Constant mean curvature surfaces with cylindrical ends

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Abstract. R. Schoen has asked whether the sphere and the cylinder are the only complete (almost) embedded constant mean curvature surfaces with finite absolute total curvature. We propose an infinite family of such surfaces. The existence of examples of this kind is supported by results of computer experiments we carried out using an algorithm developed by Oberknapp and Polthier.

The cylinder of radius $\frac{1}{2}$ is a surface with constant mean curvature 1 (a CMC surface for short). The cylinder has vanishing Gauss curvature K, and hence finite (indeed, zero) absolute total curvature $\int |K|\, dA$. It is the simplest example of an *unduloid*. These are the embedded CMC surfaces of revolution, described by Delaunay in 1841 [2] (see also [3]), which are simply periodic and have as generating curves the roulettes of ellipses (of major axis length 1). There is a one-parameter family of unduloids, depending on the eccentricity of the ellipse. For us it is more convenient to parameterize this family by the *necksize* $n \in (0, \pi]$, which is the length of the shortest closed geodesic. One extreme case of the family is the cylinder, whose necksize is π. At the other extreme, the necksize tends to 0 and the unduloids degenerate to a chain of unit spheres. Periodicity implies that every unduloid, aside from the cylinder, has infinite absolute total curvature.

The Delaunay unduloids play a significant role in the theory of embedded CMC surfaces with finite topology, that is, with finite genus g and a finite number of (necessarily annular) ends k. It is a result of Korevaar, Kusner, and Solomon [11] that each of the k ends is exponentially asymptotic to a Delaunay unduloid. Indeed, their results remain true for the slightly larger class of *almost embedded* surfaces, which are immersed surfaces whose immersion extends to the interior of the surface (see Section 1). We call any such CMC surface a *k-unduloid (of genus g)*. For $k \leq 2$ the only k-unduloids are the sphere and the unduloids themselves [11,17].

More than a decade ago R. Schoen raised the question of whether there are any complete (almost) embedded CMC surfaces with finite absolute total curvature, besides the sphere and cylinder. Such a surface must have finite topology [1]. Thus, by the asymptotics theorem [11], the question is equivalent to the problem we address in the present paper:

Problem 1. *Can a k-unduloid have all of its ends cylindrical for $k \geq 3$?*

It is worth noting that simply or doubly periodic surfaces with (an infinite number of) cylindrical ends exist [4]. Of course, these all have infinite topology and infinite absolute total curvature, though the absolute total curvature of each (non-compact) fundamental domain is finite.

1 Immersed examples and almost embeddedness

Interesting complete, non-compact immersed CMC surfaces of finite absolute total curvature are known: for example, Pinkall and Sterling depict a CMC surface with genus zero and two cylindrical ends [19]. It looks like a "two-lobed Wente torus" fused to a cylinder, and its existence was proven later (see [20], [4], and also [21] where similar surfaces can be constructed for arbitrary Delaunay ends). Since the ends of this surface are embedded they have again exponential decay to a true cylinder, so its absolute total curvature is finite. These (and many other) examples suggest that the class of all immersed CMC surfaces is too large to give much control on their geometry.

When studying minimal surfaces, embeddedness is often a natural condition to impose, especially for physically motivated problems. The maximum principle implies that, under continuous deformation, a complete minimal surface cannot suddenly stop being embedded unless self-intersections occur at the ends of the surface. The situation is different in the case of CMC surfaces, as seen in Figure 1: when we continuously deform an embedded CMC surface, embeddedness may be lost as bubbles start to overlap.

This leads us to concentrate on a natural class of immersed surfaces which arise when considering families of embedded CMC surfaces, the *almost embedded* surfaces, mentioned above. By definition, an immersed surface is almost embedded if it can be parametrized by an immersion $f : M \to \mathbb{R}^3$ which extends to an immersion $F : \Omega \to \mathbb{R}^3$, where Ω is a three-manifold with $\partial\Omega = M$. In fact for CMC surfaces of finite topology, the methods of [13] imply Ω can always be taken to be homeomorphic to a handlebody in $\mathbb{R}^3$.

The principal results on finite topology CMC surfaces are valid for this almost embedded class: for instance, ends are asymptotic to Delaunay unduloids [11], each k-unduloid remains a uniformly bounded distance from a k-ended piecewise-linear graph [10], and the moduli space of all these k-unduloids (near a surface with no L^2 Jacobi fields) is a real analytic manifold [14] of dimension $3k-6$.

2 Nonexistence results for cylindrical ends

There is evidence that k-unduloids with only cylindrical ends are rare. For example, we have proven that there are no k-unduloids of genus zero with all ends cylindrical when there are only $k=3$ ends [6]. More generally, when all the ends have their axes in one common plane — a case we call *coplanar* — there are at least two non-cylindrical ends provided $g=0$ and k is odd.

Fig. 1. Two triunduloids indicate a continuous transition from embedded to non-embedded CMC surfaces. The second surface, whose bubbles overlap, is still almost embedded.

There is further evidence for the rarity of these examples from a different perspective. Gluing constructions were introduced by Kapouleas [8], and have become a powerful and general tool to produce examples of CMC surfaces. Two unduloids, for example, can be glued together by connecting them with a small, almost catenoidal neck. Here, as in general, the resulting surface will have slightly different axis directions of the ends, and it may be necessary to perturb the necksizes, too. These changes can be made arbitrarily small, however, when the connecting neck is small enough.

For Kapouleas' construction to be applicable, the two unduloids must have small necksize, and it is natural to ask if we can similarly glue two tangent cylinders together by a small neck. The two cylinders themselves form a degenerate surface which can be naturally regarded as lying in the boundary of the moduli space of 4-unduloids; in fact there is a one-parameter family of tangent cylinders, parameterized by the angle of their axes. We might expect to find examples with only cylinder ends in the interior of moduli space within a neighborhood of these boundary points. Kapouleas, Mazzeo, and Pollack have recently announced [16, p.7] a gluing construction (inserting a small catenoidal neck between the cylinders) which apparently yields all 4-unduloids in such a neighborhood. However, on each of the surfaces constructed this way, at least two of the ends necessarily decrease their necksize, and thus are no longer cylindrical. This change can be made arbitrarily small when the gluing neck is small, but the change is always present. Similarly, when h cylinders are glued to form a $2h$-unduloid, then at least h necksizes must change. Since this type of construction always changes some necksizes, there are no examples with all cylindrical ends near these boundary points of moduli space.

3 The necksize problem

Consider a k-unduloid of genus g. Let $n_1, \ldots, n_k \in (0, \pi]$ be the asymptotic necksizes of its ends (the lengths of the shortest closed geodesics on the limiting Delaunay unduloids), and let $a_1, \ldots, a_k \in \mathbb{S}^2$ be their (outward oriented) axis directions. These asymptotic quantities satisfy the *balancing formula* [11]

$$\sum_{i=1}^{k} n_i(2\pi - n_i)a_i = 0.$$

The balancing formula has the following physical interpretation: imagine the surface being made as a soap bubble. Due to surface tension and pressure, each end exerts a force on the surface which can be measured across any cap spanning a curve which separates the end. This force is independent of the particular cap chosen and is in fact equal to $n_i(2\pi - n_i)a_i$. Note that the force of a cylindrical end has maximal modulus, while zero force is approached in the spherical bead limit. The sum of these forces represents the net force on the remaining compact domain. If this net force were nonzero, the compact piece would tend to move in that direction; but the surface is in equilibrium so balancing must hold. We remark that the surfaces we consider are not stable, and therefore only small pieces can be realized as a soap bubble. Nevertheless, the physical argument can be made rigorous using the first-variation formula that characterizes the equilibrium of a CMC surface (see [11,10]).

Balancing gives a necessary condition on the asymptotic data of any k-unduloid. We can ask to what extent balancing is also a sufficient condition, leading to a question for which Problem 1 forms an extreme case.

Problem 2. *Given potential necksizes $n_1, \ldots, n_k \in (0, \pi]$ and axis directions $a_1, \ldots, a_k \in \mathbb{S}^2$ satisfying the balancing formula, is there a k-unduloid of genus g with this asymptotic data?*

Whenever all necksizes are sufficiently small, Kapouleas's construction [8] shows that the answer is yes. On the other hand, balancing and the requirement that all necksizes lie in $(0, \pi]$ are not sufficient to guarantee the existence of a k-unduloid. This follows for instance in the case of coplanar k-unduloids of genus zero with full dihedral symmetry: here the force balancing is automatically satisfied, but the maximal necksize reached is $2\pi/k$, so that the *necksize sum* $\sum n_i$ is at most 2π [4,6].

Also, for k odd, any coplanar k-unduloid of genus zero has necksize sum at most $(k-1)\pi$, and further constraints are known [6]. Conversely, we believe that coplanar k-unduloids of genus 0 exist with any necksize sum in the interval $(0, (k-1)\pi]$ for k odd, or $(0, k\pi)$ for k even. For example, a surface like the one pictured in Fig. 2 presumably exists with four cylindrical ends and two unduloid ends of arbitrary necksize $x \in (0, \pi)$. This figure was built from four congruent pieces; using instead $k-2$ pieces suggests a k-unduloid with all but two ends cylindrical. If k is even, the necksize sum is $(k-2)\pi + 2x$, while if k is odd, the noncylindrical ends have necksizes x and $\pi - x$ so the sum is $(k-1)\pi$. Any smaller necksize sum could be obtained in a similar fashion starting from pieces with no cylindrical ends. It is an interesting question whether the bound on the necksize sum for k odd extends to noncoplanar k-unduloids of genus zero; presumably it does not hold for higher genus surfaces.

For arbitrary genus, we can quantify the rarity of examples with all ends cylindrical by a parameter count. We already noted above that, for any genus, the moduli space of all k-unduloids has dimension $3k-6$. Generically, fixing the necksize of one end gives a subspace of codimension 1, but specifying an end to be cylindrical yields codimension 2, as we see in [6] for the case $k=3$. Thus one would expect the dimension of the space of k-unduloids with all ends cylindrical to be $k-6$.

Our methods produce examples with coplanar ends. Requiring the ends to be coplanar generically reduces dimension by $k-3$. Thus we would expect the subspace of coplanar k-unduloids with all ends cylindrical to have dimension $(k-6)-(k-3)=-3$. That means, to find any examples, we need some special condition (like symmetry) to make them nongeneric. Therefore it is natural to look for surfaces with a high degree of symmetry; this is precisely where our methods apply.

4 Numerical examples

We have obtained good numerical evidence for the existence of CMC surfaces with only cylindrical ends and genus one. We carried out our computer experiments using an algorithm developed by Oberknapp and Polthier. This algorithm computes a polyhedral approximation to a minimal surface in $\mathbb{S}^3$, and then applies a discrete version of Lawson's conjugate surface construction [15,9]. It is well described in [18] so we refrain from further explanations here.

It is evident from the accompanying images that we had to search systematically for our surfaces. Indeed, as we will explain in the next section, the idea for our surfaces was found on the basis of theoretical work. The following result is illustrated in Color Plate 4 and Figure 2.

Experimental result 3. *There exist almost embedded k-unduloids of genus one having only cylindrical ends, for $k=30$ and $k=72$.*

Let us describe these surfaces in more detail. First of all, they are coplanar, so that there is a *horizontal* symmetry plane of reflection, containing the axes of all ends. Moreover, each has a large group of symmetries around the vertical axis, namely a dihedral group with 15 (or 36) vertical mirror planes. The entire surface is generated from a fundamental piece depicted in Color Plate 4, top left, by reflection in its boundary planes. This fundamental piece contains half the cylindrical end, namely the portion above the horizontal plane; moreover, it contains two (half) bubbles of different sizes. In fact, 60

Fig. 2. A 72-unduloid of genus 1 with only cylindrical ends can be obtained from a fundamental domain with a 95° angle opposite a cylinder end. The picture shows four reflected fundamental domains leaving a 20° gap. Unlike the two segments at the top (across whose boundaries further reflection can be performed), the four cylindrical ends extend to infinity, and are truncated merely for the graphics.

(or 144) copies of the fundamental domain make up the entire surface. The asymptotic axes of the ends do not meet in the center of symmetry, but instead they all have the same nonzero distance to this point. Since each fundamental domain is embedded, an immersion of the closed solid torus with 30 (or 72) boundary punctures extends the immersion to the interior. Thus our surfaces are almost embedded, even though they are evidently not embedded.

5 The fundamental domains as truncated triunduloids

We think of the fundamental domains of our surfaces with cylindrical ends as truncated, and slightly deformed, triunduloids. Here, by *triunduloid* we mean specifically an almost embedded 3-unduloid of genus zero. In [6] we classify the triunduloids by triples of points in $\mathbb{S}^2$. This classification has the following consequence for the angles enclosed by the asymptotic axes of the ends (compare Figure 3).

Fig. 3. A one-parameter family of triunduloids with one cylindrical end. Cylinders with a perpendicular string of spheres attached form the degenerate limiting surfaces of the family. The pictures indicate half of the family running from a degenerate surface with axis graph $\vdash$, to a Y-shaped (or isosceles) surface, past which it can be continued with mirror images up to another degenerate $\dashv$.

Proposition 4. *A triunduloid can have at most one end with cylindrical necksize. Furthermore, the angle opposite the cylindrical end must be in the interval* $(\pi/2, 2\arccos(2/3)]$ *while the two angles adjacent to the cylinder end are in the range* $(\pi/2, \pi)$*. The latter two angles are equal only when the angle opposite the cylindrical end is exactly* $2\arccos(2/3)$*.*

For each possible angle opposite the cylinder end, in the allowed interval $(\pi/2, 2\arccos(2/3)]$, trigonometric formulas determine the remaining two angles exactly [6]. Currently we are working on an existence proof for triunduloids whose result would in particular imply the one-parameter family suggested by Proposition 4. At present, however, the conditions stated in the proposition are only necessary conditions, and for sufficiency we take the computer experiments as support.

A triunduloid with one cylindrical end can be truncated (at some neck along each of the other two ends) to give a fundamental domain for a surface with only cylindrical ends. We want the curves along which we cut to become planar lines of curvature across which the surface can then be extended by (Schwarz) reflection. To achieve this, the surface must be slightly perturbed. In particular, the parameters of such a *truncated triunduloid*, namely the axis directions for given necksizes, must deviate from those of the complete triunduloid. While these parameters are known theoretically [6] for triunduloids, they must be determined experimentally in the truncated case in a way we will explain below. In fact, we even need to make the definition of these parameters precise in the truncated case. For the necksizes we simply take the length of the bounding curves. Since each of these curves is contained in a *vertical* plane, the axis direction can be defined to be the normal to this plane.

To see how much the parameters for truncated triunduloids differ from those of the complete surfaces, we look at the angle deviation for a given triple of necksizes. This point of view is natural since triunduloids and truncated triunduloids seem to exist for the same range of necksizes. Of course, if we truncate far out, leaving many bubbles on the truncated ends, then the angles for the truncated surface will approach those of the original triunduloid, as expected from the asymptotic convergence result [11].

Experimentally, we can determine the axis directions for truncated triunduloids by solving a *period problem*: since we need to reflect across three boundary curves in some horizontal plane, we require all three to be contained in the same plane. From the numerical construction of a fundamental domain, all we know is that they are contained in parallel planes. However, we can adjust the angle parameters to "kill" the periods, that is, to make the parallel planes coincide. As the end itself has no period, its two bounding arcs are contained in the same plane, so effectively we are left with only a single period problem.

In our computer experiments, when we use (for the truncated triunduloid) the theoretically known parameters for a complete triunduloid, we find no observable period. Therefore we may conclude that the angle parameters (for a given triple of necksizes) are almost exactly the same for a truncated triunduloid as for the original triunduloid, even if only a small number of bubbles is left on each end. (This differs from our experimental observation for other problems, like the rhombic surfaces considered in [5,7]: there the deviations in the angle parameters between the truncated and complete surfaces are significant.)

Thus it is reasonable to consider the angle range of Proposition 4 as valid for the truncated case, too. On the grounds of this assumption, we now calculate the lowest number of ends possible. The angle φ opposite a cylindrical end lies in the interval between $90°$ and $2\arccos(2/3) \approx 96.4°$. The dihedral angle of the two truncating planes is $\pi - \varphi$. Hence, if successive reflection leads to a closed surface with k ends then $(\pi - \varphi) \cdot k = 2\pi l$ for some integer l. Note that the number of ends or fundamental domains k must be even, since according to Proposition 4, only for $\varphi = 2\arccos(2/3)$ itself does the truncated triunduloid have an extra mirror symmetry; in every other case the segments between the cylindrical ends alternate in necksize. Moreover, l gives the turning number of the k-gon one sees in the horizontal symmetry plane when the axes of the ends are deleted. It is not hard to determine the lowest value for even k when φ is in the given range: with $\varphi = 96°$ we obtain $k = 30$ and $l = 7$ as in Color Plate 4. Furthermore, every even $k \geq 30$ except for $32, 36, 40, 44, 48, 52$ and 56 gives an example, while for every even $k \geq 102$ there is more than one example since different turning numbers are possible. (For $k = 72$, we have $\varphi = 95°$ and $l = 17$ as in Figure 2.)

This calculation, together with the experimental fact that truncation of a triunduloid does not significantly change the angle parameters, has the following consequence:

Experimental result 5. *Suppose that a coplanar k-unduloid of genus one has dihedral symmetry, acting transitively on its k cylindrical ends. Then k is even and $k \geq 30$ (and if k is a multiple of four, then $k \geq 60$).*

6 Conjectures

By the reasoning of Section 5 it is obvious that many further CMC surfaces with only cylinder ends must exist. First, we can increase the number of bubbles on each *segment*, that is, each annular portion of surface between successive cylindrical ends. Second, any other "rational" angle $\varphi \in (\pi/2, 2\arccos(2/3))$ will lead to a similar surface.

Conjecture 6. *For all but a finite number of even integers $k \geq 30$, there exist coplanar k-unduloids of genus 1 with only cylindrical ends and transitive dihedral symmetry on the set of ends. Moreover, for each such integer there is a countable family of these surfaces, distinguished by turning number and by the numbers of bubbles on the two distinct segments modulo symmetry.*

To find coplanar surfaces with fewer ends, we consider higher genus. For instance, we propose a 6-unduloid with genus 4 and dihedral symmetry transitive on the ends. The fundamental domain for this surface would be a coplanar 4-unduloid with one cylindrical end and two angles of $60°$ opposite this end. The axes of the segments then form an equiangular hexagon with all three of its diagonals, while the axes of the ends are six outward rays from the vertices. This surface fails to be embedded near the points where the diagonals cross. Because this surface is coplanar with simply connected fundamental domain, it is accessible by our experimental methods.

All the surfaces we have considered so far are almost embedded, but none are embedded. To find examples without self-intersection it seems we must drop the coplanarity assumption. We now propose an embedded 6-unduloid of genus 1. Consider the cycle consisting of the six edges on a cube which avoid the pair of opposite vertices along some body-diagonal direction; this cycle is a right-angled skew hexagon. Now compress the cube along the body-diagonal, until the equal angles in the hexagon increase to $2\arccos(2/3)$. Next, take an isosceles triunduloid with one cylindrical end and two truncated ends of necksize $\pi/2$. Place six copies of the triunduloid along the cycle, with the truncated ends along the edges, and a cylindrical end sticking outward at each vertex. With an appropriate edgelength for the original cube, these triunduloids should approximately match up at the truncations. The twist along each edge means there is no plane of mirror symmetry, so our present methods do not apply. Still, we believe that the truncated ends can be fused even in this case, producing an embedded example of a 6-unduloid with all ends cylindrical. In fact we suspect that this example is part of a one-parameter family of similar surfaces, using the other triunduloids from Proposition 4.

Recall that, generically, the space of k-unduloids with all ends cylindrical should have dimension $k-6$. This means that if the one-parameter family just mentioned does exist, it is nongeneric because of its symmetry. But similar symmetries fail to exist for $k = 5$, while other arguments rule out lower k. Thus the dimension count leads us to conjecture that k-unduloids with all ends cylindrical exist only for $k \geq 6$.

References

1. S. Cohn-Vossen, *Sur la courbure totale des surfaces ouvertes*, C. R. Acad. Sci. **197** (1933), 1165–1167.
2. C. Delaunay, *Sur la surface de révolution, dont la courbure moyenne est constante*, Journal de mathématiques **6** (1841), 309–320.
3. J. Eells, *The surfaces of Delaunay*, The Math. Intell. **9** (1987), 53–57.
4. K. Grosse-Brauckmann, *New surfaces of constant mean curvature*, Math. Zeit. **214** (1993), 527–565.
5. K. Grosse-Brauckmann, R. Kusner, *Moduli spaces of embedded constant mean curvature with few ends and special symmetry*, SFB 256 preprint **483**, Bonn, 1996.
6. K. Grosse-Brauckmann, R. Kusner, J. Sullivan, *Classification of embedded constant mean curvature surfaces with genus zero and three ends*, preprint, 1997.
7. K. Grosse-Brauckmann, K. Polthier, *Constant mean curvature surfaces derived from Delaunay's and Wente's examples*, p. 119–134 in: Visualization and Mathematics (H.-C. Hege, K. Pothier, eds.), Springer Verlag Heidelberg, 1997.
8. N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. **131** (1990), 239–330.

9. H. KARCHER, *The triply periodic minimal surfaces of A. Schoen and their constant mean curvature companions*, Man. math. **64** (1989), 291–357.
10. N. KOREVAAR, R. KUSNER, *The global structure of constant mean curvature surfaces*, Invent. Math. **114** (1993), 311–332.
11. N. KOREVAAR, R. KUSNER, B. SOLOMON, *The structure of complete embedded surfaces with constant mean curvature*, J. Diff. Geom. **30** (1989), 465–503.
12. R. KUSNER, *Bubbles, conservation laws, and balanced diagrams.* In: Geometric analysis and computer graphics (P. CONCUS, R. FINN, D. HOFFMAN, eds.), 103–108, Springer New York, 1990.
13. R. KUSNER, *A maximum principle at infinity and the topology of complete embedded surfaces with constant mean curvature.* In: Global differential geometry and global analysis, 108–114, Lect. Notes Math. **1481**, Springer Berlin 1991.
14. R. KUSNER, R. MAZZEO, D. POLLACK, *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. **6** (1996), 120–137.
15. H.B. LAWSON, *Complete minimal surfaces in* $\mathbb{S}^3$, Ann. of Math. **92** (1970), 335–374.
16. R. MAZZEO, D. POLLACK, *Gluing and moduli for noncompact geometric problems*, preprint, 1996.
17. W.H. MEEKS, *The topology and geometry of embedded surfaces of constant mean curvature*, J. Diff. Geom. **27** (1988), 539–552.
18. B. OBERKNAPP, K. POLTHIER, *An algorithm for discrete constant mean curvature surfaces*, p. 141–161 in: Visualization and Mathematics (H.-C. HEGE, K. POTHIER, eds.), Springer Verlag Heidelberg, 1997.
19. U. PINKALL, I. STERLING, *On the classification of constant mean curvature tori*, Ann. of Math. **130**, 407–451 (1989)
20. I. STERLING, H. C. WENTE, *Existence and classification of constant mean curvature multibubbletons of finite and infinite type*, Indiana Univ. Math. J. **42** (1993), 1239–1266.
21. H. WENTE, *Constant mean curvature immersions of Enneper type*, Memoirs of the AMS **478**, 1992.

Fig. 4. A constant mean curvature surface with finite absolute total curvature, this 30-unduloid of genus one has only cylindrical ends. At the top left, we see a fundamental domain having a 96° angle opposite the cylindrical end. At the top right, four copies leave a 24° gap. Finally, at the bottom, thirty copies close up to form the upper half of the complete surface.