CLASSIFICATION OF EMBEDDED CONSTANT MEAN CURVATURE SURFACES WITH GENUS ZERO AND THREE ENDS

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Abstract. For each embedded constant mean curvature surface in $\mathbb{R}^3$ with three ends and genus zero, we construct a conjugate cousin boundary contour in $\mathbb{S}^3$. The moduli space of such contours is parametrized by the space of triangles on $\mathbb{S}^2$. This imposes necessary conditions on these CMC surfaces, and we expect this space of triangles (upto rotation of $\mathbb{S}^2$) exactly parametrizes the moduli space of these surfaces (upto rigid motion of $\mathbb{R}^3$). Remarks on the implications for CMC surfaces with more ends and higher genus are also included.

Surfaces of constant mean curvature H in $\mathbb{R}^3$ arise naturally from minimization of surface area with a volume constraint. If $H \equiv 0$, we have a minimal surface; otherwise, by rescaling we may assume that $H \equiv 1$, and we call this a CMC surface. We are interested in classifying complete embedded CMC surfaces and in parametrizing their moduli spaces.

Embeddedness is not only physically natural, but also seems necessary to give a tractable problem. For example, Alexandrov showed the unit sphere is the only embedded compact CMC surface [A], whereas Wente constructed nonembedded compact examples [W]; many more are now known, especially by the work of Kapouleas [Kp1–3] and by an integrable systems approach [PS, Bo, EKT] which gives multidimensional families of examples. The constructions of Kapouleas also suggest that arbitrary CMC surfaces can be glued together in much more general ways (for a survey, see [MP]), showing the moduli spaces of immersed CMC surfaces must be very rich.

We will consider the slightly more general class of *almost embedded* CMC surfaces, which are those that bound an immersed three-manifold. These are the surfaces to which the Alexandrov reflection technique can be applied [A, KKS]; they share the basic structure of embedded CMC surfaces, but the interior three-manifold is allowed to overlap itself. It seems that this class is most natural mathematically, though it is difficult to determine *a priori* exactly which of these are actually embedded.

Furthermore, we want to restrict our attention to CMC surfaces of finite toplogy (excluding, for instance, triply periodic surfaces). A fundamental result [KKS] shows that each end of such a surface is asymptotically an *unduloid*, an embedded CMC surface of revolution found by Delaunay [D]. This leads us to call any almost embedded CMC surface with genus g and k ends a *k-unduloid (of genus g)*, or *triunduloid* when $k = 3$.

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When considering the set of all k-unduloids of genus g, we identify surfaces which differ by a rigid motion (preserving orientation) of $\mathbb{R}^3$, but we label the ends to distinguish them. The resulting quotient is denoted $\mathcal{M}_{g,k}$. For $k = 0$, we already noted that the sphere is the only compact example [A]. For $k = 1$, Meeks showed there there are no examples [M], while for $k = 2$, the only examples are the unduloids [KKS] themselves.

Thus we are interested in the family $\mathcal{M}_{g,k}$ for $k \geq 3$. Observe that at least three ends are needed to have positive genus; the construction of Kapouleas [Kp1] shows that for $k \geq 3$ and any g, the family $\mathcal{M}_{g,k}$ is nonempty. In general, there is a natural topology which makes this *moduli space* $\mathcal{M}_{g,k}$ into a finite dimensional real analytic variety [KMP]. Moreover, near any surface with no L^2-Jacobi fields, $\mathcal{M}_{g,k}$ is a real analytic manifold of dimension $3k - 6$ [KMP]. (Were we to ignore the end labels, the resulting moduli space would be the further quotient by the symmetric group S_k, and would have orbifold points at symmetric surfaces [G, Bl, GK, GP] which have rigid motions interchanging the ends.)

An unduloid is a CMC surface of revolution [D] whose generating curve is the trace of the focus of an ellipse (of diameter 1 to give $H = 1$) rolled along the axis of revolution (see the exposition of [Ee]). There is a one-parameter family of unduloids, depending on the eccentricity of the ellipse. We will use the *necksize*, the length of the shortest closed geodesic on the surface (or 2π times the minimum radius of revolution) to parameterize this family. Thus the space of unduloids $\mathcal{M}_{0,2}$ is naturally identified with the interval $(0, \pi]$. Here, necksize π corresponds to a cylinder (when the ellipse is a circle). Necksizes approaching zero correspond to the limit of very eccentric ellipses; these unduloids look like a string of spherical beads connected with small catenoid necks. In the almost embedded case, the Kapouleas construction gives k-unduloids with ends asymptotic to unduloids near this kind of limit.

Another useful way [KKS] to parameterize the space of unduloids is by the magnitude of their *force*: think of the unduloid physically balancing surface tension and pressure forces, and cut it through a neck; if the necksize is x, the two halves exert on the neck equal and opposite forces of magnitude $x(2\pi - x)$ directed along the axis. The cylinder is the unduloid of strongest force (magnitude π^2), and the force goes to zero in the spherical-bead limit.

More generally, one can cut an arbitrary CMC surface M along any closed curve γ, and integrate surface tension over γ minus pressure over any surface spanning γ to get a force vector. The variational characterization of CMC surfaces shows [KKS] that this force depends only on the homology class of γ. In particular, to each end of M there is an associated force, and these all *balance* (sum to zero) since all the ends "bound" M itself. For a k-unduloid, these k forces are exactly the forces of the unduloids to which the k ends are asymptotic.

Another result of [KKS] can be strengthened to show that whenever the k force vectors on the ends of a k-unduloid M lie in a plane P (which, by balancing, is always the case when $k = 3$), then P is a symmetry plane of M. The space of all balanced k-uples of coplanar force vectors, up to rotation, has dimension $2k-3$, which should also be the dimension of the space of such coplanar k-unduloids. In addition to the conditions imposed by balancing, we are interested in finding necessary conditions on the necksizes and axes directions. Under additional symmetry assumptions, bounds of this kind were first mentioned in [Ka] and proved in [GK].

Given a coplanar k-unduloid of genus zero, the symmetry plane cuts it into two simply connected pieces with cyclically labeled ends. Each symmetric half is isometric (by Lawson's theorem [L]) to a minimal surface in $\mathbb{S}^3$, its *conjugate cousin*, whose boundary consists of k great circles covered infinitely often. Karcher related the boundary of the conjugate cousin to Hopf fibrations [Ka]. In the coplanar case our key observation is that all bounding circles lie in a common Hopf fibration; projection from $\mathbb{S}^3$ to $\mathbb{S}^2$ along this fibration then gives k cyclically labeled points in $\mathbb{S}^2$, and the distances between consecutive points equal the necksizes of the original surface. Thus our new conditions on necksizes (Theorem 6 and 9) simply come from conditions on the sidelengths of spherical polygons.

For triunduloids the family of conjugate cousin boundary contours (upto rotations of $\mathbb{S}^3$) can be identified with the space of triples of points (upto rotations) in $\mathbb{S}^2$ (Theorem 10). We are currently working on the Plateau problem for this family. If (as we conjecture) it always has a unique solution, then the space $\mathcal{M}_{0,3}$ of triunduloids can also be identified with the space of triples in $\mathbb{S}^2$ (modulo rotations), and in particular our necksize bounds are sharp. Our classification predicts geometric properties, discussed in Section 4, such as which triunduloids have a cylindrical end, and the mechanism of bubble generation; this is particularly interesting, since it suggests the connectedness of $\mathcal{M}_{0,k}$ (see Section 6.2). Note that a family of symmetric triunduloids had been considered earlier by two of us [GK]; here we give simpler arguments for the more general case.

Our existence and uniqueness conjecture for triunduloids is known to be true in the fully symmetric case [G]. In general it is supported by computer experiments carried out with Oberknapp's program [OP] for CMC surfaces, built on the package Grape developed by the SFB 256 at Bonn University; our pictures of k-unduloids were generated by this program.

The situation is different for coplanar k-unduloids of genus zero when $k > 3$ (Section 5): in addition to the k ends, they can have $k-3$ finite segments. A varying number of bubbles along these segments is not distinguished by the boundary contours, and hence the space of contours does not classify these k-unduloids. Nevertheless, some information on necksize bounds and axes configurations can still be obtained.

We expect the classification of triunduloids to be directly useful in describing certain k-unduloids for $k > 3$. In particular, when all segments are long enough, any k-unduloid can be broken into *trouser* pieces which are (approximately) truncated triunduloids (see [KK]). In fact we understand the necessary perturbations to account for the truncation: the truncated triunduloids seem to exist for the same range of necksizes as the complete triunduloids but with slightly different angles of axes. Here we apply these heuristic ideas to discuss the existence of k-unduloids with only cylindrical ends in Section 6.1.

1. Preliminaries

1.1. **Lawson's theorem and symmetries.** A theorem by Lawson [L, p.364] is our essential tool. It establishes a local correspondence of euclidean CMC and spherical minimal surfaces. Here we use *surface* in the differential geometric sense, meaning an equivalence class of smooth immersions modulo diffeomorphisms of the domain; generally we assume our surfaces to be closed, possibly with boundary.

Theorem 1. *A simply connected minimally immersed surface $\widetilde{M} \subset \mathbb{S}^3$ is isometric to an immersed* CMC *surface $M \subset \mathbb{R}^3$, and vice versa. Under this isometry, great circle arcs in $\widetilde{M}$ correspond to geodesic curvature arcs in M.*

We will call such a pair of surfaces *conjugate cousins.*

A geodesic curvature arc is always contained in a plane, and Schwarz reflection extends a CMC surface smoothly across this plane. In particular, if a fundamental CMC domain with respect to reflection symmetries is bounded by geodesic curvature arcs only, then the boundary of its conjugate cousin is a great circle polygon in $\mathbb{S}^3$ and thus depends on finitely many parameters. Schwarz reflection generates a complete CMC surface from such a fundamental domain. This is the situation we consider.

The Alexandrov reflection technique was generalized to k-unduloids of genus g in [KKS]. As a result, such a surface has a reflection symmetry when the asymptotic axes of all ends are contained in a plane or, more generally, when the axes of the ends are contained in parallel planes; we call such a surface *coplanar.* In [GK], the statement of [KKS] was slightly generalized to the following.

Theorem 2. *Suppose M is a coplanar k-unduloid of genus g. Then M has a symmetry plane P and $M \setminus P$ decomposes into a union of two connected open sets M^+ and M^-, with common boundary $\partial M^+ = \partial M^- = P \cap M$. This boundary is the union of k geodesic curvature lines and at most g geodesic curvature circles. Moreover, the orthogonal projections from M^+ and M^- to P are immersions, and are embeddings in case M itself is embedded.*

Here a *line* or *circle* means a properly immersed $\mathbb{R}$ or $\mathbb{S}^1$ in P, respectively. Schwarz reflection across the plane P recovers M from M^+.

1.2. **Balancing.** We can associate to a Delaunay unduloid a *force*, which is a vector whose magnitude determines (and is determined by) the necksize of the unduloid. More precisely, suppose the unduloid has neck circumference or *necksize* $x \in (0, \pi]$ and bulge circumference $2\pi - x$, and its axis points in the direction $a \in \mathbb{S}^2$. Then we define its force

$$f = x(2\pi - x)a \ \in \mathbb{R}^3. \tag{1}$$

Each end of a k-unduloid is asymptotically a Delaunay unduloid [KKS], and hence k necksizes x_i and (outward oriented) axes a_i are defined. The corresponding forces f_i are related by the *balancing formula*

$$\sum_{i=1}^{k} f_i = 0, \tag{2}$$

which can be obtained from the first variation formula (see [KKS]). The balancing formula says that the k force vectors can be translated to form a closed k-gon in $\mathbb{R}^3$.

1.3. **The Hopf fibration.** The Hopf fibration $\Pi : \mathbb{S}^3 \to \mathbb{S}^2$ has great circle fibres. It gives rise to a unit Killing vector field on $\mathbb{S}^3$.

Let us first describe the Hopf fibration. Fix a great circle $\zeta \subset \mathbb{S}^3$. For each $d \in (0, \pi/2)$, the distance torus $T_d(\zeta) := \{x \in \mathbb{S}^3 \mid \operatorname{dist}(x, \zeta) = d\}$ is foliated by great circles whereas $T_{\pi/2}(\zeta)$ collapses to a great circle. The great circles on T_d are known as *Clifford parallels* to ζ with *Clifford distance d*. There are two great circle foliations on T_d, and thus two Clifford parallels to ζ pass through any given point of T_d. We call one of them a *left* Clifford parallel, the other a *right* parallel, and extend this notion continuously over $\mathbb{S}^3$. For the following it is convenient to select one set of parallels, say the left parallels.

We can describe in geometric terms how the Hopf fibration maps the set of left parallels onto $\mathbb{S}^2$. Consider a great sphere S meeting ζ perpendicularly at some point $p \in \zeta$. Let D be the disk of radius $\pi/2$ about p which is contained in S. Then the (left) Clifford parallels to ζ with distance $d < \pi/2$ intersect D in exactly one point. Using the inverse exponential map on D, scaling the resulting point in $\mathbb{R}^2$ by a factor of two, and exponentiating it onto $\mathbb{S}^2$, induces a bijection from the set of (left) Clifford parallels of distance less than $\pi/2$ to $\mathbb{S}^2$ minus a point. The map is onto $\mathbb{S}^2$ if we send the unique Clifford parallel of distance $\pi/2$ to this point.

To an oriented great circle there is a (unique, left or right) Clifford parallel great circle through each other point of $\mathbb{S}^3$. Thus Clifford parallelism extends a unit tangent vector in some $T_p\mathbb{S}^3$ to a unit vector field on $\mathbb{S}^3$. We call these vector fields *Hopf fields*; they form an $\mathbb{S}^2$ corresponding to the unit 2-sphere of $T_p\mathbb{S}^3$. Moreover, an orthonormal basis X, Y, Z at $T_p\mathbb{S}^3$ extends to an orthonormal frame X, Y, Z of Hopf fields. Whenever we use X, Y, Z in the following we let it stand for a fixed orthonormal frame of Hopf fields.

Given a Hopf field V, we described above that the *Hopf projection* Π_V gives a homeomorphism from the Clifford parallels tangent to V to $\mathbb{S}^2$. Thus we have a homeomorphism from the set of all oriented great circles in $\mathbb{S}^3$ (or the set of oriented two-planes in $\mathbb{R}^4$) to the product $\mathbb{S}^2 \times \mathbb{S}^2$.

1.3.1. *Hopf fields on Clifford tori.* We need an explicit description of the Hopf fields tangent to a *Clifford torus* $T_{\pi/4}(\zeta)$. One field is constant on the torus, whereas the other rotates in the following sense.

Lemma 3. *Let $X \perp Y$ be two Hopf fields, and $\zeta(s)$ be a great circle tangent to Y. If Φ_t is the integral flow of X (with $\Phi_0 = \mathrm{id}$, $d\Phi_t/dt = X$) then $(s,t) \mapsto \Phi_t(\zeta(s))$ parameterizes a*

Clifford torus. Furthermore, $s \mapsto \Phi_t(\zeta(s))$ *is tangent to the Hopf field* $\cos(2t)Y + \sin(2t)Z$ *when* X, Y, Z *are orthonormal.*

To prove this lemma, we use the fact that Hopf fields turn with unit speed with respect to parallel fields, see [G]. The set of left Clifford parallels all belongs to the same Hopf fibration, say X, and we want to express the right Clifford parallels in terms of the left Hopf fields. First of all, the great circles in the two foliations make a right angle on the Clifford torus, and thus the Hopf fields of the right Clifford parallels are orthogonal to X. The right parallels turn with unit speed with respect to parallel fields, and the parallel field again with unit speed with respect to the left Clifford parallels. It follows that the right field turns with doubled unit speed when we express it with left Hopf fields.

2. k-UNDULOIDS WITH COPLANAR ENDS

2.1. **Necksizes of coplanar k-unduloids and sets of k points in $\mathbb{S}^2$.** A symmetric half M^+ of a coplanar k-unduloid of genus g, defined by Theorem 2, is simply connected if and only if $g = 0$. To apply Lawson's theorem we consider this case from now on. Using a contribution by Karcher [Ka] to the conjugate surface construction we have:

Lemma 4. *Let* M *be a coplanar* k*-unduloid of genus* 0*, so that a symmetric half* M^+ *is bounded by* k *geodesic curvature lines. Then there is a Hopf field* X *such that the great circle lines bounding the conjugate cousin* $\widetilde{M^+}$ *are all tangent to* X*. Thus* $\Pi_X(\partial\widetilde{M^+})$ *consists of* k *points in* $\mathbb{S}^2$*.*

To analyse the set of k points in $\mathbb{S}^2$ we want to relate the necksize of an end in the CMC surface to the spherical distance of the projection of its bounding rays.

We first consider *Delaunay unduloids* whose conjugate cousins are called *spherical helicoids.* As we consider halves of surfaces it is essential to describe a *half unduloid*, that is the portion to one side of a symmetry plane containing its axis. An unduloid has geodesic curvature circles of circumference $x \in (0, \pi]$ at its necks and $2\pi - x$ at its bulges. The respective semicircles on a half unduloid have length $x/2$ and $\pi - x/2$. By Lawson's theorem they correspond to great circle arcs of the same length on the isometric piece of the spherical helicoid. These arcs are *perpendiculars* in the sense that they are great circle arcs that meet the bounding two great circle lines perpendicularly as they do on the unduloid. There is a discrete set of such perpendiculars, spaced $\pi/2$ on the great circle lines apart. These statements are also obvious from the explicit parametrization of the helicoids given in [G].

Similarly we want to consider the portion of an end to one side of a symmetry plane containing the axis of the end. To be precise, a *half end* of a k-unduloid of genus g is an open CMC disk bounded by two geodesic curvature rays and a further arc. Here a *ray* is an arc extending in one direction to infinity. The great circle rays on the cousin of a half end have a set of perpendiculars contained in the asymptotic helicoid limit exactly as in the case of Delaunay unduloids (see [GK], Lemma 3):

Lemma 5. *If M is a half end of necksize $x \in (0, \pi]$ then the boundary $\partial\widetilde{M}$ of the cousin half end contains two great circle rays which are Clifford parallel with distance $x/2$. Furthermore, the rays have perpendiculars of length $x/2$ and $\pi - x/2$ contained in the asymptotic limit of $\widetilde{M}$.*

The cylindrical necksize case $x = \pi$ is distinguished from $x < \pi$ as follows. The cylinder is foliated with geodesic curvature circles, and likewise perpendiculars to the boundary rays foliate its spherical helicoid cousin (which is a subset of a Clifford torus). On the other hand, if $x < \pi$ then the discrete set of geodesic curvature circles corresponds to a discrete set of perpendiculars contained in the helicoid cousin. These perpendiculars are unique in $\mathbb{S}^3$ up to the antipodal map. We should note that the flow of the Hopf field X creates an $\mathbb{S}^1$ orbit of isometric perpendiculars; however, from the orbit only the perpendiculars of the lemma are contained in the asymptotic limit.

We can label the ends of a coplanar k-unduloid M in cyclic order by using the fact that the closure of M^+ in $\mathbb{R}^3$ is conformally a disk with k boundary punctures [GK]. We also obtain a cyclic labelling of the curvature lines in $\partial\widetilde{M^+}$ and the points in $\Pi_X(\partial\widetilde{M^+})$. As a result of the lemma we have:

Theorem 6. *Let M^+ be a symmetric half of a k-unduloid of genus 0 with ends of necksizes $x_1, \dots, x_k \in (0, \pi]$. Then there is a Hopf field X such that the projection $\Pi_X(\partial\widetilde{M^+})$ of the conjugate cousin boundary consists of k points in $\mathbb{S}^2$ with consecutive (spherical) distances $x_1, \dots, x_k$.*

2.2. **Marking of necks and the angles of spherical k-gons.** We can join the k points in $\Pi_X(\partial\widetilde{M^+}) \in \mathbb{S}^2$ with minimizing arcs to obtain a k-gon. Unless a pair of points is antipodal, a connecting edge is unique; note that antipodal points represent an end with cylindrical necksize π. We want to determine a geometric meaning of the angles of the k-gon.

Let M be a coplanar k-unduloid of genus 0. Then $\partial\widetilde{M^+}$ consists of great circles lines tangent to X. By Lemma 5 these circles have perpendiculars of length at most $\pi/2$ contained in the asymptotic limit of $\widetilde{M^+}$. We can connect the endpoints of the perpendiculars with arcs contained in the X circles. By taking the complementary arc if necessary we can assume that the connecting arcs run in the $+X$ orientation on the resulting closed polygon. This way we obtain a closed $2k$-gon in $\mathbb{S}^3$ projecting onto a minimizing k-gon connecting $\Pi_X(\partial\widetilde{M^+})$. We call the arcs tangent to X *horizontal* and let their length be $0 < a_1, \dots, a_k \le 2\pi$, so that a_i is in between the perpendiculars of length $x_i/2$ and $x_{i+1}/2$ (indices mod k). The orientation gives the Hopf fields of the perpendiculars a sign.

We let the *Hopf twist* of two Hopf fields V, W with respect to a perpendicular field $X \in \{V, W\}^\perp$ be given by the angle of V and W in the plane $X^\perp$; we want to take the oriented angle in $[0, 2\pi)$, by requiring that X, V, W are positively oriented for angles in $(0, \pi)$. Also, in an oriented k-gon angles are defined modulo 2π.

Proposition 7. *Suppose M is a k-unduloid of genus 0 with coplanar ends of necksizes $0 < x_1, \dots, x_k \leq \pi$ and with angles $0 < \alpha_i < \pi$ between the i-th and $(i+1)$-st end.*
(i) Let Y_i be the Hopf field of the perpendicular contained in the asymptotic limit of the i-th end of M^+. Then the Hopf twist of Y_i and Y_{i+1} (perpendicular to X) is $-\alpha_i$.
(ii) Let u_i be the exterior angle the arcs with length x_i and x_{i+1} make in a minimizing k-gon connecting $\Pi_X(\partial\widetilde{M^+}) \subset \mathbb{S}^2$. Then u_i satisfies

$$u_i = 2a_i - \alpha_i \bmod 2\pi, \tag{3}$$

where a_i is the length of the horizontal arc in the $2k$-gon in $\mathbb{S}^3$.

Proof. (i) This is a consequence of Karcher's observation [Ka] that if the (interior) normal rotates with an angle φ along a curvature arc then the Hopf twist on the corresponding great circle arc of the conjugate cousin is $\varphi - \pi$. Here φ is of the form $\pi - \alpha_i$.
(ii) If two consecutive ends make an angle $\alpha_i = 0$, then on the CMC geodesic curvature arc the normal turns with an angle π according to part (i). Thus the Hopf twist is 0 and the fields Y_i and Y_{i+1} agree. If we suppose $a_i = 0 \bmod 2\pi$ then the arc with field Y_{i+1} straightly extends the arc with field Y_i, and so does its Hopf projection in $\mathbb{S}^2$. In particular, the exterior angle $u_i \bmod 2\pi$ vanishes, and this case is consistent with (3).

To deal with $a_i > 0$, we consider a Clifford torus foliated with X-fields. Under Π_X the torus is mapped to a great circle in $\mathbb{S}^2$. In particular each perpendicular great circle in the Clifford torus, being a right Clifford parallel, projects onto the same great circle. In Lemma 3 we stated the Hopf fields of the right Clifford parallels on a Clifford torus: if $\eta(t)$ is a great circle with X Hopf field, then the perpendicular great circles have the field $\cos 2t\, Y_i + \sin 2t\, \{X, Y_i\}^\perp$. Thus the projection is unchanged (and $u_i = 0$) if the Hopf twist is twice the length on η, or $2a_i = \alpha_i \bmod 2\pi$.

Finally, when the Hopf projection of the two great circle arcs makes an exterior angle u_i we see that for $a_i = 0 \bmod 2\pi$ we have $u_i + \alpha_i = 0 \bmod 2\pi$. In the general case we can add the right rotation as before to obtain (3). □

We would like to remark on a consequence of Lemma 3. The Hopf field X induces a family of isometries $\Phi_t = \exp tX \in \mathsf{SO}(4)$. The set $\partial\widetilde{M^+}$ is fixed under Φ_t for each t, but a great circle with Hopf field Y is mapped onto a circle with Hopf field $\cos 2t\, Y + \sin 2t\, Z$. For any given pair of great circle rays bounding a half end cousin this gives a family of perpendiculars with length $x/2$. Consider the case $x < \pi$. Then a pair of great circles in $\partial\widetilde{M^+}$ tangent to X does not determine the Hopf fields of the perpendiculars. Thus ∂M^+ determines only u_i, but not the angles α_i the ends of M enclose (or, equivalently the set of horizontal lengths, a_i).

3. Triunduloids and spherical triangles

Theorem 6 specifies the sets of necksizes for k-unduloids M with coplanar ends from the k points in $\Pi_X(\partial\widetilde{M^+})$. However, as we asserted in the last paragraph, in general this set

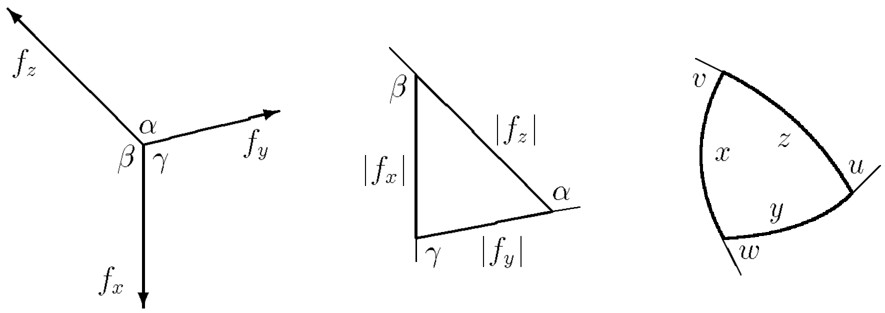

FIGURE 1. (a) The three coplanar force vectors of a triunduloid have vanishing sum. (b) The translated vectors form a triangle. (c) The boundary of a symmetric triunduloid half has the vertices of a spherical triangle as image under the Hopf projection.

does not determine the set of angles enclosed by the axes of the ends. The angles depend on (the Hopf field of) the perpendiculars contained in the asymptotic limit of the spherical minimal surfaces spanned by the boundary $\Pi_X(\partial\widetilde{M^+})$.

For a set of points $\Pi_X(\partial\widetilde{M^+})$, or given necksizes, the balancing formula (2) gives constraints on the angles of the ends of M. We will see below that in case $k = 3$ there are two constraints and these determine two and therefore all three angles completely. Thus we are able to characterize the contours of the triunduloid cousins. Some weaker statements we can make for $k \geq 4$ are stated in Section 5.

The three force vectors of the ends of a triunduloid of genus g sum up to zero. Thus a triunduloid is always coplanar, and Theorem 2 applies to give the following fact.

Proposition 8. *A triunduloid M of genus g has a plane of reflection P which decomposes M into two open symmetric halves $M^\pm$ with $\partial M^+ = \partial M^- = M \cap P$.*

Using torques it can be shown that the asymptotic axes intersect in one point of P [K1].

In the following we denote the angles the three ends of a triunduloid of genus g enclose with α, β, $\gamma = 2\pi - \alpha - \beta$; we let the opposite neck circumferences be x, y, z. As in (1) they determine three force vectors in the plane P by f_x, f_y, f_z, see Fig. 1(a). The force vectors can be combined to a *force triangle* with exterior angles angles α, β, γ, and side lengths the *weights* $|f_x|$, $|f_y|$, $|f_z|$, as in Fig. 1(b). A triunduloid of genus g can have non-degenerate ends, $x, y, z > 0$, only if the angles satisfy $0 < \alpha, \beta, \gamma < \pi$. It is clear from the triangle that the angles are continuous functions of the necksizes. The sine law for the triangle gives two equations relating angles and necksizes

$$\frac{x(2\pi - x)}{\sin\alpha} = \frac{y(2\pi - y)}{\sin\beta} = \frac{z(2\pi - z)}{\sin\gamma}. \tag{4}$$

In the general non-coplanar case (arising for $k \geq 4$) the third components of the force sum in $\mathbb{R}^3$ give a third non-trivial equation.

3.1. **A necksize bound.** From now on we want to assume genus 0. By Proposition 8 the axes of a triunduloid are coplanar and Theorem 6 applies to all triunduloids. We obtain

Theorem 9. *A triunduloid of genus* 0 *has asymptotic necksizes* $0 < x, y, z \leq \pi$ *which satisfy the circumference bound*

$$x + y + z \leq 2\pi, \tag{5}$$

and the three triangle inequalities

$$x \leq y + z, \quad y \leq z + x, \quad z \leq x + y. \tag{6}$$

Proof. By Theorem 6 the set $\Pi_X(\partial\widetilde{M^+})$ consists of three points in $\mathbb{S}^2$, whose pairwise distance is given by the necksizes x, y, z of a triunduloid. Thus we have a minimizing triangle with side lengths x, y, z, for which the inequalities hold (see, e.g. [Be, 18.6.10]). ☐

3.2. **The moduli space of triunduloid contours.** We want to study the topology of the family of cousin contours $\partial\widetilde{M^+}$ for triunduloids M of genus 0. We will always assume that the ends of a triunduloid are *labelled*, i.e. we distinguish the ends. We call the space of triples of non-oriented but labelled great circles in $\mathbb{S}^3$ with Hopf field $\pm X$, modded out by the isometries in $\mathsf{SO}(4)$ which preserve $\pm X$, the space of *triunduloid contours* $\mathcal{G}$. As a topology, we can take the (orbit) Hausdorff distance of the contours; that means the distance of $\Gamma_1, \Gamma_2 \in \mathcal{G}$ is given by $\min\{\mathrm{dist}(\Gamma_1, \Phi(\Gamma_2)) \mid \Phi \in \mathsf{SO}(4)\}$.

We let $\mathcal{T}$ be the set of *triples of (labelled) distinct points in* $\mathbb{S}^2$, modded out by the action $\mathsf{SO}(3)$ by rotations. It will be useful to represent a triple of points $p, q, r \in \mathcal{T}$ in clockwise ordering on a common latitude, and with p on a fixed meridian. Note that $\mathcal{T}$ only identifies triples that differ by a motion in $\mathsf{SO}(3)$, but not in $\mathsf{O}(3)$; so a pair of congruent triples represented in the northern and southern hemisphere is distinguished. The common latitude and the longitudes of q and r give three parameters for $\mathcal{T}$, and $\mathcal{T}$ is a topologically contractable space.

The Hopf projection Π_X induces a map from $\mathcal{G}$ to $\mathcal{T}$. A triple of Clifford distances for $\mathcal{G}$ is invariant under $\mathsf{SO}(4)$ as is a triple of spherical distances under $\mathsf{SO}(3)$. Unless one of the inequalities (5) or (6) holds with equality, the orientation (clockwise or anticlockwise) of a triangle is well-defined, and it is respected by the projection, too. Therefore $\Pi_X : \mathcal{G} \to \mathcal{T}$ is well-defined and bijective, and it is straightforward to see it is continuous.

Theorem 10. *The space* $\mathcal{G}$ *of boundary contours for triunduloids is homeomorphic to the open* 3*-ball* $\mathcal{T}$ *of triples of distinct points in* $\mathbb{S}^2$*. In particular,* $\mathcal{G}$ *is connected and contractable.*

Remark. In Theorem 11 below we will see that two triunduloids can only project to the same triple in $\mathcal{T}$ if the asymptotics of the ends, namely forces and phases, agrees. We believe that this asymptotics determines a CMC surface bounded by a contour in $\mathcal{G}$ uniquely. On the other hand, in the case of $k \geq 4$ this can no longer be expected. A 4-unduloid M with given forces can have a segment consisting of $n \in \mathbb{N}$ bubbles between two triple junctions,

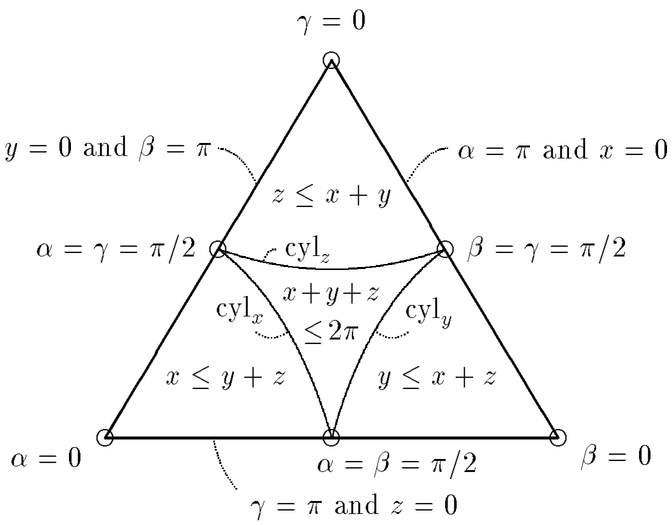

FIGURE 2. The set of admissible angles $0 < \alpha, \beta, \gamma = 2\pi - \alpha - \beta < \pi$ is represented with the interior of the bold triangle. Each point corresponds to a one-parameter family of triunduloid contours, so that the moduli space $\mathcal{G}$ can be represented by a thickened neighbourhood of this triangle taken in $\mathbb{R}^3$. There are four different subregions such that the contour with maximal necksizes in a given one-parameter family takes the indicated inequality with equality. On the three common boundary lines this holds for a pair of inequalities, and the maximal necksize contour corresponds to a triunduloid with one cylindrical end.

see [GP]. The bounding contour $\partial\widetilde{M^+}$, however, is invariant of the number of segment bubbles. Only triunduloids cannot have segments, and are thus the only class of surfaces that can be in bijection to the set of boundaries $\partial\widetilde{M^+}$.

If we do not label the ends, we must identify triples whose sidelengths are a permutation of one another. This corresponds to taking an orbifold in place of $\mathcal{T}$, which can be represented by triangles satisfying $0 < x \leq y \leq z \leq \pi$. The singular orbifold points correspond to surfaces with larger symmetry group: these are the isosceles triunduloids discussed in [GK] if one pair of necksizes is equal, and dihedrally symmetric triunduloids if all three are equal.

4. Geometric properties of triunduloid contours

With Theorem 10 we can determine many geometric properties of the associated boundary contours of triunduloid halves. In the following we want to describe how the explicit formulas for spherical triangles control the geometric parameters for triunduloids of genus 0.

4.1. **Triunduloid contours with fixed angles.** Consider a pair of congruent triangles in $\mathcal{T}$, one represented in the northern, one in the southern hemisphere. It is interesting to see that from balancing (4) it follows that triunduloids projecting onto these two triangles can be distinguished by their asymptotics. It is the spacing of necks relative to two different ends that makes the difference. By Lemma 5 the perpendiculars mark the asymptotic neck position. In Section 2.2 we described how a contour can be truncated with perpendiculars to form a $2k$-gon. For triunduloids we let a, b, c stand for the lengths modulo π of the horizontal arcs in the hexagon. We assume that a, b, c are opposite to x, y, z. These lengths represent the *asymptotic spacing between two necks* on the ends, expressed as a length modulo π on the horizontal curvature arc.

Theorem 11. *For each triple $(\alpha, \beta, \gamma = 2\pi - \alpha - \beta)$ of angles in $(0, \pi)$ there is a continuous one-parameter family of triunduloid contours in $\mathcal{G}$ such that the pairwise ratio of the weights is constant and given by (4). If a necksize triple (x, y, z) satisfies (5) or (6) with equality then there is one contour in the family (or in $\mathcal{G}$). If the inequalities are all strict then there are two different contours in the family (or in $\mathcal{G}$); these contours can only bound triunduloids with different asymptotic neck spacing.*

Proof. For a given triple (α, β, γ) balancing (4) determines weights $x(2\pi - x)$, $y(2\pi - y)$, $z(2\pi - z)$, up to a scaling factor. As we assume $x, y, z \leq \pi$ this gives a unique one-parameter family of triangle lengths (x, y, z). There are two families of congruent triangles with these lengths which coincide in a unique triangle with maximal (x, y, z) contained in the equator ((5) or (6) hold with equality).

To prove the last statement note that according to (3) for one triple the length satisfies $a = (u + \alpha)/2 \bmod \pi$ and for the other $a = (-u + \alpha)/2 \bmod \pi$. These numbers are different as $u \neq 0 \bmod 2\pi$. Similar for the lengths b, c. □

4.2. **The boundary of the moduli space.** We want to consider a compactified set of triples $\bar{\mathcal{T}}$. An obvious compactification of $\mathcal{T}$ would be to take triples of three points in $\mathbb{S}^2$ without requiring them to be distinct any longer. However, we need to distinguish coinciding points according to the direction they approach, i.e. when a limiting triangle contains an edge of length 0 we want to keep a direction for this edge. This is natural since by (3) the angles of the triangle relate to the angles of the ends of the CMC surface. Thus we want to keep this information in the limit.

Formally, we can let $\bar{\mathcal{T}}$ be the set of values $((x, y, z), (u, v, w))$, with $0 \leq x, y, z \leq \pi$ and $u, v, w \bmod 2\pi$, which satisfy besides (5) and (6) the trigonometric law

$$\sin u \sin y \sin z = \sin v \sin x \sin z = \sin w \sin x \sin y.$$

If $x = \pi$ then we must identify all angles $v = w$, and similarly if y or $z = \pi$.

We use the set $\partial\mathcal{T} := \bar{\mathcal{T}} - \mathcal{T}$ to describe the degenerate contours $\partial\mathcal{G}$. Since one necksize must be zero, and by balancing the two nonzero forces must be opposite, the triple of necksizes must be of the form $(x, x, 0)$. We distinguish the case $x = 0$ from $x \neq 0$:

(i) If all three necksizes vanish the spherical triangles are characterized by their euclidean blow-up. In particular balancing is always satisfied. There are two different cases, depending on whether the triangle is a limit of triangles converging to the north or south pole in our previous representation. The limiting surfaces arise with all triples of angles $0 \le \alpha, \beta, \gamma = 2\pi - \alpha - \beta \le \pi$.

(a) If $0 < u, v, w < \pi$ then $u + v + w = 2\pi$, and thus the horizontal lengths satisfy $a + b + c = (\alpha + \beta + \gamma)/2 + (u + v + w)/2 = 2\pi$. In $\mathbb{S}^3$ the perpendiculars must describe a full great circle and so this equation does not only hold modulo π but also modulo 2π. Therefore the corresponding triunduloids have a centre sphere, and we call this the *spheroidal boundary* component.

(b) If $0 > u, v, w > -\pi$ then $a + b + c = 0$ and we have three strings of spheres directly glued to the origin. A suitable blow-up of a limiting sequence of such surfaces converges to a minimal trinoid, and thus we call it the *trinoidal boundary* component.

(*ii*) If $x \neq 0$ the angles are $0 \le \alpha \le \pi$, $\pi - \alpha$, π (or a cyclic permutation). The corresponding surface is a Delaunay unduloid with a string of spheres attached. At the point of attachment the tangent spaces agree. An unduloid has only a limited range of slopes, with the maximum occuring at the inflection point of the generating curve. Thus there is a function $x_{\max}(\alpha)$ such that there are two different surfaces in $\partial\mathcal{G}$ for each necksize $x \in [0, x_{\max})$ and one in the equality case.

4.3. **Surfaces with maximal weights.** For each triple of angles (α, β, γ) Theorem 11 gives a unique contour in $\mathcal{G}$ with maximal necksizes $(x_{\max}, y_{\max}, z_{\max})$. We distinghuish the cases that an inequality (5) or (6) holds with equality. In both cases the triple is contained in a great circle of $\mathbb{S}^2$.

(*i*) When the circumference is maximal, $x_{\max} + y_{\max} + z_{\max} = 2\pi$, we can calculate from (4)

$$x_{\max}(\alpha, \beta, \gamma) = 2\pi \frac{\cos(\alpha/2)\sin(\beta/2)\sin(\gamma/2)}{\sin^2(\alpha/2) + \sin^2(\beta/2) + \sin^2(\gamma/2)}, \tag{7}$$

and there are similar formulas for $y_{\max}$ and $z_{\max}$ obtained by cyclic permutation.

(*ii*) On the other hand, if one of the triangle inequalities, say $x = y + z$, is satisfied then the maximal necksize $x_{\max}$ is such that $2\pi - x_{\max}$ is given by the right hand side of (7), while $y_{\max}$ and $z_{\max}$ keep the expressions from part (*i*), that is they are given by the cyclic permutations of (7).

Note that $(x_{\max}, y_{\max}, z_{\max})$ are continuous, but not differentiable, functions of (α, β, γ).

If no points of the triple in $\mathcal{T}$ are antipodal, then the triangle is unique. In case (*i*) the triangle covers the entire equator. All exterior angles vanish, $u = v = w = 0$, and by (3) the horizontal lengths are exactly the half-angles, $a = \alpha/2$, $b = \beta/2$, $c = \gamma/2$. In case (*ii*) the triangle covers a proper subset of a great circle twice, and the union of the two edges y, z forms x. Thus $u = 0$ but $v = w = \pm\pi$.

4.4. **A one-parameter family of triunduloids with cylindrical ends.** A special case arises when (5) and one of the inequalities (6) hold with equality. If we have the triangle

FIGURE 3. A one-parameter family of triunduloids with one cylindrical end. Cylinders with a perpendicular string of spheres attached form the degenerate limiting surfaces of the family. The pictures indicate half of the family running from the degenerate surface with axes graph $\vdash$ to the Y-shaped (or isosceles) surface, past which it can be continued with mirror images up to the other degenerate $\dashv$.

equality $x = y + z$ it follows from maximal circumference equation $x + y + z = 2\pi$ that in fact we have cylindrical neck circumference

$$x = \pi, \quad \text{and thus} \quad y = \pi - z, \tag{8}$$

so that the two necksizes y and z are complementary. This case is described by a triple of points in the equator containing an antipodal pair of points.

We can calculate the relation of the angles and necksizes explicitly, by using (8) in (4). Taking $0 < y < \pi$ as a parameter we obtain

$$\cos\alpha = \frac{y-\pi}{y+\pi} \cdot \frac{y}{2\pi - y} < 0, \qquad \cos\beta = \frac{y-\pi}{y+\pi} \cdot \frac{\pi + 2y}{\pi} < 0, \tag{9}$$

so that each angle is larger than $\pi/2$. Equivalently, we have

$$y = \frac{\pi}{2}\left(1 \pm \sqrt{\frac{1 + 9\cos\alpha}{1 + \cos\alpha}}\right),$$

where the plus sign holds for $\beta \le \gamma$, and the minus sign for $\beta \ge \gamma$. As a consequence, on the interval $0 < y \le \pi/2$ the function $\alpha(y)$ is monotonic in y with $\alpha(0) = \pi/2$ and $\alpha(\pi/2) = 2\arccos(2/3) \approx 96.4$ degrees. Similarly on $\pi/2 \le y < \pi$ the angle $\alpha(y)$ decreases monotonically to $\pi/2$. The function $\beta(y)$ decreases monotonically from π to $\pi/2$ on the entire interval $y \in (0, \pi)$. It follows that (9) defines a curve $\mathrm{cyl}_x = (\alpha(y), \beta(y))$ in angle space as in Fig. 2. We summarize our observations as follows, cf. Fig. 3.

Proposition 12. *There are continuous curves* $\mathrm{cyl}_x, \mathrm{cyl}_y, \mathrm{cyl}_z$ *of triunduloid contours in* $\mathcal{G}$ *with cylindrical necksize on the end* x, y, *or* z, *respectively. On these curves, the angle opposite the cylinder end attains each number in the interval* $(\pi/2, 2\arccos(2/3)]$ *whereas the*

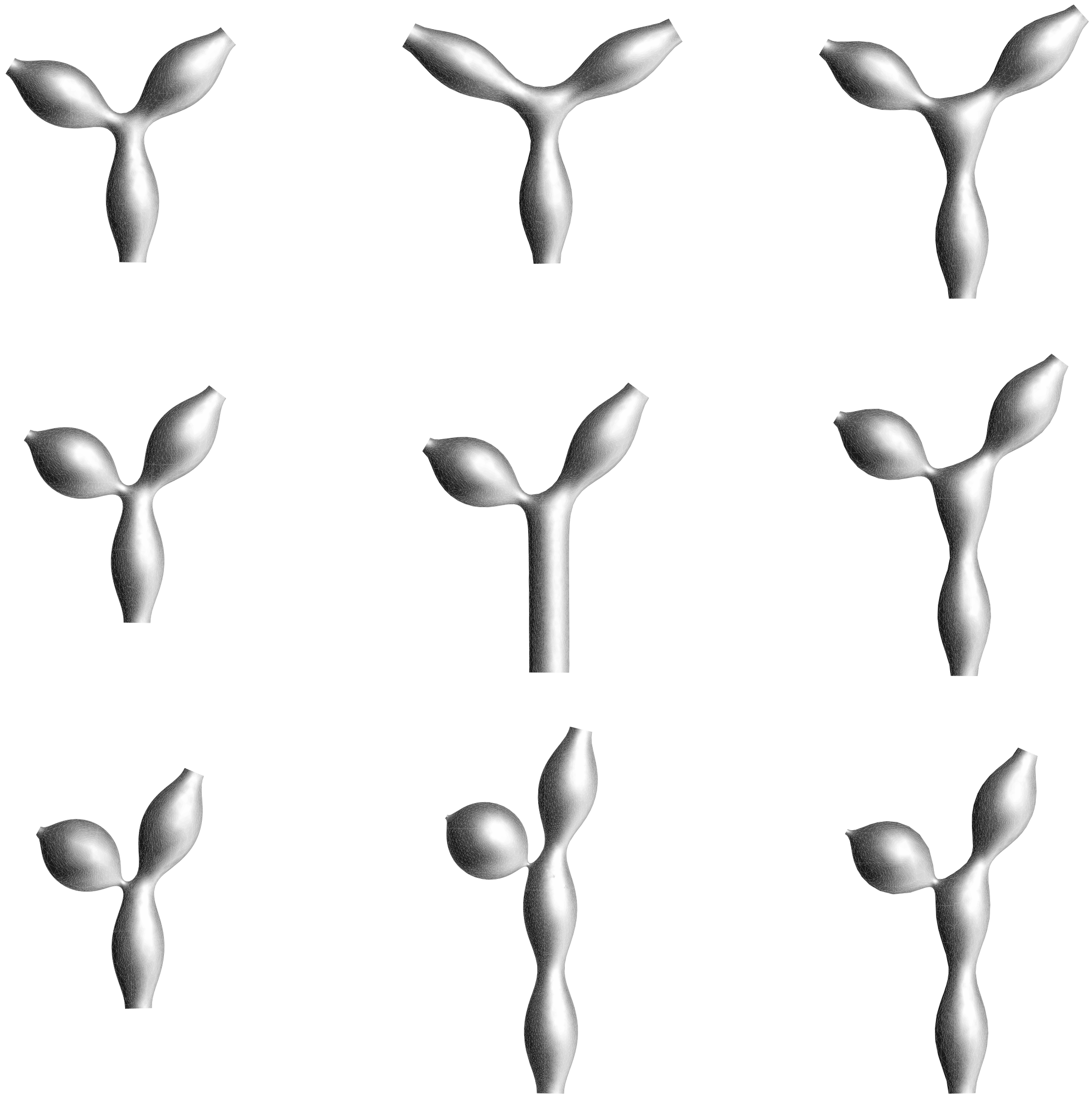

FIGURE 4. A bubble-generating loop. The eight border pictures describe a loop about the triunduloid with cylindrical end shown in the center. All shown triunduloids are truncated at a neck; the choice of truncation is discontinuous between the first two bottom pictures. On the displayed loop the bottom end has necksize $2\pi/3$ and encloses an angle $2\pi/3$ with the left end.

two adjacent angles have the range $(\pi/2, \pi)$. The limiting contours in $\partial\mathcal{G}$ on the endpoints of the paths have two cylindrical ends, and compactify $\mathrm{cyl}_x \cup \mathrm{cyl}_y \cup \mathrm{cyl}_z$ *to a closed loop of contours with one necksize cylindrical.*

4.5. **Generation of bubbles.** The space of triples of points without antipodal points $\mathcal{T}_0 := \mathcal{T} - \mathrm{cyl}_x \cup \mathrm{cyl}_y \cup \mathrm{cyl}_z$ is not simply connected. We denote its universal covering by $\widetilde{\mathcal{T}}$. The horizontal lengths a, b, c are defined modulo π on $\mathcal{T}_0$ and thus lift to real valued functions on $\widetilde{\mathcal{T}}$. This is not true for $\mathcal{T}_0$: Consider a loop η about cyl_x such that two points of the triple are fixed and a third point describes a small circle of radius ε about a point antipodal to the first one. Then one length is constant on the loop, say $x = \pi - \varepsilon$, while the other two lengths vary only slightly depending on the size of ε. Thus the angles α, β, γ determined by balancing (4) from the necksizes x, y, z on the loop vary only slightly, and do not have periods on the loop. On the other hand, the exterior angles v and w at the endpoints of the rotating arc x both increase by 2π on the loop. It follows from (3) that b and c increase by π.

The period in b and c represents a shift of the x perpendicular in $\partial\widetilde{M^+}$ by one period. On the CMC surfaces this means the (asymptotically defined) neck moves by a full period on the loop – the loop creates a bubble as in Fig. 4.

Proposition 13. *Each loop in $\mathcal{T}_0$ with winding number k about* cyl_x *generates k bubbles on the x-end of the contour.*

5. Results for coplanar k-unduloids with $k \geq 4$

Minimizing k-gons satisfy certain trigonometric relations. Nevertheless, for even k each edge can attain length π. Consequently, when $k \geq 4$ we cannot make statements on the necksizes which are as restrictive as Theorem 9 is for $k = 3$. Moreover, we cannot recover the angles of a coplanar k-unduloid M from the points $\Pi_X(\partial\widetilde{M^+})$. Finally, due to the existence of midsegments, a given contour should bound many k-unduloids, and some contours may not bound a CMC surface at all. This makes a description of the space of contours less suggestive for the moduli space of the CMC surfaces.

Only when we impose special symmetries or prescribe most necksizes we can give a meaningful characterization of the corresponding (sub)moduli spaces of boundary contours.

5.1. **$2k$-unduloids of genus 0 with dihedral symmetry of order k.** We consider $2k$-unduloids M with dihedral symmetry of order $k \geq 2$. These surfaces have alternating necksizes x, y, and consecutive ends enclose an angle $\pi/2k$. The case $2k = 4$ are the *rectangular* 4-unduloids we analyse in [GK]. The *necksize-sum* $k(x + y)$ has the same bound as for the maximally symmetric case treated in [G].

Proposition 14. *A $2k$-unduloid of genus 0 with dihedral symmetry of order $k \geq 2$ has necksize-sum at most 2π.*

Proof. The main observation is to see that the $2k$ points in $\Pi_X(\partial\widetilde{M^+})$ have dihedral symmetry of order k in $\mathbb{S}^2$. A discrete rotational symmetry (with axis the normal of some point) of M^+ induces a rotation of $\widetilde{M^+}$ (again about an axis normal to the surface). This comes from the fact that rotationally symmetric data for a surface integrate to a rotationally symmetric surface by Bonnet's theorem. Furthermore, a reflection symmetry leads to a rotation of angle π in $\mathbb{S}^3$ about a great circle arc contained in the surface. In the case of the dihedral symmetry group, the great circle axes are perpendicular to the great circles in $\partial\widetilde{M^+}$, and thus they map to great circles in $\mathbb{S}^2$ under the Hopf projection Π_X. The rotation symmetry in $\mathbb{S}^3$ of the boundary $\partial\widetilde{M^+}$ induces a reflection symmetry in $\mathbb{S}^2$.

We consider the $2k$-gon $\Pi_X(\partial\widetilde{M^+}) \in \mathbb{S}^2$ with dihedral symmetry of order k. The diagonals through opposite vertices all meet in the same point, and the $2k$-gon is tesselated with $2k$ isosceles triangles. Every other triangle has one side equal to x, and a pair of sides with length l which enclose an angle $\alpha < 2\pi/k$. The other set of triangles has one side length y, and again a pair of sides with length l, which enclose a (complementary) angle $2\pi/k - \alpha$. For the first triangle we obtain $x \leq \alpha$, and for the second one $y \leq \pi - \alpha$. Equality is obtained for $l = \pi/2$, that is when the polygon is a subset of the equator. This gives $k(x+y) \leq k(\alpha + 2\pi/k - \alpha) \leq 2\pi$. □

The moduli space of the boundary contours of these surfaces can be conveniently described in terms of the parameters $0 < l < \pi$ and $0 < \alpha < 2\pi/k$ introduced in the proof. It is a homeomorphic to a two-dimensional disk.

5.2. **Cylindrical ends.** Theorem 6 has corollaries such as the following. A coplanar k-unduloid of genus 0 cannot have one necksize $0 < x < \pi$ and all other ends cylindrical. If all but two necksizes are cylindrical then the remaining two necksizes are equal.

The case of coplanar 4-unduloids of genus 0 with cylinder ends may be worth mentioning.

- Four cylinder ends correspond to a countour projecting to two antipodal points in $\mathbb{S}^2$; we do not expect it to bound a connected minimal surface in $\mathbb{S}^3$.
- Exactly three cylindrical ends are impossible.
- If there are precisely two cylindrical ends then the quadruple of points in $\mathbb{S}^2$ contains two antipodes so that the necksizes of the remaining two ends are equal. If the forces of the two cylinder ends are opposite, and so must be the remaining two forces. If not then the pairs of equal necksizes have opposite force sum. Thus in the first case the forces have an extra rotation, and in the second case an extra reflection symmetry. We expect these symmetries of the forces to carry over to the CMC surfaces, so that these 4-unduloids look much like two copies of a triunduloid with one cylindrical necksize, truncated and glued back at one (non-cylindrical) end. The number of bubbles on the segment gives a discrete parameter, and the non-cylindrical necksize gives a continuous parameter of the families.
- If we have just one cylinder end there is a three-parameter family of contours, given by the sets of two points in $\mathbb{S}^2$ plus the antipodal pair. As far as the necksizes are concerned this space of contours can be characterized, and its boundary consists of those triunduloid contours in $\bar{\mathcal{G}}$ which have a cylindrical necksize.

6. Conjectures and computational results

To study the following problems computationally we made use of a computer program written by Oberknapp which uses ideas of Polthier [OP].

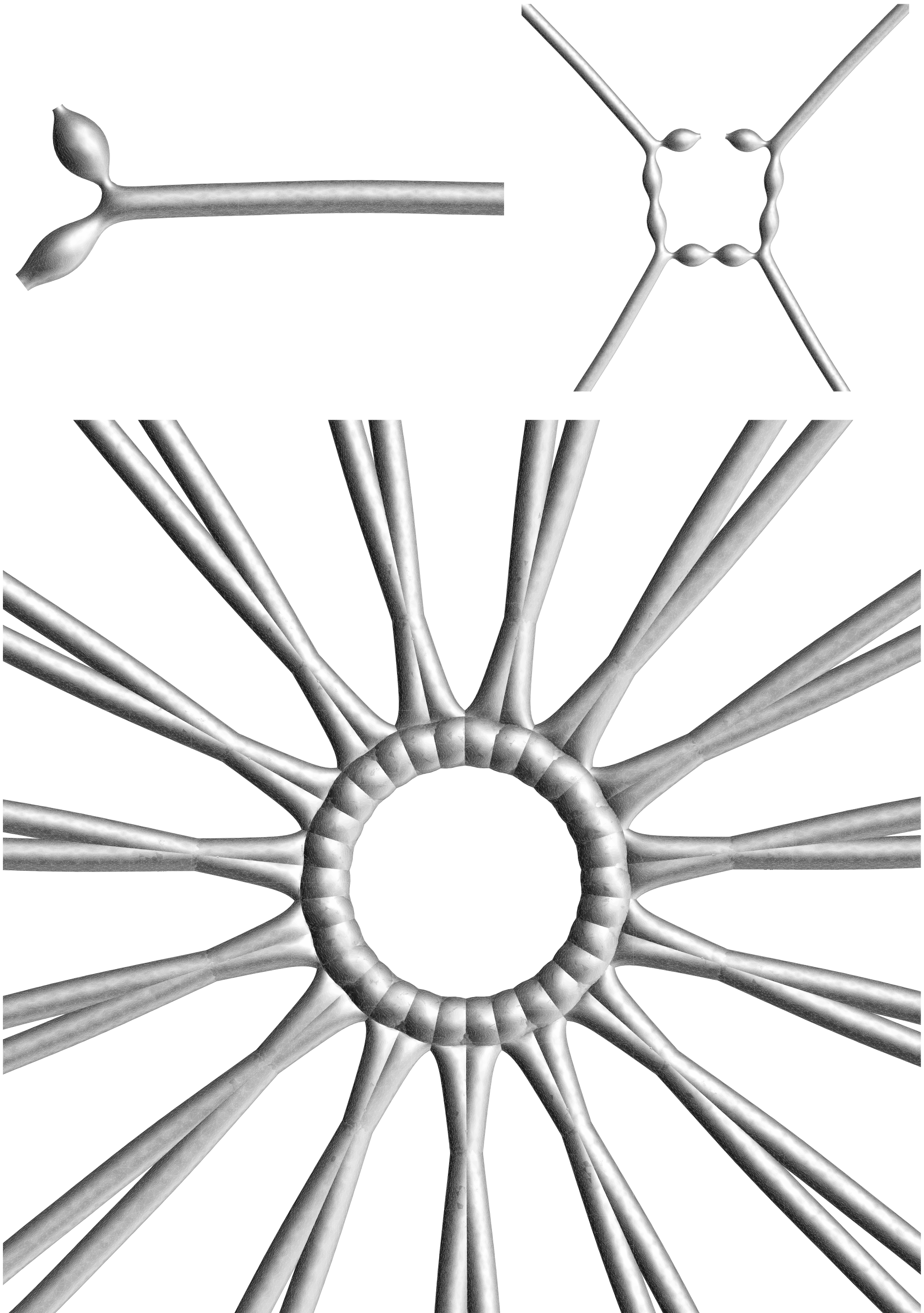

FIGURE 5. A 30-unduloid of genus 1 with cylindrical ends. The fundamental piece with 96° angle opposite the cylindrical end (top left). Four copies leave a 24° gap (top right). Thirty copies close to a complete surface (bottom).

6.1. **Coplanar k-unduloids of genus 1 with cylindrical ends only.** Rick Schoen posed the problem to determine CMC surfaces of finite topology with cylindrical ends only. By the exponentially decaying estimates on the ends given in [KKS] these surfaces can equivalently be characterized as CMC surfaces with finite total curvature.

We have shown that triunduloids do not exist with three cylindrical ends. Similarly, it seems that k-unduloids will not have cylindrical ends only. But for genus 1 we find coplanar k-unduloids with all ends cylindrical and k-fold dihedral symmetry. A half M^+ of these surfaces decomposes into k fundamental pieces, bounded by curvature arcs only. Each fundamental piece is a triunduloid with cylindrical end truncated at the two non-cylindrical ends; i.e. the spherical boundary contour of the cousin is a rightangled pentagon obtained from a cylindrical contour in $\mathcal{G}$ by two truncations.

The following result is illustrated with Fig. 5.

Experimental result 15. *There exists a 30-unduloid of genus 1 with cylindrical ends only, and $k = 30$ is the lowest possible value for a coplanar and dihedrally symmetric k-unduloid of genus 1 with only cylindrical ends. There are countably many larger integers with the same property. Also, for each k the pair of bubbles numbers on the two segments parameterizes countably many such surfaces.*

When we move the truncation of the ends to infinity the truncated surfaces converge to the untruncated ones. Thus for a large number of bubbles the angles agree for the same triple of necksizes. According to our experiments the difference is in fact tiny for any number of bubbles. (For other problems, like the rhombic surfaces considered in [GK] there are significant deviations of the truncated and complete surfaces.)

By Proposition 12 the angle φ opposite a cylindrical end lies in the interval between $90°$ and $2\arccos(2/3) \approx 96.4°$. The dihedral angle of the two truncating planes is $\pi - \varphi$. Hence, if successive reflection leads to a closed surface then there is are integers k, l such that $(\pi - \varphi) \cdot k = 2\pi l$. Here, the interger number of fundamental domains k must be even, as the necksizes alternate, and l gives the turning number of the k-gon one sees in the symmetry plane P when the axes of the ends are deleted.. It is not hard to determine the lowest values for even k when φ is in the given range: With $\varphi = 96°$ we obtain $k = 30$ and $l = 7$; for $\varphi = 95°$ we have $k = 72$ and $l = 17$.

Remark. In the next section we define necksizes for segments. With this definition (8) continues to hold for truncated ends. Thus a coplanar k-unduloid of genus 1 with cylindrical ends only has segments of alternating necksizes $0 < y < \pi$ and $\pi - y$, and k must be even.

6.2. **Connectedness of the moduli spaces for k-unduloids of genus g.** The following description might be helpful to understand the general case. A k-unduloid of genus g decomposes into $k + 2g - 2$ trousers. According to [KK] each trousers is contained in the union of three solid cylinder segments and a large ball. In the coplanar case the trousers all have a plane of symmetry and we can use spherical trigonometry to study them. The

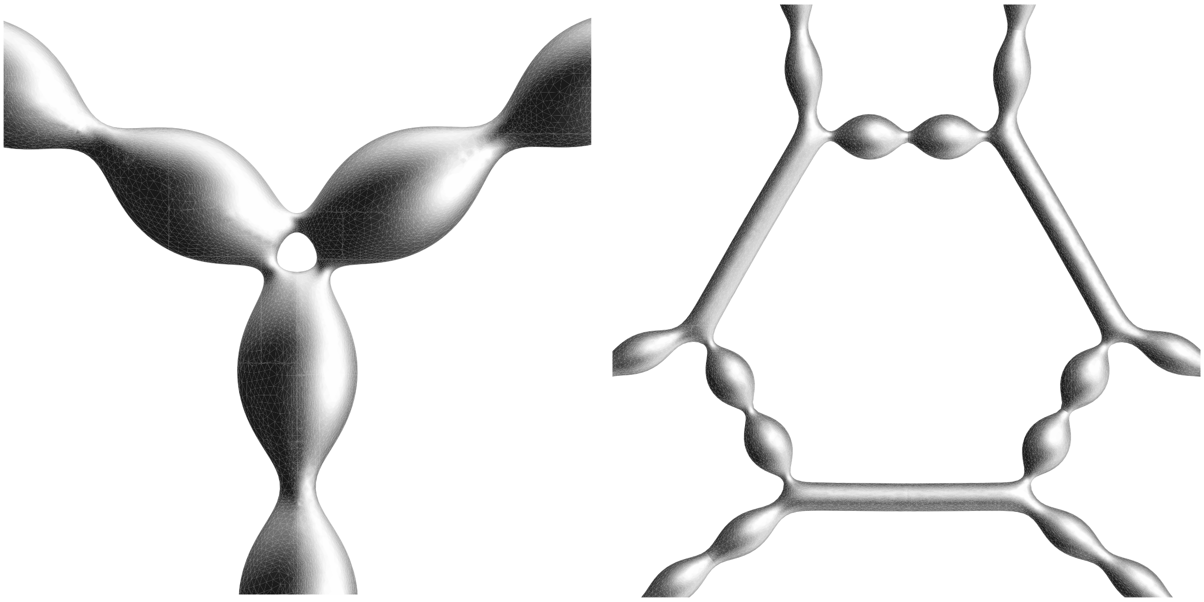

FIGURE 6. A triunduloid and a 6-unduloid of genus 1. While bubbles on the segments cannot be generated for the triunduloid, the cylindrical pieces on the 6-unduloid indicate that on every other segment bubbles can simultaneously be generated.

triunduloids give a prediction for the necksize and angle parameters of the trousers. Let us define the necksize of a segment by the Clifford distance of the bounding great circles of the cousin as we did in Lemma 5 for ends. Then it is not hard to see that the description with triunduloids is accurate for the necksizes. In particular (5) and (6) hold. To make the description more meaningful we also need the angle parameters of the trousers. These will only converge to the respective triunduloid angles when we increase the length of the segments. Another piece of information are the phases of the neck positions; for the triunduloids these are accessible numerically.

Thus, at least for coplanar CMC surfaces with long segments, we have an approximate combinatorial description at hand. In principle this description is sufficient to determine connected components of the respective moduli spaces. For genus 0 no constraints on the angles of the $k-2$ triple junctions arise, and only a pair of necksizes of adjacent junctions must be equal. Therefore the moduli space $\mathcal{M}_{0,k}$ for the k-unduloids of genus 0 is expected to be connected. On the other hand, for non-zero genus the problem is much more involved. We would like to illustrate the problem with two examples of genus one, see Fig. 6.

The axes graph of a triunduloid of genus one is a triangle with rays attached to the triangle vertices. As the triunduloid is coplanar the graph must be planar. If the moduli space $\mathcal{M}_{1,3}$ of the triunduloids of genus 1 were connected, we should be able to create bubbles on each segment (i.e. triangle edge). Then triunduloids of genus 1 with cylindrical midsegments must exist. However, in Proposition 12 we found that the axes angles adjacent to a cylindrical end are always larger than $\pi/2$ (we can justify this result also for the truncated case). As a euclidean triangle cannot have two interior angles larger than $\pi/2$ we conclude that the moduli space of triunduloids of genus 1 is not connected. It is more difficult to determine the triples of bubble numbers which triunduloids of genus 1 can have on their segments. A necessary condition should be a triangle inequality on the number of bubbles induced from the triangle inequalities for the central triangle.

The other example is a 6-unduloid of genus 1. Due to the larger (interior) angles hexagons have compared with triangles, in this case cylindrical segments are possible. Thus bubbles can be generated. However, many involved side conditions must be satisfied, and it is hard to predict which 6-unduloids can be deformed through CMC surfaces to others.

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