1)} df_- (\lambda - 1),$$

which indeed defines an immersion $f : M \to \mathbb{R}^3$ since $df_- \wedge \mathrm{d}\lambda = 0$. Similarily, $df \wedge df^* = 0$ so that f is isothermic with dual f^*.

□

Remark 4.
1. Choices of real ϵ and imaginary quaternionic a give a 4-parameter family of Bonnet pairs to each isothermic surface. For $\epsilon = 0$ the pair degenerates to a 3-paramter family of isothermic surfaces $d\tilde{f} = \overline{(f^* + a)} df (f^* + a)$. Compactifying this $\mathbb{R}^4$ to S^4 adds f (up to congruence) at infinity. Thus the above construction provides an S^4-worth of Bonnet pairs with an equatorial S^3 of isothermic surfaces.
2. It is known that there exists at most two compact embedded surfaces in $\mathbb{R}^3$ with prescribed mean curvature and metric [9]. Understanding the period problem in our local classification could provide an affirmative solution to the question of whether a given compact embedded surface allows a Bonnet partner.

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