CONVERGENCE AND ERROR ESTIMATES OF RELAXATION SCHEMES FOR MULTIDIMENSIONAL CONSERVATION LAWS

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Abstract. We study discrete and semidiscrete relaxation schemes for multidimensional scalar conservation laws. We show convergence of the relaxation schemes to the entropy solution of the conservation law and derive error estimates that exhibit the precise interaction between the relaxation time and the space/time discretization parameters of the schemes.

1. Introduction

In this paper we construct and analyze semidiscrete and fully discrete relaxation schemes for the approximation of the unique global weak solution of the scalar multidimensional conservation law,

$$\begin{cases} \partial_t u + \sum_{i=1}^{N} \partial_{x_i} F_i(u) = 0, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\,, \end{cases} \tag{1.1}$$

satisfying the Kruzhkov entropy conditions [13],

$$\partial_t |u-k| + \sum_{i=1}^{N} \partial_{x_i} \big[\, (F_i(u) - F_i(k))\mathrm{sign}(u-k) \,\big] \leq 0\ , \quad \text{in } \mathcal{D}', \text{ for all } k \in \mathbb{R}. \tag{1.2}$$

Our schemes are based on the relaxation approximation of (1.1) proposed recently by Katsoulakis and Tzavaras [11]:

$$\begin{aligned} \partial_t w^\varepsilon + \sum_{i=1}^{N} \tilde{A}_i \partial_{x_i} w^\varepsilon &= \frac{1}{\varepsilon} \sum_{i=1}^{N} (h_i(w^\varepsilon) - z_i^\varepsilon) \\ \partial_t z_i^\varepsilon - A_i \partial_{x_i} z_i^\varepsilon &= \frac{1}{\varepsilon}(h_i(w^\varepsilon) - z_i^\varepsilon)\,, \qquad i = 1, ..., N. \end{aligned} \tag{1.3}$$

In (1.3), the quantities z_i are convected with velocities $-A_i e_i$ (e_i are the unit coordinate vectors), while the quantity w is convected with velocity $(\tilde{A}_1, \ldots, \tilde{A}_N)$, where $\tilde{A}_i = \omega_i A_i > 0$, $A_i > 0$. The functions $h_i(w)$, $i = 1, \cdots, N$ describing the interaction rates between w and $Z = (z_1, \cdots z_N)$, are smooth and strictly decreasing. Furthermore the system (1.3) is equipped with the conservation law,

$$\partial_t (w^\varepsilon - \sum_{i=1}^{N} z_i^\varepsilon) + \sum_{i=1}^{N} \partial_{x_i} A_i (\omega_i w^\varepsilon + z_i^\varepsilon) = 0\,. \tag{1.4}$$

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It is shown in [11] that, as $\varepsilon \to 0$, the local equilibrium, $z_i = h_i(w)$, $i = 1, \ldots, N$, is enforced and that the limiting dynamics of (1.3) is described by weak entropy solutions of

$$\partial_t (w - \sum_{i=1}^{N} h_i(w)) + \sum_{i=1}^{N} \partial_{x_i} A_i(\omega_i w + h_i(w)) = 0 \,. \tag{1.5}$$

Conversely, for a given conservation law (1.1), one can find functions h_i, $i = 1, \ldots, N$ (h_i decreasing) and positive constants $\omega_i\,, A_i\,, i = 1, \cdots, N$ such that $u^\varepsilon = w^\varepsilon - \sum_{i=1}^{N} h_i(w^\varepsilon)$ will converge as $\varepsilon \to 0$ to the unique solution of (1.1-2), where $(w^\varepsilon, Z^\varepsilon)$ solve (1.3).

The goal of this paper is the study of *relaxation schemes* induced by discretizing (1.3), yielding approximations to the solution of the multidimensional scalar conservation law (1.1), as $\varepsilon \to 0$. We use this problem as a benchmark for understanding stability, convergence and error estimates for such schemes and in particular, the interrelation between the relaxation parameter ε and the space/time discretization parameters. Relaxation schemes to systems of conservation laws were first introduced by Jin and Xin [9], based on relaxation models similar (but not equivalent, except for $N = 1$) to (1.3). One of their principal advantages is their simplicity: due to the linear convection in the relaxation model, relaxation schemes are Riemann solver–free. In addition, semidiscrete relaxation schemes such as the ones studied in this paper turn out to be an important tool in the derivation of macroscopic conservation laws as a hydrodynamic limit of Interacting Particle Systems induced by a kinetic interpretation of (1.3), [12].

Convergence of the schemes. We consider upwind semidiscrete and fully discrete schemes for the discretization of (1.3), cf. Section 3. In the discrete case the stiff nonlinear term is discretized implicitly, [15], [9]. Let h and τ be the space and time discretization parameters and $(W^{h,\varepsilon}, Z^{h,\varepsilon})$, $(W^{\tau,h,\varepsilon}, Z^{\tau,h,\varepsilon})$ piecewise constant interpolations of the semidiscrete and fully discrete approximations of w in (1.3). We also set

$$U^{h,\varepsilon} = W^{h,\varepsilon} - \sum_{i=1}^{N} h_i(W^{h,\varepsilon})\,, \quad U^{\tau,h,\varepsilon} = W^{\tau,h,\varepsilon} - \sum_{i=1}^{N} h_i(W^{\tau,h,\varepsilon})\,.$$

Assuming that the grid in the fully discrete scheme satisfies an appropriate CFL condition, cf. Section 4b, we show that our schemes have the properties:

- they are L^1–contractive and Total Variation Diminishing (TVD).
- If the initial data u_0, $(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon})$ are in $BV(\mathbb{R}^n)$ and additionally the initial approximations satisfy $\sum_{i=1}^{N} ||h_i(w_0^{h,\varepsilon}) - z_{0i}^{h,\varepsilon}||_{L^1} = O(\varepsilon)$, then for $t > 0$,

$$||U^{h,\varepsilon}(t) - u(t,\cdot)||_{L^1(\mathbb{R}^n)} \le ||U^{h,\varepsilon}(0) - u(0,\cdot)||_{L^1(\mathbb{R}^n)} + O(\sqrt{\varepsilon + h})\,. \tag{1.5}$$

In the fully discrete case and under the same assumptions on the initial data we have

$$||U^{\tau,h,\varepsilon}(t) - u(t,\cdot)||_{L^1(\mathbb{R}^n)} \le ||U^{\tau,h,\varepsilon}(0) - u(0,\cdot)||_{L^1(\mathbb{R}^n)} + O(\sqrt{\varepsilon + \tau + h})\,. \tag{1.6}$$

These estimates yield the compactness and convergence properties of both schemes claimed earlier, under elementary hypotheses on the approximating initial data. In (1.5) and (1.6), the O–terms depend on the L^∞- norm and the total variation of the initial data and, on the terminal time T. It is important to note that our error estimate does not interrelate the parameters ε and h, τ in a restrictive way. Note also that the results obtained herein imply also the convergence, and the corresponding error estimate, for the first order schemes of [9] for $N = 1$. Similar error estimates, but of order $O(\varepsilon^{1/3} + \sqrt{\tau})$, were obtained by Schroll, Tveito and Winther [20], for discrete approximations to a one dimensional relaxation model with nonlinear convection arising in chromatography; see also [21] for a similar result in two space dimensions. Stability and convergence properties of splitting schemes obtained by discretizing a 2×2 relaxation approximation of a one dimensional scalar conservation law, were studied in [1].

An approximation theorem. We begin, in Section 2, by considering an approximation theorem which is the discrete analogue of the main estimate in [3]. The method of doubling of variables introduced by Kruzhkov [13] and the ideas of Kuznetsov [14] to apply such a method to numerical approximations, have been extensively used by several authors to obtain error estimates for approximations to the entropy solution of (1.1), cf. e.g. [14], [19], [17]. This technique was further developed in the case of finite volume or finite element approximations [5], [22], [6], see also [7]. In this case, due to the lack of BV bounds for the discrete schemes even the convergence of the approximations is a rather technical task. Kuznetsov's method along with DiPerna's theory [8] are the main available tools to prove convergence in this case. Recently, Bouchut and Perthame [3], see also [2], proposed a compact form for deriving error estimates to conservation laws by revising the approach of [14], [13]. Their theorem can be applied directly without doubling the variables and thus avoiding much of the technical work, needed up to now, to obtain estimates for functions satisfying an approximate entropy inequality, cf. [3]. In section 2 we propose a variation of this result, which is what we call a discrete version of the main theorem in [3]; this is our Theorem 2.1. In the approximation estimate of Theorem 2.1 we explicitly include terms that typically arise in any numerical scheme approximating the scalar conservation law. This result is our main tool in obtaining the estimates (1.5), (1.6) for the semidiscrete and fully discrete relaxation approximations in section 5, cf. Theorems 5.1 and 5.2. The same estimate is also used by Katsaounis and Makridakis [10], to obtain converegence and error estimates to the entropy solution of (1.1), for a finite volume relaxation scheme based on the system (1.3).

The structure of the paper is as follows. We begin, in Section 2, by considering the approximation theorem mentioned above. In section 3 we present the discrete and semi-discrete relaxation schemes.

In section 4 we study the properties of the schemes, in particular we show that under structural conditions on the relaxation functions h_i, both schemes are L^1 contractive with diminishing associated total variation (TVD) and satisfy an entropy condition. In section 5 we prove the convergence results and give the rate of convergence based on the theorem presented in section 2.

2. **Error Bound for Approximating Schemes**

In this section we establish an error estimate for approximating schemes for conservation laws, which is the discrete analogue of the theorem by Bouchut and Perthame [3]. In the following theorem, u_h stands for any function approximating the solution u of (1.1-2).

Theorem 2.1. *Let $u_h, u \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^N))$ be right continuous in t, with values in $L^1_{loc}(\mathbb{R}^N)$. Assume that u is the entropy solution of a given conservation law, i.e., it satisfies* (1.1) *and* (1.2). *Let Ψ a nonegative test function $\Psi \in C^\infty_c([0,\infty) \times \mathbb{R}^N)$ and assume that u_h satisfies,*

$$
\begin{aligned}
&-\iint_{(0,\infty)\times\mathbb{R}^N} \big(|u_h - k|\partial_t\Psi + \operatorname{sign}(u_h - k)[f(u) - f(k)]\cdot\nabla_x\Psi\big)dtdx \\
&\le \iint_{(0,\infty)\times\mathbb{R}^N} \Big(\alpha_K|\Psi| + \alpha_G|\partial_t\Psi| + \sum_j \alpha_H^j|\frac{\partial\Psi}{\partial x_j}| + \sum_{1\le i,\, j\le N} \alpha_L^{ij}|\frac{\partial^2\Psi}{\partial x_i\partial x_j}|\Big)dxdt \\
&\quad + \iint_{(0,\infty)\times\mathbb{R}^N} \Big(\beta_G B_G(\partial_t\Psi) + \sum_j \beta_H^j B_H^j(\frac{\partial\Psi}{\partial x_j}) + \sum_{i,j} \beta_L^{ij} B_L^{ij}(\frac{\partial^2\Psi}{\partial x_i x_j})\Big)dxdt \\
&\quad \textit{for all } k \in \mathbb{R},
\end{aligned}
\tag{2.1}
$$

where $f = (F_1, \dots, F_N)$ and α_G, α_H^j, α_K, α_L^{ij}, β_G, β_H^j, β_L^{ij} are nonnegative k-independent functions in $L^1_{loc}([0,\infty)\times\mathbb{R}^N)$ and

$$
\alpha_G,\ \ \beta_G \in L^\infty_{loc}([0,\infty), L^1_{loc}(\mathbb{R}^N))\,. \tag{2.2}
$$

In addition, the operators B_G, B_H^j, $B_L^{ij} : C^\infty([0,\infty)\times\mathbb{R}^N) \to L^\infty_{loc}([0,\infty)\times\mathbb{R}^N)$ satisfy the properties: For $\Delta, \delta > 0$, let $\mathcal{T}_h = \{K\}$ be an element decomposition of supp g, $g \in C^\infty([0,\infty)\times\mathbb{R}^N)$ into elements K, such that

$$
\begin{aligned}
&\operatorname{diam}(K_t) \le \Delta\,, \quad \textit{if either } B_H^j, \textit{ or } B_L^{ij} \textit{ is not zero, and} \\
&|K_x| \le \delta\,, \quad \textit{if the term } B_G \textit{ is not zero,}
\end{aligned}
\tag{2.3}
$$

where $K_x = \{t : (t,x) \in K\}$ and $K_t = \{x : (t,x) \in K\}$. We assume that for all $(t,x) \in K$, $1 \le i,j \le N$,

$$
\begin{aligned}
|B_G(g)(t,x)| &\le C \sup_{t\in K_x} |g(t,x)| \\
|B_H^j(g)(t,x)| &\le C \sup_{x\in K_t} |g(t,x)| \\
|B_L^{ij}(g)(t,x)| &\le C \sup_{x\in K_t} |g(t,x)|\,,
\end{aligned}
\tag{2.4}
$$

where C is a uniform constant independent of g and the element decomposition $\mathcal{T}_h$.

Then the following estimate holds: for any $T \geq 0$, $x_0 \in \mathbb{R}^N$, $R > 0$, $\Delta > 0$, $\delta > 0$, $\nu \geq 0$, denoting by $M = Lip(f)$, $B_t = B(x_0, R + M(T-t) + \Delta + \nu)$, we have:

$$\begin{aligned}\int_{|x-x_0|<R} |u_h(T,x) - u(T,x)|dx \leq \int_{B_0} |u_h(0,x) - u(0,x)|dx \\ + C(E^t + E^x + E^G + E^H + E^K + E^L + \tilde{E}_G + \tilde{E}_H + \tilde{E}_L)\,.\end{aligned} \tag{2.5}$$

Here

$$\begin{aligned}
E^t &= \sup_{t=0,T\,,0<s-t<\delta} \int_{B_t} |u(s,x) - u(t,x)|dx\;, E^x = \sup_{t=0,T\,,0<|x-y|<\Delta} \int_{B_t} |u(t,y) - u(t,x)|dx \\
E^K &= \iint_{0\leq t\leq T\, x\in B_t} \alpha_K(t,x)dxdt, \quad E^H = \frac{1}{\Delta}\sum_{j=1}^{N} \iint_{0\leq t\leq T\, x\in B_t} \alpha_H^j(t,x)dxdt, \\
E^L &= \frac{1}{\Delta^2} \sum_{1\leq i,j\leq N} \iint_{0\leq t\leq T\,, x\in B_t} \alpha_L^{ij}(t,x)dxdt \\
E^G &= (1 + \frac{T}{\delta} + \frac{MT}{\Delta+\nu}) \sup_{0\leq t\leq 2T} \int_{B_t} \alpha_G(t,x)dx
\end{aligned}$$

and the $\tilde{E}$–terms are the same as the E–terms with $\alpha's$ replaced by $\beta's$.

The Lipschitz hypothesis on the function f can be removed if, for example, it is known that both u and u_h are uniformly bounded. Notice also that if the initial datum u_0 in (1.1) is in $BV(\mathbb{R}^N)$ then

$$E^t \leq M\delta TV(u_0)\,, \quad E^x \leq \Delta TV(u_0)\,,$$

where $TV(v)$ denotes the total variation of a function $v \in BV(\mathbb{R}^N)$. The proof of Theorem 2.1 follows along the lines of the basic theorem in [3]. The novelty here is the explicit inclusion in (2.1) and the bound yielding (2.5), of the error $\beta-$terms that typically arise in any discrete or semidiscrete approximation of the scalar conservation law. The importance of such terms will become more clear in the sequel, when we apply Theorem 2.1 in the relaxation schemes.

Proof. For the sake of completeness we describe the basic steps of the proof, following [1], and then we estimate in detail the new $\beta-$terms.

1. Given two nonnegative functions $\Phi, \zeta \in C_c^\infty((0,\infty)\times\mathbb{R}^n)$ that will be specified later, we set

$$\phi(t,x,s,y) = \Phi(t,x)\zeta(t-s, x-y)\,. \tag{2.6}$$

We now consider (2.1) for $\Psi = \phi(\cdot,\cdot,s,y)$ with fixed $(s,y) \in (0,\infty)\times\mathbb{R}^N$ and $k = u(s,y)$; similarly we also consider the entropy inequality (1.2) for the test function $\Psi = \phi(t,x,\cdot,\cdot)$, fixing $(t,x) \in$

$(0,\infty)\times\mathbb{R}^N$ and picking $k=u_h(t,x)$. We add the two relations, integrate with respect to all variables and using $\partial_t\zeta=-\partial_s\zeta$, $\nabla_x\zeta=-\nabla_y\zeta$, we obtain

(2.7)
$$-\iiiint\Big[|u_h(t,x)-u(s,y)|\partial_t\Phi(t,x)+\mathrm{sign}(u_h(t,x)-u(s,y))[f(u_h(t,x))-f(u(s,y))]\cdot\nabla_x\Phi(t,x)\Big]$$
$$\times\,\zeta(t-s,x-y)dsdtdydx\le R^\alpha+R^\beta\,,$$

where

$$\begin{aligned}R^\alpha=\iiiint &\alpha_K(t,x)|\phi(t,x,s,y)|+\alpha_G(t,x)|\partial_t\phi(t,x,s,y)|\\ &+\sum_j\alpha_H^j(t,x)|\partial_{x_j}\phi(t,x,s,y)|\\ &+\sum_{1\le i,\,j\le N}\alpha_L^{ij}(t,x)|\partial^2_{x_ix_j}\phi(t,x,s,y)|dsdtdydx\end{aligned}$$

and

$$\begin{aligned}R^\beta=\iiiint &\beta_G(t,x)B_G(\partial_t\phi(t,x,s,y))\\ &+\sum_j\beta_H^j(t,x)B_H^j(\partial_{x_j}\phi(t,x,s,y))\\ &+\sum_{1\le i,\,j\le N}\beta_L^{ij}(t,x)B_L^{ij}(\partial^2_{x_ix_j}\phi(t,x,s,y))dsdtdydx\\ &=:R_G^\beta+\sum_j R_H^{\beta,j}+\sum_{1\le i,\,j\le N}R_L^{\beta,(i,j)}\,.\end{aligned}$$

2. Now we select the functions Φ and ζ in (2.6). First, for any positive constants δ and Δ, we define ζ as follows:

$$\zeta(t,x)=\zeta^t(t)\zeta^x(x),\ \ \zeta^t,\zeta^x\in C_c^\infty,\ \ge 0,\ \ \int\zeta^t dt=\int\zeta^x dx=1$$

$$\zeta^t(t)=\frac{1}{\delta}\zeta_1^t(\frac{t}{\delta}),\ \ \mathrm{supp}\zeta_1^t\subset(-1,0),$$

$$\zeta^x(x)=\frac{1}{\Delta^N}\zeta_1^x(\frac{x}{\Delta}),\ \ \mathrm{supp}\zeta_1^x\subset B(0,1/4),$$

where ζ_1^t,ζ_1^x are given smooth functions independent of the partition, and of δ,Δ. The constants in the estimates of β terms, will depend also on

$$C_t^{(j)}=||\partial_t^j\zeta_1^t||_{L^\infty},\ \ C_x^{(j)}=||\zeta_1^x||_{W^{j,\infty}}\,.$$

For $\theta>0$, we define $Y_\theta(t)$ so that $Y_\theta(-\infty)=0$ and $Y_\theta'(t)=\frac{1}{\theta}Y'(\frac{t}{\theta})$, where $Y'\in C_c^\infty,\ \ge 0$ and $\int Y'=1$. We intorduce yet another parameter $\varepsilon>0$ and set $\chi(t)=Y_\varepsilon(t)-Y_\varepsilon(t-T)\ \in C_c^\infty((0,T+\varepsilon)),\ \ge 0$. Finally we define $\psi(t,x)=1-Y_\theta\big(|x-x_0|-R-\Delta/2-M(T-t)\big)\ge 0$. We now set

$$\Phi(t,x)=\chi(t)\psi(t,x)\,.$$

Notice that $\Phi \in C^\infty$ as long as $M\varepsilon \le R + \Delta/2$. Inserting Φ in (2.7) and using the Lipschitz condition on f, we get, cf. [1],

$$-\iiiint |u_h(t,x) - u(s,y)|\chi'(t)\psi(t,x)\zeta(t-s,x-y)dsdtdydx \le R^\alpha + R^\beta\,.$$

Therefore

$$0 \le I + R^t + R^x + R^\alpha + R^\beta\,, \tag{2.8}$$

where

$$\begin{aligned}
I &= \iiiint |u_h(t,x) - u(t,x)|\chi'(t)\psi(t,x)\zeta(t-s,x-y)dsdtdydx \\
R^x &= \iiiint |u_h(t,x) - u(t,y)|\chi'(t)\psi(t,x)\zeta(t-s,x-y)dsdtdydx \\
R^t &= \iiiint |u_h(t,y) - u(s,y)|\chi'(t)\psi(t,x)\zeta(t-s,x-y)dsdtdydx\,.
\end{aligned}$$

Furthermore notice that for θ fixed and ε small,

$$1_{B(x_0,R+M(T-t)+\Delta/2)} \le \psi(t,x) \le 1_{B(x_0,R+M(T-t)+\Delta/2+\theta}\,. \tag{2.9}$$

We now intend to pass to the $\varepsilon \to 0$ limit in (2.8), and with a suitable choice θ, obtain (2.5).

3. Using (2.9), the properties of ζ and the right continuity of v (see [3] for more details), we get,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} I \le \int_{|x-x_0|<R+MT+\Delta/2+\theta} |u_h(0,x) - u(0,x)|dx - \int_{|x-x_0|<R+\Delta/2} |u_h(T,x) - u(T,x)|dx \\
&\limsup_{\varepsilon\to 0} R^t \le 2E^t, \quad \limsup_{\varepsilon\to 0} R^x \le 2E^x\,,
\end{aligned}$$

provided we choose $\theta = \Delta/4 + \nu$. We turn now to the terms R^α and R^β. First we have the following bounds for Φ,

$$\begin{aligned}
&|\Phi(t,x)| \le C, \quad |\nabla_x\Phi(t,x)| \le \frac{C}{\theta}, \\
&|\partial_t\Phi(t,x)| \le |\chi'(t)| + C\frac{M}{\theta}, \quad |\partial_{x_ix_j}\Phi| \le \frac{C}{\theta^2}\,,
\end{aligned} \tag{2.10}$$

and let

$$\begin{aligned}
\Omega &= \{(t,x) : 0 \le t < T+\varepsilon, |x-x_0| < R + M(T-t) + \Delta/2 + \theta\}\,, \\
\Omega_t &= \{x : |x-x_0| < R + M(T-t) + \Delta/2 + \theta\}\,.
\end{aligned}$$

Then (2.10) and the definition of ζ imply

$$\begin{aligned}
R^\alpha \le \iint_\Omega \Big[&\alpha_K(t,x) + C\Big(\frac{1}{\theta} + \frac{1}{\Delta}\Big)\sum_j \alpha_H^j(t,x) + \frac{C}{\delta}\alpha_G(t,x) + \\
&C\Big(\frac{1}{\Delta^2} + \frac{1}{\theta\Delta} + \frac{1}{\theta^2}\Big)\sum_{ij} \alpha_L^{ij}(t,x)\Big]dtdx + C\big(1 + M(T+\varepsilon)/\theta\big)\sup_{0<t<T+\varepsilon}\int_{\Omega_t} \alpha_G(t,x)dx\,.
\end{aligned}$$

Keeping in mind that $\theta = \Delta/4 + \nu$, we get

$$\limsup_{\varepsilon\to 0} R^\alpha \le C(E^K + E^G + E^H + E^L)\,. \tag{2.11}$$

4. Finally we estimate the term R^β. Let $\mathcal{T}_h = \{K\}$ a decomposition of the support of Ψ, then $\mathcal{T}_{h,t} = \{K_t\}$ is a partition of the space domain.

To estimate the second term of R^β we note that (2.3) and supp$\Psi \subset \Omega$ imply

$$\begin{aligned} R_H^{\beta,j} \le & C \iiint \int \sum_{K_t\in\mathcal{T}_{h,t}} \int_{K_t} \beta_H^j(t,x)\Big(\sup_x |(\partial_{x_j}\zeta(t-s,x-y))\,\Phi(t,x)| \\ & + \sup_x |\zeta(t-s,x-y)\,\partial_{x_j}\Phi(t,x)|\Big)dxdtdyds =: I_H^{1,j} + I_H^{2,j}. \end{aligned}$$

Then, (2.9-10) yield

$$I_H^{1,j} \le C\iiint \int \sum_{K_t\in\mathcal{T}_{h,t}} \int_{K_t} 1_\Omega(t,x)\beta_H^j(t,x)\sup_x |\partial_{x_j}\zeta^x(x-y)|\,\zeta^t(t-s)dxdtdyds\,.$$

Since supp$\partial_j\zeta^x \subset B(0,\Delta/4)$, then if $|x-y| > \Delta/4$ we have $\partial_j\zeta^x(x-y) = 0$. Therefore if $K_{t,\Delta} = \{y \in \mathbb{R}^N : \mathrm{dist}(y,K_t) \le \Delta/4\}$ then for $x \in K_t, y \in (K_{t,\Delta})^C$ we will have that $\partial_{x_j}\zeta^x(x-y) = 0$, whence $\sup_{x\in K_t, y\in(K_{t,\Delta})^C} |\partial_{x_j}\zeta^x(x-y)| = 0$. Hence,

$$\begin{aligned} &\int\int_{K_t}\int_{\mathbb{R}^N} 1_\Omega(t,x)\beta_H^j(t,x)\sup_x|\partial_{x_j}\zeta^x(x-y)|\,\zeta^t(t-s)dydxds \\ &\le \int_{K_t} 1_\Omega(t,x)\beta_H^j(t,x)\int_{K_{t,\Delta}} C_x^{(1)}\frac{1}{\Delta^{N+1}}dydx \\ &\le C\,C_x^{(1)}\int_{K_t} 1_\Omega(t,x)\beta_H^j(t,x)|K_{t,\Delta}|\frac{1}{\Delta^{N+1}}dydx. \end{aligned}$$

Now

$$|K_{t,\Delta}| \le C\,(\mathrm{diam}(K_t)^N + \Delta^N),$$

therefore, by (2.3),

$$\begin{aligned} I_H^{1,j} &\le C\frac{1}{\Delta}\int \sum_{K_t\in\mathcal{T}_{h,t}} \frac{\mathrm{diam}\ (K_t)}{\Delta}\int_{K_t} 1_\Omega(t,x)\beta_H^j(t,x)dxdt \\ &\le C\,\frac{1}{\Delta}\iint_\Omega \beta_H^j dxdt. \end{aligned}$$

Similarly using $(2.3-4)$, (2.10) and the fact that supp$|\partial_{x_j}\Phi| \subset \Omega$, we have,

$$\begin{aligned} I_H^{2,j} &\le \iiint \int \sum_{K_t\in\mathcal{T}_{h,t}} \int_{K_t} 1_\Omega(t,x)\beta_H^j(t,x)\frac{1}{\theta}\sup_x|\zeta(t-s,x-y)|dxdtdyds \\ &\le C\frac{1}{\theta}\int \sum_{K_t\in\mathcal{T}_{h,t}} \frac{\mathrm{diam}(K)}{\Delta}\int_{K_t} C_x^{(0)} 1_\Omega(t,x)\beta_H^j(t,x)dxdt \\ &\le \frac{C}{\theta}\iint_\Omega \beta_H^j dxdt. \end{aligned}$$

For the terms with the time derivatives we first observe that for each fixed x, $\mathcal{T}_{h,x} = \{K_x\}$ defines a decomposition of the time domain. Therefore

$$
\begin{aligned}
&\iint \iint \beta_G |B_G(\partial_t \phi)| dxdtdyds \\
&\quad \le C \iint \int_{\mathbb{R}^N} \sum_{K_x \in \mathcal{T}_{h,x}} \int_{K_x} 1_\Omega(t,x) \beta_G(x,t) \Big[\sup_t |\Phi(t,x) \partial_t \zeta(t-s, x-y)| \\
&\qquad + \sup_t |\partial_t \Phi(t,x) \zeta(t-s,x-y)| \Big] dxdtdyds =: I_G^1 + I_G^2.
\end{aligned}
$$

Now:

$$
\begin{aligned}
I_G^1 &\le \iint \int_{\mathbb{R}^N} \sum_{K_x \in \mathcal{T}_{h,x}} \int_{K_x} 1_\Omega(t,x) \beta_G(x,t) \sup_t |\Phi(t,x)| \sup_t |\partial_t \zeta(t-s,x-y)| dtdxdyds \\
&\le C \iint \int_{\mathbb{R}^N} \sum_{K_x \in \mathcal{T}_{h,x}} \int_{K_x} 1_\Omega(t,x) \beta_G(x,t) \sup_t |\partial_t \zeta^t(t-s)| \zeta^x(x-y) dtdxdyds.
\end{aligned}
$$

Denoting by $K_x^\delta = \{t : t \in K_x + \delta\}$ we observe that since $\operatorname{supp} \partial_t \zeta^t \subset (-\delta, 0)$, if $t \in K_x, s \in (K_x^\delta)^C$, then $\partial_t \zeta^t(t-s) = 0$. Therefore,

$$
\begin{aligned}
&\iint \int_{\mathbb{R}^N} \int_{K_x} 1_\Omega(t,x) \beta_G(t,x) \sup_t |\partial_t \zeta^t(t-s)| \zeta^x(x-y) dtdxdsdy \\
&\quad \le C \int_{\mathbb{R}^N} \int_{K_x} 1_\Omega(t,x) \beta_G(t,x) \int_{K_x^\delta} C_t^{(1)} \frac{1}{\delta^2} dsdtdx \\
&\quad \le \frac{C}{\delta} \int_{\mathbb{R}^N} \int_{K_x} 1_\Omega(t,x) \beta_G(t,x) C_t^{(1)} \frac{|K_x^\delta|}{\delta} dtdx \\
&\quad \le \frac{C}{\delta} \int_{\mathbb{R}^N} \int_{K_x} 1_\Omega(t,x) (1 + \frac{|K_x|}{\delta}) \beta_G(t,x) dtdx.
\end{aligned}
$$

Using (2.3) we obtain:

$$
I_G^1 \le \frac{C}{\delta} \iint_\Omega \beta_G(t,x) dtdx.
$$

We now estimate I_G^2 using the inequality $|\partial_t \Phi(t,x)| \le |\chi'(t)| + C\frac{M}{\theta}$ (cf. (2.10)):

$$
\begin{aligned}
I_G^2 &= \iint \int_{\mathbb{R}^N} \sum_{K_x \in \mathcal{T}_{h,x}} \int_{K_x} 1_\Omega(t,x) \beta_G(t,x) \sup_t |\partial_t \Phi(t,x)| \sup_t \zeta^t(t-s) \zeta^x(x-y) dtdxdsdy \\
&\le \int \int_{\mathbb{R}^N} \sum_{K_x \in \mathcal{T}_{h,x}} \int_{K_x} 1_\Omega(t,x) \beta_G(t,x) \sup_t |\partial_t \Phi(t,x)| \sup_t \zeta^t(t-s) dtdxds \\
&\le C \iint 1_\Omega(t,x) \beta_G(t,x) \frac{M}{\theta} \sup_t \zeta^t(t-s) dtdsdx \\
&+ \int_{\Omega_x} \int \sum_{K_x \in \mathcal{T}_{h,x}} \int_{K_x \cap \big((0,\varepsilon) \cup (T, T+\varepsilon)\big)} 1_\Omega(t,x) \beta_G(t,x) |\chi'(t)| \sup_t \zeta^t(t-s) dtdsdx.
\end{aligned}
$$

The first term is bounded as before by

$$
C\frac{M}{\theta} \iint_\Omega \beta_G(t,x) dtdx \le C\frac{M}{\theta}(T+\varepsilon) \sup_{0<t<T+\varepsilon} \int_{|x-x_0|<R+M(T-t)+\Delta/2+\theta} \beta_G(t,x) dx.
$$

To estimate the second term recall also that $\chi(t) = Y_\varepsilon(t) - Y_\varepsilon(t-T)$ hence $\mathrm{supp}|\chi'| \subset (0,\varepsilon) \cup (T, T+\varepsilon)$ and $|\chi'| \leq \frac{C}{\varepsilon}$. Then using similar arguments as above, we bound this term by

$$\begin{aligned}
&C \int_{\mathbb{R}^N} \sum_{K_x} \int_{K_x \cap \big((0,\varepsilon)\cup(T,T+\varepsilon)\big)} (1 + \frac{|K_x|}{\delta}) 1_\Omega(t,x) \beta_G(t,x) |\chi'(t)| dt dx \\
&\leq C \int_{\mathbb{R}^N} \sum_{K_x} \int_{K_x \cap \big((0,\varepsilon)\cup(T,T+\varepsilon)\big)} 1_\Omega(t,x) \beta_G(t,x) \frac{C}{\varepsilon} dt dx \\
&\leq C \sup_{T \leq t \leq T+\varepsilon,\, 0 \leq t \leq \varepsilon} \int_{\Omega_t} \beta_G(t,x) dx.
\end{aligned}$$

The above estimates yield the corresponding $\tilde{E}$ terms by using that $\theta = \Delta/4 + \nu$. Finally, we estimate $R_L^{\beta,(i,j)}$ employing similar arguments as in the estimate of $R_H^{\beta,j}$, and the bound of the second derivatives of Φ in (2.10). We omit the details. $\square$

Remark 2.1 The results of Theorem 2.1 hold if we replace assumptions (2.3-4) by

$$\mathrm{diam}(K) \leq \Delta\,, \quad \Delta = \delta \tag{2.3$'$}$$

and

$$\|B(g)\|_{L^\infty(K)} \leq C \|g\|_{L^\infty(K)}, \tag{2.4$'$}$$

where B stands for any of B_G, B_H^j or B_L^{ij}. These assumptions are probably better suited in the case of space-time partitions.

3. Relaxation Approximations

In this section we present the schemes for the relaxation system that will be analysed in the sequel. First we discuss the relation between the conservation law (1.1) and the relaxation system (1.3) and in particular whether any given multidimensional, scalar conservation law can be realized as a zero-relaxation limit of solutions to (1.3). In view of the relation between (1.3) and (1.5), the question is rephrased whether (1.1) can be transformed to the form (1.5), with the functions h_i, describing the curve of local equilibria, being strictly decreasing. It turns out (see [11], Lemma 4.1) that it is possible to construct such functions $h_i : \mathbb{R} \to \mathbb{R}$, with the properties $\frac{dh_i}{dw} < 0$, $h_i(0) = 0$ and $\lim_{w \to \pm\infty} h_i(w) = \mp\infty$, whenever $\omega_i\,, A_i > 0\,, i = 1, \cdots, N$ are selected so that the fluxes $F_i(u)$ satisfy the conditions

$$\begin{aligned}
&1 + \sum_i \frac{1}{A_i} \frac{dF_i}{du} > 0\,, \\
&-1 + \omega + c < \frac{1}{\frac{1}{1+\omega}\big(1 + \sum_i \frac{1}{A_i}\frac{dF_i}{du}\big)} \frac{1}{A_i} \frac{dF_i}{du} < \omega_i\,, \qquad \text{for } u \in \mathbb{R}\,.
\end{aligned} \tag{3.1}$$

Here $\omega = \sum_i \omega_i$ and c a positive constant. Note that the second equation implies

$$\sum_i \frac{1}{A_i} \frac{dF_i}{du} < \omega \,,$$

so (3.1) is a multi-dimensional analogue of the subcharacteristic condition (cf. Liu [16], Chen, Levermore, Liu [4]). In addition, the constructed functions h_i have the property

$$1 - \sum_i |\frac{dh_i}{dw}| > c \,, \qquad \text{for } w \in \mathbb{R} \,. \tag{3.2}$$

c being the constant of (3.1). This property is essential for the converegence of the relaxation schemes to the conservation law because it provides an estimate on the distance $\sum_i ||h_i(w) - z_i||_{L^1}$ of the solution (w, Z) of (1.3) from the line of equilibria

$$\{(w, Z) \in \mathbb{R} \times \mathbb{R}^N : h_i(w) = z_i\} \,. \tag{3.3}$$

Notice also that since we are dealing with bounded solutions of (1.1) and (1.3), properties (3.1) and (3.2) need not hold for all $u, w \in \mathbb{R}$ but only for a bounded interval where the solutions lie. For now on we assume that we are given the functions h_i satisfying the above properties and for notational convenience we let

$$G_i(w, z_i) := h_i(w) - z_i.$$

3.a Semi-discrete relaxation schemes. For a space discretization parameter $h > 0$ and $\varepsilon > 0$, we consider approximations of the solution (w, Z) of the system (1.3), $w_q(t) \cong w(t, hq)$, $z_{i,q} \cong z_i(t, hq)$, $q \in \mathbb{Z}^N$, defined by:

$$\begin{aligned} &\partial_t w_q + \frac{1}{h} \sum_i \tilde{A}_i (w_q(t) - w_{q-e_i}(t)) = \frac{1}{\varepsilon} \sum_i G_i(w_q(t), z_{i,q}(t)) \\ &\partial_t z_{i,q} - \frac{1}{h} A_i (z_{i,q+e_i}(t) - z_{i,q}(t)) = \frac{1}{\varepsilon} G_i(w_q(t), z_{i,q}(t)) \,, \; i = 1, \cdots, N, \end{aligned} \tag{3.4}$$

with given initial approximations $w_q(0), z_{i,q}(0)$. In addition, $A_i > 0$, $\tilde{A}_i = A_i \omega_i > 0$, cf. §1, and e_i is the is the x_i-unit coordinate vector. This semi-discrete scheme was considered in [11], and in [12] an $l^1(\mathbb{Z}^N)$–contraction and a TVD property were shown.

3.b Discrete relaxation schemes. In addition, let $\tau > 0$ be the time discretization parameter. Then the fully discrete relaxation scheme is defined by

$$\begin{aligned} &w_q^{n+1} - w_q^n + \frac{\tau}{h} \sum_i \tilde{A}_i (w_q^n - w_{q-e_i}^n) = \frac{\tau}{\varepsilon} \sum_i G_i(w_q^{n+1}, z_{i,q}^{n+1}) \\ &z_{i,q}^{n+1} - z_{i,q}^n - \frac{\tau}{h} A_i (z_{i,q+e_i}^n - z_{i,q}^n) = \frac{\tau}{\varepsilon} G_i(w_q^{n+1}, z_{i,q}^{n+1}), \; i = 1, \cdots, N. \end{aligned} \tag{3.5}$$

Here $w_q^n \cong w(hq, n\tau)$ and $z_{i,q}^n \cong z_i(hq, n\tau)$, and $w_q^0, z_{i,q}^0$ are given approximations of the initial data. In the sequel we will use the notation (W, Z) to denote the set of values of the schemes (3.4), (3.5) on all grid points and typically supress the dependence on the parameters ε, h, τ, unless necessary.

4. **Properties of the schemes**

4.a Semi-discrete Scheme. The first proposition shows that the semi-discrete scheme is l^1-contractive and Total Variation Diminishing (TVD); the total variation of a function $v : \mathbb{Z}^N \mapsto \mathbb{R}$ is defined as:

$$TV(v) = \sum_{i=1}^{N} TV^i(v) = \sum_{i=1}^{N} \sum_{q \in \mathbb{Z}^N} h^{N-1} |v_{q+e_i} - v_q| \,.$$

Proposition 4.1. (L^1-contraction, TVD property) *Let* (W, Z), $(\bar{W}, \bar{Z})$ *be two solutions of* (3.4), *with corresponding initial data* (W_0, Z_0), $(\bar{W}_0, \bar{Z}_0)$ *such that* $W_0, Z_0, \bar{W}_0, \bar{Z}_0 \in l^1(\mathbb{Z}^N) \cap l^\infty(\mathbb{Z}^N)$. *Then, for all* $t > 0, \varepsilon > 0$,

$$\sum_{q \in \mathbb{Z}^N} \Big(|w_q(t) - \bar{w}_q(t)| + \sum_i |z_{i,q}(t) - \bar{z}_{i,q}(t)| \Big) \\ \leq \sum_{q \in \mathbb{Z}^N} \Big(|w_q(0) - \bar{w}_q(0)| + \sum_i |z_{i,q}(0) - \bar{z}_{i,q}(0)| \Big) \,.$$

Furthermore the semidiscrete scheme (3.4) *is TVD:*

$$\sum_{q \in \mathbb{Z}^N} \Big(|w_{q+e_k}(t) - w_q(t)| + \sum_i |z_{i,q+e_k}(t) - z_{i,q}(t)| \Big) \\ \leq \sum_{q \in \mathbb{Z}^N} \Big(|w_{q+e_k}(0) - w_q(0)| + \sum_i |z_{i,q+e_k}(0) - z_{i,q}(0)| \Big) \,,$$

for $t > 0$ *and all directions* e_k, $k = 1, ..., N$.

Proof. The essential ingredient of this proof is the monotonicity of G_i. To get an error estimate equation we first subtract the corresponding equations for (W, Z) and $(\overline{W}, \overline{Z})$. Then we multiply the equation for $w_q - \bar{w}_q$ by $\operatorname{sign}(w_q(t) - \bar{w}_q(t))$ and that for $z_{i,q} - \bar{z}_{i,q}$ by $\operatorname{sign}(z_{i,q}(t) - \bar{z}_{i,q}(t))$ and we add them up to obtain:

$$\partial_t (|w_q - \bar{w}_q| + \sum_i |z_{i,q} - \bar{z}_{i,q}|) + \frac{1}{h} \sum_i A_i (\omega_i |w_q - \bar{w}_q| + |z_{i,q} - \bar{z}_{i,q}|) \\ - \frac{1}{h} \sum_i A_i \Big[\omega_i \operatorname{sign}(w_q - \bar{w}_q)(w_{q-e_i} - \bar{w}_{q-e_i}) + \operatorname{sign}(z_{i,q} - \bar{z}_{i,q})(z_{i,q+e_i} - \bar{z}_{i,q+e_i}) \Big] \\ = \frac{1}{\varepsilon} \sum_i \Big[G_i(w_q(t), z_{i,q}(t)) - G_i(\bar{w}_q(t), \bar{z}_{i,q}(t)) \Big] \Big(\operatorname{sign}(w_q(t) - \bar{w}_q(t)) + \operatorname{sign}(z_{i,q}(t) - \bar{z}_{i,q}(t)) \Big) \leq 0 \,.$$

The l^1-contraction property follows from the above inequality by summing over q, provided we show first that $(W(t), Z(t))$, $(\bar{W}(t), \bar{Z}(t)) \in l^1$ for any $t > 0$. For the details see [12]. The TVD estimate property follows by the translation invariance property of the scheme. □

We next prove a comparison principle which implies the discrete entropy inequality and an l^∞ bound:

Proposition 4.2. *Under the assumptions of Proposition* 4.1 *we have: For all $t > 0, \varepsilon > 0$,*

(i)
$$\partial_t\Big((w_q - \bar{w}_q)^+ + \sum_i (z_{i,q} - \bar{z}_{i,q})^-\Big) + \frac{1}{h}\sum_i \tilde{A}_i\Big[(w_q - \bar{w}_q)^+ - (w_{q-e_i} - \bar{w}_{q-e_i})^+\Big]$$
$$- \frac{1}{h}\sum_i A_i\Big[(z_{i,q+e_i} - \bar{z}_{i,q+e_i})^- - (z_{i,q} - \bar{z}_{i,q})^-\Big] \le 0,$$

(ii)
$$\sum_{q\in\mathbb{Z}^N}\Big[(w_q(t) - \bar{w}_q(t))^+ + \sum_i (z_{i,q}(t) - \bar{z}_{i,q}(t))^-\Big] \le \sum_{q\in\mathbb{Z}^N}\Big[(w_q^0 - \bar{w}_q^0)^+ + \sum_i (z_{i,q}^0 - \bar{z}_{i,q}^0)^-\Big],$$

(iii) *If for some $a < b$ we have, $a \le w_q(0) \le b,\ h_i(b) \le z_{i,q}(0) \le h_i(a),\ i = 1, \cdots, N, q \in \mathbb{Z}^N$, then*

$$a \le w_q(t) \le b,\ h_i(b) \le z_{i,q}(t) \le h_i(a),\ q \in \mathbb{Z}^N,\ i = 1, \cdots, N,$$

i.e. the region $\mathcal{R}^{a,b} = [a,b] \times \prod_{i=1}^N [h_i(b), h_i(a)]$ is positively invariant.

Proof. Let $\chi_{g>0}$ stand for the characteristic function of the set $\{s : g(s) > 0\}$ where g is an arbitrary function and set $\eta_q = (w_q - \bar{w}_q)$ and $\zeta_{i,q} = z_{i,q} - \bar{z}_{i,q}$. Then multiplying the error equations for η_q, $\zeta_{i,q}$ by $\chi_{\eta_q>0}$, $-\chi_{\zeta_{i,q}<0}$ respectively and summing over i we get:

$$\partial_t(\eta_q^+ + \sum_i \zeta_{i,q}^-) + \frac{1}{h}\sum_i \tilde{A}_i(\eta_q - \eta_{q-e_i})\chi_{\eta_q>0} + \frac{1}{h}\sum_i A_i(\zeta_{i,q+e_i} - \zeta_{i,q})\chi_{\zeta_{i,q}<0}$$
$$= \frac{1}{\varepsilon}\sum_i \Big[G_i(w_q, z_{i,q}) - G_i(\bar{w}_q, \bar{z}_{i,q})\Big]\Big[\chi_{\eta_q>0} - \chi_{\zeta_{i,q}<0}\Big].$$

Observe now that the monotonicity properties of G imply

$$\Big[G_i(w_q, z_{q,i}) - G_i(\bar{w}_q, \bar{z}_{i,q})\Big](\chi_{\eta_q>0} - \chi_{\zeta_{i,q}<0}) \le 0. \tag{4.2}$$

Also

$$(\eta_q - \eta_{q-e_i})\chi_{\eta_q>0} \ge \eta_q^+ - \eta_{q-e_i}^+ \text{ and } (\zeta_{i,q+e_i} - \zeta_{i,q})\chi_{\zeta_{i,q}<0} \ge -\zeta_{i,q+e_i}^- + \zeta_{i,q}^-. \tag{4.3}$$

Thererefore for each $q \in \mathbb{Z}^N$,

$$\partial_t(\eta_q^+ + \sum_i \zeta_{i,q}^-) + \frac{1}{h}\sum_i \tilde{A}_i(\eta_q^+ - \eta_{q-e_i}^+) - \frac{1}{h}\sum_i A_i(\zeta_{i,q+e_i}^- - \zeta_{i,q}^-) \le 0,$$

i.e., (i) holds. By summing over $q \in \mathbb{Z}^N$ ($W(t), Z(t)$ are in $l^1(\mathbb{Z}^N)$ by proposition 4.1), we get (ii).

For (iii), we note that $\bar{w}_q = b,\ \bar{z}_{i,q} = h_i(b),\ q \in \mathbb{Z}^N$ is a solution of (3.4). Then (ii) implies $w_q(t) - b \le 0$, $z_{i,q}(t) - h_i(b) \ge 0$ for all $q \in \mathbb{Z}^N,\ t > 0$. A similar argument gives the lower bounds. □

Discrete Entropies. For any $k \in \mathbb{R}$ consider the solution $(\bar{W}, \bar{Z})$ of (4.1), where $\bar{w}_q = k$, $\bar{z}_{i,q} = h_i(k)$, $q \in \mathbb{Z}^N$. It is also clear that Proposition 4.2(i) is valid if we interchange positive with negative parts, thus after summation we get

$$
\begin{aligned}
&\partial_t(|w_q - k| + \sum_i |z_{i,q} - h_i(k)|) + \frac{1}{h}\sum_i \tilde{A}_i\Big(|w_q - k| - |w_{q-e_i} - k|\Big) \\
&\qquad - \frac{1}{h}\sum_i A_i(|z_{i,q+e_i} - h_i(k)| - |z_{i,q} - h_i(k)| \le 0,\ q \in \mathbb{Z}^N,\ k \in \mathbb{R}.
\end{aligned}
\tag{4.4}
$$

Finally we have the following proposition regarding the distance of a solution (W, Z) of (3.4) from the line of equilibria (3.3).

Proposition 4.3. *In addition to the assumptions of Proposition 4.4, we assume that (3.2) holds. Let (W, Z) be a solution of (3.4) emanating from data with finite total variation and lying in an (invariant) region $\mathcal{R}^{a,b}$. Then, the following estimate holds:*

$$
\frac{1}{\varepsilon}\sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q(t), z_{i,q}(t))| \le K + \frac{e^{-\frac{c}{\varepsilon}t}}{\varepsilon}\sum_{q\in\mathbb{Z}^N} h^N \sum_i G_i(w_q^0, z_{i,q}^0),
$$

wherew c is the constant in (3.2) *and K depends on $\mathcal{R}^{a,b}$ and the total variation of the initial data W_0, Z_0.*

Proof. We observe

$$
\begin{aligned}
\partial_t G_i(w_q(t), z_{i,q}(t)) =& \frac{\partial G_i}{\partial w}\Big[-\frac{1}{h}\sum_j \tilde{A}_j(w_q(t) - w_{q-e_j}(t)) + \frac{1}{\varepsilon}\sum_j G_j(w_q(t), z_{j,q}(t))\Big] \\
&+ \frac{\partial G_i}{\partial z}\Big[\frac{1}{h}A_i(z_{i,q+e_i}(t) - z_{i,q}(t)) + \frac{1}{\varepsilon}G_i(w_q, z_{i,q})\Big]
\end{aligned}
$$

Multiplying by $\mathrm{sign} G_i$ and adding, we obtain

$$
\begin{aligned}
\partial_t \sum_i |G_i| + \frac{1}{\varepsilon}\sum_i(-\frac{\partial G_i}{\partial z})|G_i| =& \frac{1}{\varepsilon}\sum_i \frac{\partial G_i}{\partial w}\mathrm{sign}G_i \sum_j G_j \\
&+ \sum_i \mathrm{sign}G_i\Big(\frac{\partial G_i}{\partial w}\Big[-\frac{1}{h}\sum_j \tilde{A}_i(w_q - w_{q-e_j})\Big] + \frac{\partial G_i}{\partial z}\Big[\frac{1}{h}A_i(z_{i,q+e_i} - z_{i,q})\Big]\Big)
\end{aligned}
$$

Then, in view of the fact that W, Z are bounded (Proposition 4.2) and the TVD property of the scheme, we have upon summing with respect to $q \in \mathbb{Z}^N$,

$$
\begin{aligned}
&\frac{d}{dt}\sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i| + \frac{1}{\varepsilon}\sum_{q\in\mathbb{Z}^N} h^N \sum_i \Big(-\frac{\partial G_i}{\partial z} - \sum_j |\frac{\partial G_j}{\partial w}|\Big)|G_i| \\
&\qquad \le C\sum_{q\in\mathbb{Z}^N} h^{N-1}\sum_i(|w_q^0 - w_{q-e_i}^0| + |z_{i,q}^0 - z_{i,q+e_i}^0|) \le K
\end{aligned}
$$

We conclude the proof by integrating the above equality and using (3.2). □

4.b Discrete upwind scheme. In this section we present the properties of the discrete scheme (3.5).

Proposition 4.4. (L^1-contraction, TVD property) *Let* (W, Z), $(\bar{W}, \bar{Z})$ *be two solutions of* (3.5) *with corresponding initial data* (W_0, Z_0), $(\bar{W}_0, \bar{Z}_0)$ *such that* $W_0, Z_0, \bar{W}_0, \bar{Z}_0 \in l^1(\mathbb{Z}^N) \cap l^\infty(\mathbb{Z}^N)$. *If the CFL condition*

$$\tau_h = \frac{\tau}{h} \leq \min_i \{1/N\tilde{A}_i,\ 1/A_i\} \tag{4.5}$$

is satisfied, then for all $n \in \mathbb{N}$, $\varepsilon > 0$,

$$\sum_{q \in \mathbb{Z}^N} \Big(|w_q^n - \bar{w}_q^n| + \sum_i |z_{i,q}^n - \bar{z}_{i,q}^n| \Big) \leq \sum_{q \in \mathbb{Z}^N} \Big(|w_q^0 - \bar{w}_q^0| + \sum_i |z_{i,q}^0 - \bar{z}_{i,q}^0| \Big).$$

Furthermore the fully discrete scheme (3.5) *is TVD:*

$$\sum_{q \in \mathbb{Z}^N} \Big(|w_{q+e_k}^n - w_q^n| + \sum_i |z_{i,q+e_k}^n - z_{i,q}^n| \Big) \leq \sum_{q \in \mathbb{Z}^N} \Big(|w_{q+e_k}^0 - w_q^0| + \sum_i |z_{i,q+e_k}^0 - z_{i,q}^0| \Big),$$

for all $n \in \mathbb{N}$, $\varepsilon > 0$ *and all directions* $e_k,\ k = 1, ..., N$.

Proof. Without loss of generality we may assume that the initial data vanish outside a ball B_M of radius M: due to the finite speed of propagation in (3.5), the solution (W_n, Z_n) at time $t = n\tau$ vanishes outside a ball $B_{M+Vn\tau}$, where $V = max\{A_i, \tilde{A}_i\,, i = 1, \cdots, N\}$. Therefore all the summations below are finite, and the statement of the Proposition will follow by eventually sending $M \to \infty$.

Set $U_q^n = w_q^n - \bar{w}_q^n$, $V_q^n = z_q^n - \bar{z}_q^n$. Multiplying the error equations for U_q^n and V_q^n by $\mathrm{sign} U_q^{n+1}$ and $\mathrm{sign} V_q^{n+1}$ respectively, we obtain:

$$\begin{aligned} &|U_q^{n+1}| + \sum_i \tilde{A}_i ((\tau_h - \frac{1}{N\tilde{A}_i}) U_q^n - \tau_h U_{q-e_i}^n) \mathrm{sign} U_q^{n+1} \\ &\qquad = \frac{\tau}{\varepsilon} \sum_i \Big[G_i(w_q^{n+1}, z_i^{n+1}) - G_i(\bar{w}_q^{n+1}, \bar{z}_{i,q}^{n+1}) \Big] \mathrm{sign} U_q^{n+1}\,, \end{aligned}$$

$$\begin{aligned} &|V_{i,q}^{n+1}| + \Big[(\tau_h A_i - 1) V_{i,q}^n - \tau_h A_i V_{i,q+e_i}^n \Big] \mathrm{sign} V_{i,q}^{n+1} \\ &\qquad = \frac{\tau}{\varepsilon} \Big[G_i(w_q^{n+1}, z_{i,q}^{n+1}) - G_i(\bar{w}_q^{n+1}, \bar{z}_{i,q}^{n+1}) \Big] \mathrm{sign} V_{i,q}^{n+1}\,, \end{aligned}$$

where $\tau_h = \frac{\tau}{h}$. We add the two relations and sum over q. Due to the monotonicity of G_i the right hand side of the resulting equality is nonpositive which yields:

$$\begin{aligned} &\sum_{q \in \mathbb{Z}^N} (|U_q^{n+1}| + \sum_i |V_{i,q}^{n+1}|) \\ &+ \sum_i \sum_{q \in \mathbb{Z}^N} \Big[(\tau_h \tilde{A}_i - \frac{1}{N}) U_q^n \mathrm{sign} U_q^{n+1} - \tau_h \tilde{A}_i U_{q-e_i}^n \mathrm{sign} U_q^{n+1} \Big] \\ &+ \sum_i \sum_{q \in \mathbb{Z}^N} \Big[(\tau_h A_i - 1) V_{i,q}^n \mathrm{sign} V_{i,q}^{n+1} - \tau_h A_i V_{i,q+e_i}^n \mathrm{sign} V_{i,q}^{n+1} \Big] \leq 0\,. \end{aligned} \tag{4.6}$$

By selecting $\tau_h \le \min_i\{1/N\tilde{A}_i,\ 1/A_i\}$ and using the inequalities

$$(\tau_h\tilde{A}_i - \frac{1}{N})U_q^n \mathrm{sign} U_q^{n+1} \ge (\tau_h\tilde{A}_i - \frac{1}{N})|U_q^n|,\ (\tau_h A_i - 1)V_{i,q}^n \mathrm{sign} V_{i,q}^{n+1} \ge ((\tau_h A_i - 1)|V_{i,q}^n|,$$

$$\tau_h \tilde{A}_i U_{q-e_i}^n \mathrm{sign} U_q^{n+1} \le \tau_h\tilde{A}_i |U_{q-e_i}^n|,\ \tau_h A_i V_{i,q+e_i}^n \mathrm{sign} V_{i,q}^{n+1} \le \tau_h A_i |V_{i,q+e_i}^n|\,,$$

equation (4.6) yields

$$\begin{aligned}\sum_{q\in\mathbb{Z}^N} (|U_q^{n+1}| {+} \sum_i |V_{i,q}^{n+1}|) - \sum_{q\in\mathbb{Z}^N} (|U_q^m| + \sum_i |V_{i,q}^m|) \\ + \sum_i \tau_h \tilde{A}_i \sum_{q\in\mathbb{Z}^N} (|U_q^m| - |U_{q-e_i}^m|) + \sum_i \tau_h A_i \sum_{q\in\mathbb{Z}^N} (|V_q^m| - |V_{q+e_i}^m| \le 0\,.\end{aligned}$$

The l^1-contraction follows as indicated at the beginning of the proof; the TVD property is a consequence of the l^1-contraction and the translation invariance of (3.5). □

As in the case of the semi-discrete scheme, (3.5) satisfies a monotonicity property and an entropy condition. The proof follows along the lines of that of Proposition 4.2.

Proposition 4.5. *Under the assumptions of Proposition* 4.4 *the discrete scheme* (3.5) *satisfies:*

(i) $$\sum_{q\in\mathbb{Z}^N} \Big((w_q^{n+1} - \bar{w}_q^{n+1})^+ + \sum_i (z_{i,q}^{n+1} - \bar{z}_{i,q}^{n+1})^- \Big) \le \sum_{q\in\mathbb{Z}^N} \Big((w_q^n - \bar{w}_q^n)^+ + \sum_i (z_{i,q}^n - \bar{z}_{i,q}^n)^- \Big)$$

where the same inequality holds with positive and negative parts interchanged.

(ii) *(entropy condition). For any $k \in \mathbb{R}$,*

$$\begin{aligned}&|w_q^{n+1} - k| - |w_q^n - k| + \sum_i \Big(|z_{i,q}^{n+1} - h_i(k)| - |z_{i,q}^n - h_i(k)| \Big) \\ &+ \frac{\tau}{h} \sum_i \tilde{A}_i (|w_q^n - k| - |w_{q-e_i}^n - k|) - \frac{\tau}{h} \sum_i A_i (|z_{i,q+e_i}^n - h_i(k)| - |z_{i,q}^n - h_i(k)|) \le 0\end{aligned}$$

(iii) *If for some $a < b$ the initial approximations satisfy $a \le w_q^0 \le b,\ h_i(b) \le z_{i,q}^0 \le h_i(a),\ i = 1, \cdots, N,\ q \in \mathbb{Z}^N$ then*

$$a \le w_q^n \le b,\ h_i(b) \le z_{i,q}^n \le h_i(a),\ q \in \mathbb{Z}^N,\ n \in \mathbb{N},\ i = 1, \cdots, N,$$

i.e. the region $\mathcal{R}^{a,b} = [a,b] \times \prod_{i=1}^n [h_i(b), h_i(a)]$ is positively invariant. □

Finally we prove an analogous result to Proposition 4.3:

Proposition 4.6. *In addition to the assumptions of Proposition* 4.4, *we assume that* (3.2) *holds. Let* (W, Z) *be a solution of (3.4) emanating from data with finite total variation and lying in an (invariant) region* $\mathcal{R}^{a,b}$. *Then for all* $n \in \mathbb{N}, \varepsilon > 0$,

(i) $$\sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^n, z_{i,q}^n)| \le (1 + c\frac{\tau}{\varepsilon})^{-n} \sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^0, z_{i,q}^0)| + \varepsilon K$$

(ii) $$\frac{1}{\tau}\sum_{q\in\mathbb{Z}^N} h^N \Big(|w_q^{n+1} - w_q^n| + \sum_i |z_{i,q}^{n+1} - z_{i,q}^n|\Big) \le \frac{1}{\varepsilon}(1 + c\frac{\tau}{\varepsilon})^{-n} \sum_{q\in\mathbb{Z}^n} h^N \sum_i |G_i(w_q^0, z_{i,q}^0)| + K$$

where c *is the constant in* (3.2) *and* K *depends on* $\mathcal{R}^{a,b}$ *and the total variation of the initial data.*

Proof. We have

$$\begin{aligned}
G_i(w_q^{n+1}, & z_{i,q}^{n+1}) - G_i(w_q^n, z_{i,q}^n) = \\
& (w_q^{n+1} - w_q^n)\int_0^1 \frac{\partial G_i}{\partial w}\Big(w_q^n + (w_q^{n+1} - w_q^n)s, z_{i,q}^n + (z_{i,q}^{n+1} - z_{i,q}^n)s\Big)ds \\
& + (z_{i,q}^{n+1} - z_{i,q}^n)\int_0^1 \frac{\partial G_i}{\partial z}\Big((w_q^n + (w_q^{n+1} - w_q^n)s, z_{i,q}^n + (z_{i,q}^{n+1} - z_{i,q}^n)s\Big)ds \\
& = \Big(\int_0^1 \frac{\partial G_i}{\partial w}ds\Big)\Big[-\frac{\tau}{h}\sum_j \tilde{A}_j(w_q^n - w_{q-e_j}^n) + \frac{\tau}{\varepsilon}\sum_j G_j(w_q^{n+1}, z_{j,q}^{n+1})\Big] \\
& + \Big(\int_0^1 \frac{\partial G_i}{\partial z}ds\Big)\Big[\frac{\tau}{h}A_i(z_{i,q+e_i}^n - z_{i,q}^n) + \frac{\tau}{\varepsilon}G_i(w_q^{n+1}, z_{i,q}^{n+1})\Big]
\end{aligned}$$

We multiply the above equality by $\operatorname{sign} G_i(w_q^{n+1}, z_{i,q}^{n+1})$:

$$\begin{aligned}
|G_i(w_q^{n+1}, z_{i,q}^{n+1})| - |G_i(w_q^n, z_{i,q}^n)| - \frac{\tau}{\varepsilon}\Big(\int_0^1 \frac{\partial G_i}{\partial z}ds\Big)|G_i(w_q^{n+1}, z_{i,q}^{n+1})| \\
- \frac{\tau}{\varepsilon}\Big(\int_0^1 |\frac{\partial G_j}{\partial w}|\Big)\sum_j |G_i(w_q^{n+1}, z_{j,q}^{n+1})| \\
\le C\frac{\tau}{h}\sum_j |w_q^n - w_{q-e_j}^n| + \tilde{c}\frac{\tau}{h}|z_{i,q}^n - z_{i,q+e_i}|\,,
\end{aligned}$$

where C depends on $\mathcal{R}^{a,b}$. We again multiply the above inequality by h^N, we sum over i, q and use (3.2) to obtain

$$\begin{aligned}
(1 + c\frac{\tau}{\varepsilon})\sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^{n+1}, z_{i,q}^{n+1})| \le \sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^n, z_{i,q}^n)| \\
+ C\tau\sum_{q\in\mathbb{Z}^N} h^{N-1}\sum_i |w_q^n - w_{q-e_i}^n| + C\tau\sum_{q\in\mathbb{Z}^N} h^{N-1}\sum_i |z_{i,q}^n - z_{i,q+e_i}|
\end{aligned}$$

where c is given in (3.3). We use the above inequality and the TVD property of the discrete scheme to get

$$(1 + c\frac{\tau}{\varepsilon})\sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^{n+1}, z_{i,q}^{n+1})| \le \sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^n, z_{i,q}^n)| + \tau K$$

where K depends on $\mathcal{R}^{a,b}$ and the total variation of the initial data. By means of the last inequality we conclude

$$(1+c\frac{\tau}{\varepsilon})^n \sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^n, z_{i,q}^n)| \le \sum_{q\in\mathbb{Z}^N} h^N \sum_i |G_i(w_q^0, z_{i,q}^0)| + \tau K \sum_{k=0}^{n-1} (1+c\frac{\tau}{\varepsilon})^k$$

which implies (i). We obtain inequality (ii) directly from (3.5),

$$\frac{1}{\tau}|w_q^{n+1} - w_q^n| \le \frac{V}{h} \sum_i |w_q - w_{q-e_i}| + \frac{1}{\varepsilon} \sum_i |G_i(w_q^n, z_{i,q}^n)|\,,$$

the corresponding estimate for the $z_{i,q}^n$ terms, (i) and the TVD property of the scheme. $\square$

5. **Convergence**

We first consider the semi-discrete scheme defined by (3.4). Let $\Omega_q \subset \mathbb{R}^N$ the rectangular region with side length h and center at the $q \in \mathbb{Z}^N$ grid point and let (W^h, Z^h) the piecewise constant approximation function defined by

$$(W^{h,\varepsilon}(t,\cdot)|_{\Omega_q},\, Z^{h,\varepsilon}(t,\cdot)|_{\Omega_q}) = (w_q(t), z_{1,q}(t), \cdots, z_{n,q}(t))\,. \tag{5.1}$$

We consider initial data $(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon})$ lying close to the line of equilibria, in the sense

$$\sum_i ||G_i(W_0^{h,\varepsilon}, Z_{i0}^{h,\varepsilon})||_{L^1} = O(\varepsilon)\,. \tag{5.2}$$

We then have the following theorem:

Theorem 5.1. *Let $(W^{h,\varepsilon}, Z^{h,\varepsilon})$ be the piecewise constant functions* (5.1) *obtained by solutions of* (3.4)*, with initial data $(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon}) \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and satisfying assumption (5.2). Let u be the entropy solution of (1.1) and $U^{h,\varepsilon} = W^{h,\varepsilon} - \sum_i Z_i^{h,\varepsilon}$. Then, for any fixed $T > 0$ and all $t \le T$,*

$$||U^{h,\varepsilon}(t,\cdot) - u(t,\cdot)||_{L^1} \le ||U^{h,\varepsilon}(0,\cdot) - u(0,\cdot)||_{L^1} + \hat{C}\sqrt{\varepsilon + h}\,. \tag{5.3}$$

Here, $\hat{C}$ is a positive constant depending on the fluxes F^i, the L^∞ norms and the total variation of $(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon})$ and $u_0 = u(0,\cdot)$.

Proof. We first show that $U^{h,\varepsilon}$ satisfies an approximate entropy condition similar to (2.1). For all $k \in \mathbb{R}$, define $\kappa \in \mathbb{R}$ such that $k = \kappa - \sum_i h_i(\kappa)$. Since the functions h_i, $i = 1, \cdots, N$ are nonincreasing, we readily see that

$$|U^{h,\varepsilon} - k|\Big|_{\Omega_q} = |w_q - \kappa| + \sum_i |h_i(w_q) - h_i(\kappa)| = |w_q - \kappa| + \sum_i |z_{i,q} - h_i(\kappa)| + J^{h,\varepsilon}(x,t)\,,$$

where $|J^{h,\varepsilon}(t,x)| \le \sum_i |G_i(W^{h,\varepsilon}(t,x), Z_i^{h,\varepsilon}(t,x))|$. Similarly,

$$\begin{aligned}(F_i(U^{h,\varepsilon}) - F_i(k))\mathrm{sign}(U^{h,\varepsilon} - k)\Big|_{\Omega_q} &= \Big[A_i\omega_i(w_q - \kappa) + A_i(h_i(w_q) - h_i(\kappa))\Big]\mathrm{sign}(w_q - \kappa)\\ &= A_i\omega_i|w_q - \kappa| - A_i|h_i(w_q) - h_i(\kappa)|\\ &= A_i\omega_i|w_q - \kappa| - A_i|z_{i,q} - h_i(\kappa)| + H_i^{h,\varepsilon}\,,\end{aligned}$$

where $\sum_i |H_i^{h,\varepsilon}(t,x)| \le V\sum_i |G_i(W^{h,\varepsilon}(t,x), Z_i^{h,\varepsilon}(t,x))|$. If $\Psi \in C_c^\infty([0,\infty)\times\mathbb{R}^n)$, $\Psi \ge 0$, then

$$\begin{aligned}I(k,\Psi) = &-\iint \Big(|U^{h,\varepsilon} - k|\Psi_t + \sum_i (F_i(U^{h,\varepsilon}) - F_i(k))\mathrm{sign}(U^{h,\varepsilon} - k)\Psi_{x_i}\Big)dxdt\\
= &-\sum_q \int dt\Big((|w_q - \kappa| + \sum_i |z_{i,q} - h_i(\kappa)|)\int_{\Omega_q}\Psi_t dx\Big) - \iint J^{h,\varepsilon}\Psi_t dxdt\\
&-\sum_i\Big[\sum_q \int dt(A_i\omega_i|w_q - \kappa| - A_i|z_{i,q} - h_i(\kappa)|)\int_{\Omega_q}\Psi_{x_i}dx\Big] - \sum_i \iint H_i^{h,\varepsilon}(x,t)\Psi_{x_i}dxdt\\
= &-\sum_q \int dt\Big((|w_q - \kappa| + \sum_i |z_{i,q} - h_i(\kappa)|)\int_{\Omega_q}\Psi_t dx\Big)\\
&-\sum_i \int dt A_i \sum_q \Big((\omega_i|w_q - \kappa| - |z_{i,q} - h_i(\kappa)|)\frac{1}{h}\int_{\Omega_q}(\Psi\Big|_{x_i = h(q+\frac{1}{2}e_i)} - \Psi\Big|_{x_i = h(q-\frac{1}{2}e_i)})dx\Big)\\
&-\sum_i \iint H_i^{h,\varepsilon}(x,t)\Psi_{x_i}dxdt - \iint J^{h,\varepsilon}\Psi_t dxdt\\
= &-\sum_q \int_{\mathbb{R}} dt\Big((|w_q - \kappa| + \sum_i |z_{i,q} - h_i(\kappa)|)\int_{\Omega_q}\Psi_t dx\Big)\\
&+\sum_i \int dt\Big(A_i\omega_i\sum_q (|w_q - \kappa| - |w_{q-e_i} - \kappa|)\frac{1}{h}\int_{\Omega_q}\Psi\Big|_{x_i = h(q-\frac{1}{2}e_i)}dx\Big)\\
&-\sum_i \int dt\Big(A_i\sum_q (|z_{i,q+e_i} - \kappa| - |z_{i,q} - \kappa|)\frac{1}{h}\int_{\Omega_q}\Psi\Big|_{x_i = h(q+\frac{1}{2}e_i)}dx\Big)\\
&-\sum_i \iint H_i^{h,\varepsilon}\Psi_{x_i}dxdt - \iint J^{h,\varepsilon}\Psi_t dxdt\end{aligned}$$

Using the discrete entropy inequality we obtain

$$\begin{aligned}I(\kappa,\Psi) \le &\sum_i A_i\omega_i\int dt\Big(\sum_q (|w_q - \kappa| - |w_{q-e_i} - \kappa|)\frac{1}{h}\int_{\Omega_q}(\Psi\Big|_{x_i = h(q-\frac{1}{2}e_i)} - \Psi)dx\Big)\\
&-\sum_i A_i\int dt\Big(\sum_q (|z_{i,q+e_i} - \kappa| - |z_{i,q} - \kappa|)\frac{1}{h}\int_{\Omega_q}(\Psi\Big|_{x_i = h(q+\frac{1}{2}e_i)} - \Psi)dx\Big)\\
&-\sum_i \iint H_i^{h,\varepsilon}(x,t)\Psi_{x_i}dxdt - \iint J^{h,\varepsilon}\Psi_t dxdt\,.\end{aligned}$$

The first term on the right hand-side of the inequality is bounded by

$$\sum_i A_i\omega_i\int dt\Big(\sum_q (|w_q - w_{q-e_i}|)\frac{1}{h}\int_{\Omega_q}|\Psi\Big|_{x_i = h(q-\frac{1}{2}e_i)} - \Psi|dx\Big),$$

and the second term by

$$\sum_i A_i \int dt \Big(\sum_q (|z_{i,q+e_i} - z_{i,q}|) \frac{1}{h} \int_{\Omega_q} |\Psi\Big|_{x_i = h(q+\frac{1}{2}e_i)} - \Psi| dx \Big).$$

To apply Theorem 2.1 note that the partition of $[0,\infty)\times\mathbb{R}^n$ consisting by $[0,\infty)\times\Omega_q$, $q\in\mathbb{Z}^N$ defines a partition of $\operatorname{supp}\Psi$, $\mathcal{T}_h = \{K^q\}$, simply by taking $K^q = (\,[0,\infty)\times\Omega_q)\cap\operatorname{supp}\Psi$. Then we define for $V = \max_i\{A_i, A_i\omega_i\}$,

$$B^i_H(\partial_{x_i}\Psi)\Big|_{K^q} = \frac{V}{h}\Big(|\Psi\Big|_{x_i = h(q-\frac{1}{2}e_i)} - \Psi| + |\Psi\Big|_{x_i = h(q+\frac{1}{2}e_i)} - \Psi| \Big),$$

and

$$\beta^i_H(t,x)\Big|_{K^q} = |w_q - w_{q-e_i}| + |z_{i,q+e_i} - z_{i,q}|\,.$$

Then B^i_H, satisfies (2.4). The TVD property of the semidiscrete approximation implies

$$\sum_i \iint \beta^i_H dtdx \le ThTV(W^{h,\varepsilon}_0, Z^{h,\varepsilon}_0)$$

where $TV(W^{h,\varepsilon}_0, Z^{h,\varepsilon}_0) = TV(W^{h,\varepsilon}_0) + TV(Z^{h,\varepsilon}_0)$ and $TV(Z^{h,\varepsilon}_0) = \sum_i\sum_q h^{N-1}|z^0_{i,q+e_i} - z^0_{i,q}|$, $TV(W^{h,\varepsilon}_0) = \sum_i\sum_q h^{N-1}|w^0_{q+e_i} - w^0_q|$. Since all the hypotheses of Theorem 2.1 are satisfied, sending $\delta\to 0$, $R\to\infty$, delivers

$$\begin{aligned}||U^{h,\varepsilon}(\cdot,T) - u(\cdot,T)||_{L^1(\mathbb{R}^N)} \le ||U^{h,\varepsilon}(\cdot,T) - u(\cdot,T)||_{L^1(\mathbb{R}^N)} + C\Big[\Delta TV(u_0) \\ + \frac{Th}{\Delta}TV(W^{h,\varepsilon}_0, Z^{h,\varepsilon}_0) + (1 + \frac{(M+1)T}{\Delta})\sup_{0\le t\le 2T}\int_{\mathbb{R}^N}\sum_i |G_i(W^{h,\varepsilon}(x,t), Z^{h,\varepsilon}_i(x,t)|dxdt \\ + \frac{1}{\Delta}\sum_i \iint_{0\le t\le T,\, x\in\mathbb{R}^N} |G_i(W^{h,\varepsilon}(x,t), Z^{h,\varepsilon}_i(x,t)|dxdt\Big]\,.\end{aligned}$$

Setting $I^{h,\varepsilon} = \sup_{[0,2T]}\sum_i ||G_i(W^{h,\varepsilon}(\cdot,t), Z^{h,\varepsilon}_i(\cdot,t)||_{L^1}$, we get

$$\begin{aligned}||U^{h,\varepsilon}(\cdot,T) - u(\cdot,T)||_{L^1} \le & ||U^{h,\varepsilon}(\cdot,0) - u(\cdot,0)||_{L^1} \\ & + C\Big[\Delta TV(u_0) + \frac{Th}{\Delta}TV(W^{h,\varepsilon}_0, Z^{h,\varepsilon}_0) + I^{h,\varepsilon} + \frac{(M+2)TI^{h,\varepsilon}}{\Delta}\Big].\end{aligned}$$

By considering the minimum of the right-hand side of (4.6) over Δ we get

$$\begin{aligned}||U^{h,\varepsilon}(\cdot,T) - u(\cdot,T)|| \le & ||U^{h,\varepsilon}(\cdot,0) - u(\cdot,0)||_{L^1(\mathbb{R}^N)} \\ & + C\Big[I^{h,\varepsilon} + T^{1/2}\sqrt{TV(W^{h,\varepsilon}_0, Z^{h,\varepsilon}_0)TV(u_0)h + (M+2)TV(u_0)I^{h,\varepsilon}}\,\Big],\end{aligned}$$

In view of the above inequality and Proposition 4.3 we conclude the proof of the theorem. □

Remark. 5.1 The previous analysis applies directly to the semidiscrete upwind relaxation scheme introduced in [9] for the scalar one dimensional conservation law, based on the relaxation approximation,

$$
(5.4) \qquad \begin{cases} u_t + v_x = 0\,, & (t,x) \in (0,\infty) \times \mathbb{R} \\ v_t + c^2 u_x = -\frac{1}{\varepsilon}[v - f(u)]\,, & (t,x) \in (0,\infty) \times \mathbb{R}\,. \end{cases}
$$

As the relaxation parameter $\varepsilon \to 0$, the local equilibrium $v = f(u)$ is enforced, yielding the scalar equation [18],

$$
u_t + f(u)_x = 0\,, (t,x) \in (0,\infty) \times \mathbb{R}\,,
$$

provided the following *subcharacteristic condition* holds:

$$
-c < f'(u) < c\,.
$$

The approximating scheme for (5.4), introduced in [9] is

$$
\begin{aligned}
&\frac{\partial}{\partial t} u_j + \frac{1}{2\gamma}[v_{j+1} - v_{j-1}] - \frac{1}{2\gamma} c[u_{j+1} - 2u_j + u_{j-1}] = 0, \\
&\frac{\partial}{\partial t} v_j + \frac{1}{2\gamma} c^2[u_{j+1} - u_{j-1}] - \frac{1}{2\gamma} c[v_{j+1} - 2v_j + v_{j-1}] = -\frac{1}{\epsilon}[v_j - f(u_j)]\,.
\end{aligned}
$$

Rewriting the semidiscrete system in Riemann invariants,

$$
w_j = v_j + c u_j, \qquad z_j = v_j - c u_j\,,
$$

we have

$$
\begin{aligned}
&\frac{\partial}{\partial t} w_j + \frac{c}{\gamma}[w_j - w_{j-1}] = \frac{1}{\epsilon} G(w_j, z_j) \\
&\frac{\partial}{\partial t} z_j - \frac{c}{\gamma}[z_{j+1} - z_j] = \frac{1}{\epsilon} G(w_j, z_j)\,,
\end{aligned}
$$

where $G(w,z) = f(\frac{w-z}{2c}) - \frac{w+z}{2}$. If the subcharacteristic condition is met, then G is separately nonincreasing in both variables and the previous results hold for this case also.

We next obtain a convergence rate for the fully discrete scheme (3.5). We first define the piecewise constant approximation function given by

$$
(5.5) \qquad (W^{\tau,h,\varepsilon}(\cdot,\cdot)|_{\Omega_q \times [n\tau,(n+1)\tau)},\, Z^{\tau,h,\varepsilon}(\cdot,\cdot)|_{\Omega_q \times [n\tau,(n+1)\tau)}) = (w_q^n, z_{1,q}^n, \cdots, z_{N,q}^n).
$$

Since the proof follows the lines of Theorem 5.1 we only consider the extra terms that appear in (3.5) due to the time discretization.

Theorem 5.2. *Let $(W^{\tau,h,\varepsilon}, Z^{\tau,h,\varepsilon})$ be the piecewise constant functions* (5.5) *obtained by solutions of* (3.5)*, with initial data $(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon}) \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and satisfying assumption* (5.2)*. Let u be the entropy solution of* (1.1) *and $U^{\tau,h,\varepsilon} = W^{\tau,h,\varepsilon} - \sum_i Z_i^{\tau,h,\varepsilon}$. Then, if the CFL condition* (4.5) *is satisfied, then for any fixed $T > 0$ and all $t \leq T, \varepsilon > 0$,*

$$||U^{\tau,h,\varepsilon}(t,\cdot) - u(t,\cdot)||_{L^1} \leq ||U^{\tau,h,\varepsilon}(0,\cdot) - u(0,\cdot)||_{L^1} + \tilde{C}\sqrt{\varepsilon + \tau + h}\,, \tag{5.6}$$

Here, $\tilde{C}$ is a positive constant depending on the fluxes F^i, the L^∞ norms and the total variation of $(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon})$ and $u_0 = u(0,\cdot)$.

Proof. We only consider in the sequel the contribution of the time derivative in (2.1). We have,

$$\begin{aligned}
&-\iint |U^{\tau,h,\varepsilon} - k|\Psi_t dtdx = \\
&= -\sum_q \sum_n \Big(|w_q^n - \kappa| + \sum_i |z_{i,q}^n - h_i(k)|\Big) \int_{\Omega_q}\int_{I_n} \Psi_t dtdx - \iint J^{\tau,h,\varepsilon}\Psi_t dtdx \\
&= -\sum_q \sum_n \Big(|w_q^n - \kappa| + \sum_i |z_{i,q}^n - h_i(k)|\Big) \int_{\Omega_q} [\Psi((n+1)\tau,\cdot) - \Psi(n\tau,\cdot)]dx \\
&\quad - \iint J^{\tau,h,\varepsilon}\Psi_t dtdx \\
&= \sum_q \sum_n \Big((|w_q^{n+1} - \kappa| - |w_q^n - \kappa|) + \\
&\quad + \sum_i \big(|z_{i,q}^{n+1} - h_i(k)| - |z_{i,q}^n - h_i(k)|\big)\Big) \int_{\Omega_q} \Psi((n+1)\tau,\cdot)dx - \iint J^{\tau,h,\varepsilon}\Psi_t dtdx \\
&= \sum_q \sum_n \Big((|w_q^{n+1} - \kappa| - |w_q^n - \kappa|) + \\
&\quad + \sum_i \big(|z_{i,q}^{n+1} - h_i(k)| - |z_{i,q}^n - h_i(k)|\big)\Big) \frac{1}{\tau}\int_{\Omega_q}\int_{I_n} \Psi(t,x)dtdx \\
&\quad + \sum_q \sum_n \Big((|w_q^{n+1} - \kappa| - |w_q^n - \kappa|) + \\
&\quad + \sum_i (|z_{i,q}^{n+1} - h_i(k)| - |z_{i,q}^n - h_i(k)|)\Big) \frac{1}{\tau}\int_{\Omega_q}\int_{I_n} \Big(\Psi(t,x)\Big|_{t=(n+1)\tau} - \Psi(t,x)\Big)dtdx \\
&\quad - \iint J^{\tau,h,\varepsilon}\Psi_t dtdx,
\end{aligned}$$

where $I_n = [n\tau, (n+1)\tau)$ and $|J^{\tau,h,\varepsilon}(t,x)| \leq \sum_i |G_i(W^{\tau,h,\varepsilon}(t,x), Z_i^{\tau,h,\varepsilon}(t,x))|$. That is, the time discretization will contribute the extra term

$$\sum_q \sum_n \Big[\frac{1}{\tau}|w_q^{n+1} - w_q^n| + \frac{1}{\tau}\sum_i |z_{i,q}^{n+1} - z_{i,q}^n|\Big] \int_{\Omega_q}\int_{n\tau}^{(n+1)\tau} |\Psi(t,x)\Big|_{t=(n+1)\tau} - \Psi(t,x)|dtdx$$

to the bound of the discrete entropy $I(k,\Psi)$. Following the notation of Theorem 2.1, for $K^{n,q} = (I_n \times \Omega_q) \cap \operatorname{supp}\Psi$, define the function

$$B_G(\partial_t\Psi)\Big|_{K^{n,q}} = \frac{1}{\tau}|\Psi(t,x)\Big|_{t=(n+1)\tau} - \Psi(t,x)|,$$

and

$$\beta_G(t,x)\Big|_{K^{n,q}} = |w_q^{n+1} - w_q^n| + \sum_i |z_{i,q}^{n+1} - z_{i,q}^n|.$$

It is clear that $B_G(\partial_t \Psi)$ satisfies (2.4). By Proposition 4.6(ii), we have that β_G satisfies (2.2) and

$$\sup_{0\le t\le 2T} \int \beta_G \, dx \le \tau \Big[\frac{1}{\varepsilon}(1 + c\frac{\tau}{\varepsilon})^{-n} \sum_{q\in\mathbb{Z}^n} h^N \sum_i |G_i(w_q^0, z_{i,q}^0)| + K\Big] \le \tau \hat{K}\,,$$

where $\hat{K}$ is a positive constant depending on the L^∞ norm and the total variation of the data, as well as the assumption (5.2).

The spatial terms (that give rise to the β_H^j terms in Theorem 2.1) are handled as in Theorem 5.1. Following the lines of the proof of Theorem 5.1 we conclude by minimizing over Δ the quantity (we take $\delta = \Delta$)

$$\frac{C}{\Delta}\Big[ThTV(W_0^{h,\varepsilon}, Z_0^{h,\varepsilon}) + (M+2)TI^{\tau,h,\varepsilon} + \hat{K}(M+1)T\tau\Big] + C(M+1)\Delta TV(u_0) + CI^{\tau,h,\varepsilon} + \tau\hat{K}\,,$$

where $I^{\tau,h,\varepsilon}$ is the corresponding quantity to $I^{h,\varepsilon}$ in the proof of Theorem 5.1. $\square$

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