Periodic discrete conformal maps.

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1 Introduction.

Recently there has been much interest in the theory of discrete surfaces in 3-space and its connection with the discretization of soliton equations (see e.g. [4, 11] and references therein). In this article we study a discrete geometry which is the simplest example for both theories. Following [1, 3] we will define a discrete conformal map (DCM) to be a map $z : \mathbf{Z}^2 \to \mathbf{P}^1$ with the property that the cross-ratio of each fundamental quadrilateral is the same. Specifically, for four points a, b, c, d on $\mathbf{P}^1$ define their cross-ratio to be

$$[a : b : c : d] = \frac{(a-b)(c-d)}{(b-c)(d-a)}.$$

Then $z : \mathbf{Z}^2 \to \mathbf{P}^1$ is discrete conformal when

$$[z_{k,m+1} : z_{k,m} : z_{k+1,m} : z_{k+1,m+1}] = q \tag{1}$$

for some constant $q \neq 0, 1, \infty$ for all k, m.

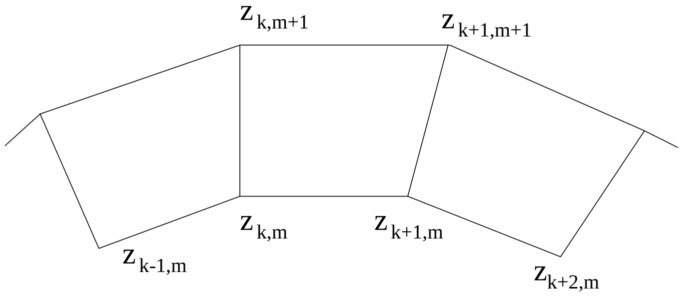

Figure 1: The points about $z_{k,m}$ with neighbours joined by edges.

The motivation for this definition is that if $z : \mathbf{R}^2 \to \mathbf{C}^1$ is smooth then it is (weakly) conformal precisely when

$$\lim_{\epsilon \to 0} [z(x, y+\epsilon) : z(x, y) : z(x+\epsilon, y) : z(x+\epsilon, y+\epsilon)] = -1,$$

i.e. when $z_y^2/z_x^2 = -1$. Moreover, Nijhoff & Capel [11] have shown that one can think of the equations (1) as being a discretization of the Schwarzian KdV (SKdV) equations, hence this geometry should be 'integrable' in some appropriate sense. From another perspective, Bobenko [1] has shown that all the circle patterns of Schramm [13] correspond to DCM's with cross-ratio -1. However, as we shall see, we achieve greater insight by allowing the cross-ratio to be any complex value.

Our main aim is to show that all periodic DCM's (i.e. $z_{k+n,m} = z_{k,m}$ for some n and all k, m) can be constructed using methods which are straight from integrable systems theory, viz, by relating each such map to a linear flow on the Jacobi variety of a compact Riemann surface Σ (or more generally, an algebraic curve). Recall that this is the moduli space of degree zero holomorphic line bundles over the curve: it is a complex manifold with the structure of an abelian group and its dimension equals the genus of Σ. In our case the flow is discrete so by 'linear' we mean the flow is a map of $\mathbf{Z}^2$ into this Jacobian which is essentially a homomorphism (generically the map is a *zigzag* i.e. a homomorphism on a subgroup of index two, but each of these is just a deformation of a homomorphism). We show that every periodic DCM is determined, uniquely up to Möbius equivalence, by its *spectral data*, which consists of: a compact hyperelliptic Riemann surface Σ (which may be singular) equipped with three marked points O, S, Q; a degree two rational function λ on Σ for which $\lambda(O) = 0$, $\lambda(S) = 1$, $\lambda(Q) = q$; and, a degree $g + 1$ line bundle $\mathcal{L}$ over Σ satisfying a non-speciality condition, where g is the genus of Σ.

The spectral data arises by considering the DCM as the 'conformal flow' of a periodic discrete curve i.e. each of the discrete periodic curves $\Gamma_{k,m} = (z_{k,m}, z_{k+1,m}, \ldots, z_{k+n-1,m})$ is considered to be the evolution of the initial curve $\Gamma_{0,0}$ according to the cross-ratio condition. Given $\Gamma_{0,0}$ and a cross-ratio q one asks the question: what condition must a point $z \in \mathbf{P}^1$ satisfy for it to be a neighbour of $z_{0,0}$ in this flow? This is a question about the fixed points of a composite of Möbius transformations as we go around $\Gamma_{0,0}$. We call this composite the holonomy $H_{0,0}$ of the closed curve $\Gamma_{0,0}$. By treating the cross-ratio as a parameter (which we re-label λ) the holonomy becomes a rational function of λ with values in $\mathbf{P}GL_2$. The fixed points of $H_{0,0}$ are the eigenlines of its matrix representation: these vary with λ. The characteristic polynomial of this matrix determines Σ while $\mathcal{L}$ is the *dual* of its bundle of eigenlines. As a result, the $\mathbf{P}^1$ in which the discrete map takes values gets identified with the projective space $\mathbf{P}\Gamma(\mathcal{L})^*$ of hyperplanes (i.e. dual lines) in $\Gamma(\mathcal{L})$, the space of globally holomorphic sections of $\mathcal{L}$.

When we do the same construction for $\Gamma_{k,m}$ we obtain another holonomy matrix, $H_{k,m}$, with its spectral curve and line bundle $\mathcal{L}_{k,m}$. But $H_{k,m}$ is conjugate to $H_{0,0}$ by a matrix of rational functions of λ, so the spectral curves are isomorphic. Moreover, since the conjugacy maps eigenlines to eigenlines we obtain a rational section of the degree zero line bundle $\mathrm{Hom}(\mathcal{L}_{k,m}, \mathcal{L}_{0,0}) \simeq \mathcal{L}_{0,0} \otimes \mathcal{L}_{k,m}^{-1}$. We can explicitly compute the divisor $D_{k,m}$ of poles and zeroes of this section. In the simplest case, where $\lambda = 0$ is a branch point, $D_{k,m}$ is the divisor $k(S - O) + m(Q - O)$ whence the conformal flow 'linearises' on the Jacobian of Σ.

However, the periodicity condition requires a little more: $n(S - O)$ must be the divisor of a rational function on the singularisation Σ' of Σ obtained by identifying the two points over $\lambda = \infty$. This suggests that we should think of the linearised flow as taking place on the generalised Jacobian J' for this singular curve. This leads us to a fairly elegant formula for periodic DCM's involving the θ-function for Σ pulled back to J'. This is analogous to the formula found in [5] for discrete surfaces of negative Gaussian curvature.

This much is contained in sections 2 and 3. Section 2 treats the holonomy matrix for a discrete curve and derives the spectral data. For simplicity we assume that Σ is a non-singular curve (we show in the Appendix that this is the generic case). Section 3 applies this to the construction of periodic DCM's and proves that the spectral data $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$ characterizes the DCM uniquely in its Möbius equivalence class. We give the explicit formula for $z_{k,m}$ in terms of the θ-function and show that these maps will have singularities whenever the flow passes through (a certain translate of) the θ-divisor. These singularities manifest as the collapse of all four neighbours of a point $z_{k,m}$ onto that point: in this case the cross-ratio breaks down in the adjacent quadrilaterals. We also give a geometric interpretation for the θ-function formula which supports the view that the Schwarzian KdV equations are (one) continuum limit of the equations (1). Geometrically this limit is very easy to describe. Let $\mathcal{A}' : \Sigma' \to J'$ be the Abel map for Σ', then the SKdV limit arises as the secant $\vec{OS}$ (on $\mathcal{A}'(\Sigma')$) tends to the tangent line at O while $\vec{OQ}$ tends to the third derivative $\partial^3\mathcal{A}'/\partial\zeta^3$ at O (where ζ is a local parameter about O). Finally, we compute some examples in the case where Σ is a rational nodal curve. These we interpret as the soliton solutions for this theory: recall that the soliton solutions of the KdV equation have rational nodal spectral curves. Indeed, computer investigations show that the multi-soliton solutions behave like superposed 1-solitons (in this geometry 1-solitons can be distinguished by their rotational symmetries).

Section 4 addresses a different aspect of the integrable nature of this geometry: the existence of the Lax pair found by Nijhoff & Capel [11]. This, together with the fact that DCM's naturally occur in 1-parameter families indexed by λ, suggests that we can apply the loop group dressing action theory known for KdV [14, 16] and SKdV [10]. This is achieved without much difficulty and we show that the appropriate dressing action preserves the cross-ratio of the map. We concentrate on describing the corresponding 'dressing orbit of the vacuum solution'. The term 'vacuum solution' comes from soliton theory and means the most elementary solution. In our case there is a vacuum solution for every cross-ratio: each is the tiling of the plane by a given parallelogram. We show that each dressing orbit of a vacuum solution is infinite dimensional and their union contains every periodic DCM which is not too great a perturbation of its continuum limit (by comparison, one knows from [14] that every solution of KdV arising from a spectral curve lies in the dressing orbit of the vacuum). The dressing construction also gives many DCM's which are not periodic and we compute a simple example: the discrete cubic.

Finally, we prove a result about the connection between Darboux (Bäcklund) transforms of the KdV equation and DCM's. In [2] Bobenko argues that whenever a smooth integrable geometry possesses a discrete infinite group of Bäcklund transforms this family can be taken to be an integrable discretization of the smooth geometry. We show that each DCM in the dressing orbit of the vacuum solution corresponds to a $\mathbf{Z}^2$ family of Darboux transforms of a solution of the KdV equation. In the case of finite type (i.e. those DCM's possessing a spectral curve) these Darboux transforms are precisely the ones described in [6] as those preserving the spectral curve of the KdV solution.

Acknowledgements. We thank Nick Schmitt for designing a computing package which allowed us to investigate the moduli space of one and two solitons. Figures 5 and 6 come from this package.

2 Discrete Curves in $\mathbf{P}^1$.

2.1 Preliminaries.

Let $\Gamma = (z_0, \ldots, z_{n-1})$ be distinct ordered points in $\mathbf{P}^1$ (and to avoid trivialities we take $n \geq 4$). We will call Γ a periodic discrete curve with base point z_0. To each edge (z_k, z_{k+1}) (with $z_n = z_0$) we associate a rational map of the

λ-sphere $\mathbf{P}^1_\lambda$ (i.e. $\mathbf{P}^1$ with an affine coordinate λ) into the Möbius group,

$$\mathbf{P}^1_\lambda \to \mathbf{P}GL_2 \quad , \quad \lambda \mapsto T^\lambda_k, \tag{2}$$

by requiring that, for all $z \in \mathbf{P}^1 \backslash \{z_k, z_{k+1}\}$, the cross-ratio condition

$$[z : z_k : z_{k+1} : T^\lambda_k(z)] = \lambda$$

is satisfied. A simple calculation shows that T^λ_k is invertible except at $\lambda = 0, 1$ and that we can represent it in $\mathfrak{gl}_2$ by

$$T^\lambda_k = I - \lambda^{-1} A_k \tag{3}$$

where A_k is the projection matrix with kernel z_k and image z_{k+1} (thinking of these as lines in $\mathbf{C}^2$).

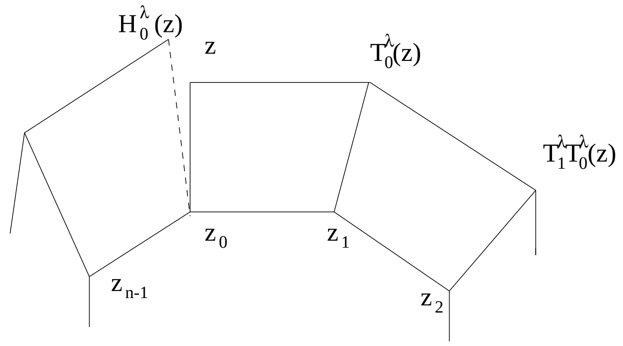

Figure 2: $T^\lambda_0(z)$ and $H^\lambda_0(z)$.

We now introduce the holonomy of Γ at the base point z_0 (see Figure 2):

$$H^\lambda_0 = T^\lambda_{n-1} \circ T^\lambda_{n-2} \circ \ldots \circ T^\lambda_0. \tag{4}$$

We will use this notation for both the map into $\mathbf{P}GL_2$ and the matrix representation corresponding to (3).

Lemma 1 *Let d denote the degree of H^λ_0 (in λ^{-1}). Then $d \leq n/2$ for n even and $d \leq (n+1)/2$ for n odd.*

Proof. Since $A_{k+1} A_k = 0$ the highest order term that can possibly appear in the expansion $H^\lambda_0 = I + h_1 \lambda^{-1} + \ldots + h_d \lambda^{-d}$ is given by

$$h_d = \begin{cases} \lambda^{-n/2}(A_{n-2}B_1A_0 + A_{n-1}B_2A_1 + A_{n-1}B_3A_0) & \text{for } n \text{ even;} \\ \lambda^{-(n+1)/2}(A_{n-1}A_{n-3}\ldots A_0) & \text{for } n \text{ odd;} \end{cases}$$

where the B_j are some composites of the A_k.$\square$

2.2 Spectral Data.

The spectral data will be computed from the trace free part of the holonomy matrix. Set $p(\lambda) = \mathrm{Tr}(H_0^\lambda)/2$; then one readily sees that $p(\lambda)$ is a polynomial in λ^{-1} of degree d with $p(\lambda) = 1 + p_1\lambda^{-1} + \ldots + p_d\lambda^{-d}$.

Remark. For n odd we observe from the formula above that z_0 is both the kernel and image of h_d, whence $h_d^2 = 0$ i.e. it is nilpotent. Therefore $p_d = 0$ for n odd.

Define

$$M_0^\lambda = \lambda(H_0^\lambda - p(\lambda)I) \tag{5}$$

This is a trace-free matrix polynomial in λ^{-1} of degree $\leq d-1$. Set $m(\lambda) = \det(M_0^\lambda)$: this is a polynomial in λ^{-1} with leading order term $\lambda^{2-2d}\det(h_d - p_d I)$, so by the previous remark we see that $\deg(m(\lambda))$ is at most $2d-2$ for n even and $2d-3$ for n odd. By the lemma this means that for any n this degree is at most $n-2$. From now on we will make a genericity assumption: we will assume that $\deg(m(\lambda)) = n-2$, that $m(\infty) \neq 0$ and that $m(\lambda)$ has distinct roots. Since the map $(z_0, \ldots, z_{n-1}) \mapsto \det(M_0^\lambda)$ is algebraic it is clear that the generic discrete curves occupy a Zariski open subset of $\{(z_0, \ldots, z_{n-1}) : z_i \neq z_j\}$. We will show in the appendix that it is not empty, so such discrete curves exist and are indeed generic. With these assumptions we have $d = n/2$ for n even, $d = (n+1)/2$ for n odd.

Define the spectral curve to be the isomorphism class Σ of the curve

$$\Sigma_0 = \{(\lambda, [v]) \in \mathbf{P}^1 \times \mathbf{P}^1 : M_0^\lambda[v] = [v]\}.$$

In this notation $[v]$ denotes the line through $v \in \mathbf{C}^2$.

Proposition 1 *This construction makes Σ a complete non-singular hyperelliptic curve of genus $g = d-2$ (equal to $(n-4)/2$ for n even, $(n-3)/2$ for n odd). This curve comes equipped with a rational function λ of degree two and a degree $g+1$ map $f : \Sigma \to \mathbf{P}^1$. The function λ is unbranched at 1 and ∞ but when n is odd it is branched at 0.*

Proof. By the genericity assumption Σ is modelled by the non-singular completion of the affine curve with equation $\det(\mu I - M_0^\lambda) = \mu^2 + m(\lambda) = 0$. This is clearly a hyperelliptic curve with hyperelliptic cover $\lambda : \Sigma \to \mathbf{P}^1$. Since we have assumed $m(\infty) \neq 0$ there is no branch point at ∞. Since $\deg(m(\lambda)) = n-2$ this cover is branched at $\lambda = 0$ (i.e. $\lambda^{-1} = \infty$) when n is odd and we read off the genus from $\deg m(\lambda) = 2g+2$ for n even and

$\deg m(\lambda) = 2g+1$ for n odd, giving $g = d-2$. To show λ is unbranched at 1 it suffices to observe that z_1, z_{n-1} are distinct eigenlines of M_0^1. To see this, simply note that $H_0^1 = (I - A_{n-1}) \circ \ldots \circ (I - A_0)$ and $z_1 = \ker(I - A_0)$ while $z_{n-1} = \mathrm{im}(I - A_{n-1})$.

In $\Sigma \times \mathbf{C}^2$ we have the kernel line bundle of $\mu I - M_0^\lambda$, which is clearly holomorphic. Its projectivisation is a holomorphic map $\Sigma \to \Sigma \times \mathbf{P}^1$ and we obtain $f : \Sigma \to \mathbf{P}^1$ by composing this with projection on the second factor. Clearly Σ_0 is the image of $\lambda \times f : \Sigma \to \mathbf{P}^1 \times \mathbf{P}^1$. We have to show that this is an embedding. Certainly it is injective, since the eigenlines of M_0^λ are distinct away from branch points of λ. Moreover it is an embedding, for when $d\lambda = 0$ we are at ramification points, which lie over the roots of $m(\lambda)$ (and the point over $\lambda = 0$ when n is odd). By the genericity assumption at these points M_0^λ is tranverse to the determinant conic, whence its eigenlines have distinct tangents i.e. $df \neq 0$. Finally, since the image curve has genus g in the quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$ it must be of type $(2, g+1)$ (see e.g. [8]) i.e. the degree of f is $g+1$. □

Let us define two triples (Σ, λ, f) and (Σ', λ', f'), of data of the type in the previous proposition, to be isomorphic if there is an isomorphism $\Sigma \simeq \Sigma'$ which *identifies* λ with λ' and equates f with a Möbius transform of f'. Then we have the following lemma.

Lemma 2 *Any Möbius transformation of* Γ *leaves the isomorphism class of* (Σ, λ, f) *fixed.*

Proof. Let $\Gamma' = (gz_0, \ldots, gz_{n-1})$ for some Möbius transformation g. Clearly all the maps/matrices $T^\lambda, H^\lambda, M^\lambda$ are conjugated by g and so we have

$$\Sigma_0' = \{(\lambda, [v]) \in \mathbf{P}^1 \times \mathbf{P}^1 : gM_0^\lambda g^{-1}[v] = [v]\}$$

whence the map $(\lambda, [v]) \mapsto (\lambda, g[v])$ is an isomorphism between Σ_0 and Σ_0' which identifies λ with λ' and equates f with a Möbius transform of f'.□

Henceforth we will use (Σ, λ, f) to denote this isomorphism class. The *spectral data* for the based curve Γ is the quintuple $(\Sigma, \lambda, f, O, S)$ where O, S are points on Σ such that $(\lambda, f)(O) = (0, z_0)$ and $(\lambda, f)(S) = (1, z_1)$. That such points exist follows from:

Lemma 3 *The points* $(0, z_0)$ *and* $(1, z_1)$ *both lie on* Σ_0.

Proof. We have already shown that $(1, z_1)$ lies on Σ_0 in the proof of the previous proposition. Now, given the calculations in the proof of lemma 1 we want to show that z_0 is an eigenline of $h_d - \mathrm{Tr}(h_d)/2$. When n is odd, h_d is nilpotent with kernel z_0, so $(0, z_0)$ is the ramification point over $\lambda = 0$. When n is even we know h_d is a sum of three matrices: the first has kernel z_0 and image z_{n-2}, the second has kernel z_1 and image z_0 while the third has kernel z_0 and image z_{n-1}. It follows that h_d maps the line z_0 to itself i.e. it is an eigenline.$\square$

It is much more useful to take, in place of the Möbius class of f, the line bundle $\mathcal{L} = f^*\mathcal{O}_{\mathbf{P}}(1)$ i.e. the pullback of the hyperplane line bundle over $\mathbf{P}^1$. Indeed, by definition $\mathcal{L}$ is the dual to the bundle of eigenlines over Σ and therefore contains exactly the information we require. It clearly has degree $g+1$ since f does. As a result of the next lemma we may recover f up to isomorphism from $\mathcal{L}$ as the map $\Sigma \to \mathbf{P}\Gamma(\mathcal{L})^*$ in which P is mapped to the hyperplane $\Gamma(\mathcal{L}(-P))$ of all sections vanishing at P.

Lemma 4 $\Gamma(\mathcal{L}(-P - \tilde{P})) = 0$ *for each point* $P \in \Sigma$ *(where* $\tilde{P}$ *denotes the hyperelliptic involute of* P*). Thus* $\dim \Gamma(\mathcal{L}) = 2$ *and the map* $\Sigma \to \mathbf{P}\Gamma(\mathcal{L})^*$*;* $P \mapsto \Gamma(\mathcal{L}(-P))$ *separates points in hyperelliptic involution.*

Proof. Let $V = \mathbf{P}^1 \times \mathbf{C}^2$, we will show that this is isomorphic to the direct image $\lambda_*\mathcal{L}$. It follows that $\Gamma(\mathcal{L}) = \Gamma(V)$ and a global section of $\mathcal{L}$ vanishes at $P + \tilde{P}$ precisely when the corresponding section of V vanishes at $\lambda(P)$. Since all global sections of V are constant this will prove the lemma. Observe that the sheaf of local sections of the dual, $V^* \simeq V$, is an $\mathcal{O}_\Sigma$-module: for any local section σ of V^* and locally regular function $r(\lambda, \mu)$ on Σ we define $r(\lambda, \mu)\sigma = \sigma \circ r(\lambda I, M_0^\lambda)$. Now let $\mathcal{E} \subset \Sigma \times \mathbf{C}^2$ denote the eigenline bundle whose dual is $\mathcal{L}$, then the natural pairing gives rise to an injective $\mathcal{O}_\Sigma$-module homomorphism of V^* into $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_\Sigma)$. Therefore as an $\mathcal{O}_\Sigma$-module $V^* \simeq \mathcal{L}(-D)$ for some positive divisor D of degree d. But as an $\mathcal{O}_{\mathbf{P}}$-module $V^* \simeq \lambda_*\mathcal{L}(-D)$ and V^* has Euler characteristic $\chi(V^*) = 2$, which gives $\chi(\mathcal{L}(-D)) = 2$. But in fact D must be the trivial divisor, since by Riemann-Roch $\chi(\mathcal{L}(-D)) = (g+1-d)+1-g$ so $d = 0$, whence $V^* \simeq \lambda_*\mathcal{L}$.$\square$

2.3 Change of base point.

Given the spectral data we wish ultimately to recover the discrete curve Γ. We have seen that the point O corresponds to the base point z_0 of Γ via the line bundle $\mathcal{L}$. Here we will show that the change of base point corresponds to moving only the line bundle $\mathcal{L}$, not the other spectral data. We examine what happens when the based curve $\Gamma_0 = \Gamma$ is subjected to a cyclic permutation to give $\Gamma_k = (z_k, z_{k+1}, \dots, z_{k-1})$. For Γ_k we have the corresponding holonomy H_k^λ with base point z_k and its trace free part M_k^λ. Clearly we have the relationship

$$M_{k+1}^\lambda = T_k \circ M_k^\lambda \circ T_k^{-1}. \tag{6}$$

Let the spectral data for Γ_k be $(\Sigma_k, \lambda_k, \mathcal{L}_k, O_k, S_k)$. In particular, $f_k(O_k) = z_k$ and $f_k(S_k) = z_{k+1}$. Recall that when n is odd O_k is a ramification point of λ and therefore a fixed point of the hyperelliptic involution $P \mapsto \tilde{P}$.

Proposition 2 *For each k we have an isomorphism $(\Sigma_k, \lambda_k) \simeq (\Sigma, \lambda)$ such that S_k is mapped to S but O_k maps to O for k even and $\tilde{O}$ for k odd. Further*

$$\mathcal{L}_{k+1} \otimes \mathcal{L}_k^{-1} \simeq \mathcal{O}_\Sigma(\tilde{O}_k - S).$$

Proof. We will construct the isomorphisms $(\Sigma_k, \lambda_k) \simeq (\Sigma_{k+1}, \lambda_{k+1})$ and then deduce the result from these. Fix k and consider the map

$$\begin{array}{ccc} \Sigma_k & \to & \Sigma_{k+1} \\ (\lambda, [v]) & \mapsto & (\lambda, [T_k v]) \end{array} \tag{7}$$

which we deduce from (6). Since T_k is invertible except at $\lambda = 0, 1$ this map is certainly biholomorphic off the points over $\lambda = 0, 1$ and equates λ_k with λ_{k+1}. Now we consider T_k^λ about $\lambda = 0, 1$, where it has at most simple zeroes or poles.

Set $\eta = \lambda^{-1} - 1$: this is a local coordinate about both points $S_k, \tilde{S}_k$ over $\lambda = 1$. Let $v_\eta = v_0 + \eta v_1 + \dots$ be the expansion for a locally holomorphic family of eigenvectors for M_k^λ about $\eta = 0$. Then from $T_k^\lambda = (I - A_k) - \eta A_k$ we see that

$$T_k^\lambda v_\eta = (I - A_k) v_0 + \eta[(I - A_k) v_1 - A_k v_0] + O(\eta^2).$$

If $[v_0] = z_{k+1} = \mathrm{im} A_k$ this has a simple zero, otherwise it has no zero. Since we may rescale T_k without changing the map (7) we see that replacing T_k^λ by $\eta^{-1} T_k^\lambda$ exhibits (7) as a biholomorphic map about $S_k = (1, z_{k+1})$. Further, to see that S_k is mapped to $S_{k+1} = (1, z_{k+2})$ it is enough to see that $\tilde{S}_k$ is not mapped to it. But $\mathrm{im}(I - A_k) = z_k$ so from the expression above $\tilde{S}_k$ is mapped to $(1, z_k)$.

To perform a similar calculation about $\lambda = 0$ we have to consider the two cases: $\lambda = 0$ is a branch or is not a branch. In the latter case λ is a local parameter about each point $O_k, \tilde{O}_k$. Any locally holomorphic family of eigenvectors for M_k^λ has expansion $v_\lambda = v_0 + \lambda v_1 + \ldots$ about $\lambda = 0$, whence

$$T_k^\lambda v_\lambda = (I - \lambda^{-1} A_k) v_\lambda = -\lambda^{-1} A_k v_0 + (v_0 - A_k v_1) + O(\lambda).$$

This has a simple pole unless $[v_0] = z_k = \ker A_k$ i.e. a simple pole only at $\tilde{O}_k$. By replacing T_k with λT_k about $\tilde{O}_k$ we see that (7) is biholomorphic here also. Moreover, since $\mathrm{im} A_k = z_{k+1}$ we see that (7) maps $\tilde{O}_k$ to O_{k+1} and therefore O_k maps to $\tilde{O}_{k+1}$. When $\lambda = 0$ is a branch we choose $\zeta = \sqrt{\lambda}$ to be a local parameter. A locally holomorphic family $v_\zeta = v_0 + \zeta v_1 + \ldots$ of eigenvectors now yields

$$T_k^{\zeta^2} v_\zeta = -\zeta^{-2} A_k v_0 - \zeta^{-1} A_k v_1 + O(1).$$

But $[v_0] = z_k$ since there is only one point over $\lambda = 0$ hence T_k has a simple pole at O_k. Again, the image of O_k under (7) is O_{k+1} since $z_{k+1} = \mathrm{im} A_k$.

Finally, since T_k maps eigenlines to eigenlines it represents a rational section of $\mathcal{L}_k \otimes \mathcal{L}_{k+1}^{-1}$ since $\mathcal{L}_k$ is the dual of the eigenline bundle of M_k. By the discussion above this section has divisor $S_k - \tilde{O}_k$ so $\mathcal{L}_k \otimes \mathcal{L}_{k+1}^{-1} \simeq \mathcal{O}_\Sigma(S_k - \tilde{O}_k)$. But (7) maps $(O_k, \tilde{O}_k)$ to $(\tilde{O}_{k+1}, O_{k+1})$ so we find that O_k is O for k even and $\tilde{O}$ for k odd. This completes the proof.□

Let us define a periodic map $L : \mathbf{Z} \to Jac(\Sigma)$ into the Jacobi variety (i.e. the group of isomorphism classes of line bundles of degree zero over Σ) by setting $L_0 = \mathcal{O}_\Sigma$ and $L_{k+1} \otimes L_k^{-1} = \mathcal{L}_{k+1} \otimes \mathcal{L}_k^{-1}$ (with $\mathcal{L}_{k+n} = \mathcal{L}_k$). The previous proposition shows that when n is odd this is a homomorphism, whereas when n is even we call it a *zigzag* since it is only a homomorphism on $2\mathbf{Z}$. In fact in either case

$$\mathcal{O}_\Sigma \simeq L_{2n} \simeq \mathcal{O}_\Sigma(O - S + \tilde{O} - S)^n \simeq \mathcal{O}_\Sigma(\tilde{S} - S)^n \tag{8}$$

using the fact that $S + \tilde{S} \sim O + \tilde{O}$ (linear equivalence). Therefore the divisor $\tilde{S} - S$ is a torsion divisor (in which case S is called a division point on Σ). Indeed $\tilde{S} - S$ satisfies a slightly stronger condition.

Lemma 5 *The divisor $n(\tilde{S} - S)$ is the divisor of a rational function on Σ which takes the same value over the two points $P_\infty, \tilde{P}_\infty$ over ∞.*

Another way of saying this is to say that $\tilde{S} - S$ is a torsion divisor on the singular curve Σ' obtained from Σ by identifying the two points at infinity to obtain an ordinary double point.

Proof. By the proof of the previous proposition $2S-(O+\tilde{O})$ is the divisor of the rational section of $\mathcal{L}_{2j}\otimes\mathcal{L}_{2j+2}^{-1}$ represented by $T_{2j+1}\circ T_{2j}$, therefore $S-\tilde{S}$ is the divisor of $(1-\lambda^{-1})^{-1}T_{2j+1}\circ T_{2j}$. Observe that

$$(H_0^\lambda)^2=\prod_{j=n-1}^{0}T_{2j+1}\circ T_{2j}$$

(with the indices counted modulo n), so $(1-\lambda^{-1})^{-n}(H_0^\lambda)^2$ is a rational section of $\mathcal{L}_0\otimes\mathcal{L}_0^{-1}$ with divisor $n(S-\tilde{S})$. But H_0^λ is itself a section of $\mathcal{L}_0\otimes\mathcal{L}_0^{-1}\simeq\mathcal{O}_\Sigma$ and from (5) we see that any section v of $\mathcal{L}_0^{-1}$ satisfies

$$H_0^\lambda v=(p+\lambda^{-1}\mu)v,$$

where we recall that $p(\lambda)=\mathrm{Tr}(H_0^\lambda)/2$. Therefore H_0^λ represents $p+\lambda^{-1}\mu$, which takes the value 1 at any point where $\lambda^{-1}=0$ since $p(\infty)=1$. Therefore $n(S-\tilde{S})$ is the divisor for $(1-\lambda^{-1})^{-n}(p+\lambda^{-1}\mu)^2$, which also takes the value 1 wherever $\lambda^{-1}=0$.$\square$

2.4 Recovery of the discrete curve from its spectral data.

The spectral data $(\Sigma,\lambda,\mathcal{L},O,S)$ determines each $\mathcal{L}_k$ by proposition 2 if we take $\mathcal{L}_0=\mathcal{L}$. This is enough to give each map $f_k:\Sigma\to\mathbf{P}^1$ up to a Möbius transform: we take it from the natural map $\Sigma\to\mathbf{P}\Gamma(\mathcal{L}_k)^*$ which assigns to each point P the hyperplane $\Gamma(\mathcal{L}_k(-P))$. By lemma 3 we know $f_k(O_k)$ gives z_k upon an appropriate identification of $\mathbf{P}\Gamma(\mathcal{L}_k)$ with $\mathbf{P}^1$. So to recover the curve Γ we need only understand how this identification is fixed. Indeed it is clear that since Γ is only to be determined up to Möbius transformation what we really want to see is how each $\mathbf{P}\Gamma(\mathcal{L}_k)$ is identified with, say, $\mathbf{P}\Gamma(\mathcal{L}_0)$. This is achieved by first identifying each space $\Gamma(\mathcal{L}_k)$ with the sum of the two fibres of $\mathcal{L}_k$ over $\lambda=\infty$. We then interpret $T_k^\infty=I$ as identifying these fibres for $\mathcal{L}_k$ with the fibres for $\mathcal{L}_{k+1}$.

More precisely, for each k let $\mathcal{E}_k\subset\Sigma\times\mathbf{C}^2$ denote the eigenline bundle with dual $\mathcal{L}_k$. It follows that any linear form $e\in(\mathbf{C}^2)^*$, being a global section of $\Sigma\times(\mathbf{C}^2)^*$, induces a global section σ_k of $\mathcal{L}_k$. Now let τ_k denote the section of $\mathrm{Hom}(\mathcal{E}_k,\mathcal{E}_{k+1})\simeq\mathcal{L}_k\otimes\mathcal{L}_{k+1}^{-1}$ corresponding to the map T_k^λ. Then $e\circ T_k^\lambda$ represents the rational section $\sigma_{k+1}\otimes\tau_k$ of $\mathcal{L}_k$. Since $T_k^\infty=I$ we have the identity

$$(\sigma_{k+1}\otimes\tau_k)|P=\sigma_k|P\quad\text{for }\lambda(P)=\infty.$$

Here $\sigma|P$ denotes the section σ restricted to P. This uniquely determines σ_{k+1} given σ_k since no global section vanishes at both points over $\lambda = \infty$ (by lemma 4). Thus we have maps

$$t_k : \Gamma(\mathcal{L}_{k+1}) \to \Gamma(\mathcal{L}_k) \quad \text{where } t_k(\sigma)|\infty = (\sigma \otimes \tau_k)|\infty \tag{9}$$

and $\sigma|\infty$ denotes $(\sigma|P_\infty, \sigma|\tilde{P}_\infty)$. This uses the identification

$$\Gamma(\mathcal{L}_k) \to \mathcal{L}_k|P_\infty \oplus \mathcal{L}_k|\tilde{P}_\infty$$

which restricts sections to the two fibres over infinity.

For simplicity let V_k denote the sum of fibres on the right. Notice that to each point P on Σ we have a line in V_k, by evaluating the section vanishing at P at the two fibres over infinity. The lines for P_∞ and $\tilde{P}_\infty$ are independent and we choose a third point P to fix the identification of $\mathbf{P}V_0$ with $\mathbf{P}^1$ by sending these three lines to $0, \infty$ and 1 respectively. Combining this with the map t_k from (9) gives the identification of $\mathbf{P}V_k$ with $\mathbf{P}^1$ for every k (notice this only depends on the divisor of τ and not its scale). Since $\mathcal{L}$ has no sections which vanish at both $O, \tilde{O}$ any globally holomorphic section of $\mathcal{L}_k(-O_k)$ has divisor $D_k + O_k$ where D_k is a positive and non-special divisor of degree g.

Lemma 6 *Given a discrete curve Γ with spectral data as above, let ψ_k be the (unique up to scaling) non-zero rational function on Σ with divisor $D_k + E_k - D_0$ where $E_k = \sum_{j=0}^{k-1}(S - O_j)$. Then we recover Γ, up to a Möbius transform, as the image of the map $z : \mathbf{Z} \to \mathbf{P}^1$ given by $z_k = \psi_k(P_\infty)/\psi_k(\tilde{P}_\infty)$.*

Proof. Let σ_k generate $\Gamma(\mathcal{L}_k(-O_k))$, then σ_k has divisor $D_k + O_k$. According to (9) it determines a line in V_0 by evaluating the section $s_k = \sigma_k \otimes \tau_{k-1} \otimes \ldots \otimes \tau_0$ at P_∞ and $\tilde{P}_\infty$. The resulting line $[s_k|P_\infty, s_k|\tilde{P}_\infty]$ is then mapped to the line in $\mathbf{P}^1$ with homogeneous coordinates $[(s_k/\sigma)|P_\infty, (s_k/\sigma)|\tilde{P}_\infty]$ where σ is any section generating the line $\Gamma(\mathcal{L}_0(-P))$ corresponding to our third point P, according to the identification of $\mathbf{P}V_0$ with $\mathbf{P}^1$ fixed above. Since $\psi_k = s_k/s_0 = (s_k/\sigma)/(s_0/\sigma)$ this rational function determines a Möbius equivalent discrete curve. This function has divisor

$$D_k + O_k + \sum_{j=0}^{k-1}(S - \tilde{O}_j) - D_0 - O_0 = D_k + \sum_{j=0}^{k-1}(S - O_j) - D_0,$$

since $\tilde{O}_j = O_{j+1}$.$\square$.

Notice that we will not have $\psi_k(P_\infty) = 0 = \psi_k(\tilde{P}_\infty)$ since we have assumed that $\mathcal{L}(-P_\infty - \tilde{P}_\infty)$ has no global sections. We will postpone the explicit computation of the z_k until we have introduced discrete conformal maps.

3 Discrete Conformal Maps.

A discrete conformal map is a map $z : \mathbf{Z}^2 \to \mathbf{P}^1$ with the property that

$$[z_{k,m+1} : z_{k,m} : z_{k+1,m} : z_{k+1,m+1}] = q \tag{10}$$

for some constant $q \neq 0, 1, \infty$ for all k, m. We will be principally concerned with discrete conformal maps with one period i.e. we will assume there is an n such that $z_{k+n,m} = z_{k,m}$ for all $k, m \in \mathbf{Z}^2$. In that case we can also think of the map as describing the conformal flow of the discrete curve $\Gamma_{0,0} = (z_{0,0}, \ldots, z_{n-1,0})$.

To each discrete curve $\Gamma_{k,m}$ in this flow let us assign its spectral data $(\Sigma_{k,m}, \lambda_{k,m}, \mathcal{L}_{k,m}, O_{k,m}, S_{k,m})$.

Lemma 7 *The point $Q_{k,m} = (q, z_{k,m+1})$ lies on $\Sigma_{k,m}$.*

Proof. By (10) we see that

$$T^q_{k,m}(z_{k,m+1}) = z_{k+1,m+1}$$

for all k, m. It follows that $H^q_{k,m}(z_{k,m+1}) = z_{k,m+1}$.□

As earlier, we use (Σ, λ) to denote the isomorphism class of $(\Sigma_{0,0}, \lambda_{0,0})$. We define $Q \in \Sigma$ to be the point corresponding to $Q_{0,0}$ on $\Sigma_{0,0}$.

Proposition 3 *For each k, m there is an isomorphism $(\Sigma_{k,m}, \lambda_{k,m}) \simeq (\Sigma, \lambda)$ such that $S_{k,m}$ is mapped to S, $Q_{k,m}$ is mapped to Q but $O_{k,m}$ is mapped to O for $k + m$ even and $\tilde{O}$ for $k + m$ odd. Further:*

$$\begin{array}{c} \mathcal{L}_{k+1,m} \otimes \mathcal{L}^{-1}_{k,m} \simeq \mathcal{O}_\Sigma(\tilde{O}_{k,m} - S); \\ \mathcal{L}_{k,m+1} \otimes \mathcal{L}^{-1}_{k,m} \simeq \mathcal{O}_\Sigma(\tilde{O}_{k,m} - Q). \end{array} \tag{11}$$

The proof of this is identical to the proof of proposition 2 given the next lemma, which tells us how the holonomy changes under the conformal flow. Let us introduce the map $\hat{T}_{k,m} : \mathbf{P}^1 \to \mathbf{P}GL_2$ characterised by

$$[z : z_{k,m} : z_{k,m+1} : \hat{T}^\lambda_{k,m}(z)] = \lambda.$$

By earlier remarks this has matrix representation

$$\hat{T}^\lambda_{k,m} = I - \lambda^{-1} \hat{A}_{k,m}$$

where $\hat{A}_{k,m}$ is the projection matrix with kernel $z_{k,m}$ and image $z_{k,m+1}$. The following lemma tells us how the holonomy evolves as we change the base point (cf. [9] for a similar result about discrete isothermic nets).

Lemma 8 *The trace free part $M^{\lambda}_{k,m}$ of the holonomy for $\Gamma_{k,m}$ evolves according to*

$$\begin{aligned} M^{\lambda}_{k+1,m} &= T^{\lambda}_{k,m} \circ M^{\lambda}_{k,m} \circ (T^{\lambda}_{k,m})^{-1}; \\ M^{\lambda}_{k,m+1} &= \hat{T}^{\lambda/q}_{k,m} \circ M^{\lambda}_{k,m} \circ (\hat{T}^{\lambda/q}_{k,m})^{-1}. \end{aligned} \tag{12}$$

Proof. The first identity we know from earlier. To prove the second identity it suffices to show that

$$\hat{T}^{\lambda/q}_{k+1,m} \circ T^{\lambda}_{k,m} = T^{\lambda}_{k,m+1} \circ \hat{T}^{\lambda/q}_{k,m},$$

when (10) holds. If we expand the matrix representations for these maps as functions of λ^{-1} we see that this is equivalent to showing that:

$$\begin{aligned} &\text{(a) } \hat{A}_{k+1,m} A_{k,m} = A_{k,m+1} \hat{A}_{k,m}, \text{ and} \\ &\text{(b) } A_{k,m} + q\hat{A}_{k+1,m} = A_{k,m+1} + q\hat{A}_{k,m}. \end{aligned}$$

In (a) it is clear that on both sides the image of the first matrix is the kernel of the second, hence both sides are identically zero. For (b) we can compute the matrices explicitly. But this can be made easier by first mapping $(z_{k,m+1}, z_{k,m}, z_{k+1,m}, z_{k+1,m+1})$ to $(\infty, 1, 0, q)$ by Möbius transform. If we lift $z \in \mathbf{P}^1$ to $(z,1)^t$ (or $(1,0)^t$ when $z = \infty$), then elementary calculations show that:

$$\begin{aligned} A_{k,m} &= \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}; \quad q\hat{A}_{k+1,m} = \begin{pmatrix} q & 0 \\ 1 & 0 \end{pmatrix}; \\ A_{k,m+1} &= \begin{pmatrix} 0 & q \\ 0 & 1 \end{pmatrix}; \quad q\hat{A}_{k,m} = \begin{pmatrix} q & -q \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The identity required follows immediately. □

By arguments identical to those in the proof of proposition 2 we see that $\hat{T}^{\lambda}_{k,m}$ represents a rational section of $\mathcal{L}^{-1}_{k,m+1} \otimes \mathcal{L}_{k,m}$ with divisor $Q_{k,m} - \tilde{O}_{k,m}$ and combining this with proposition 2 we deduce proposition 3. Naturally this means the complete spectral data for a periodic discrete conformal map is the sextuple $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$. We will see later that any sextuple $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$ possessing the properties of proposition 1 is spectral data for a discrete conformal map.

3.1 Explicit formula for the discrete conformal map.

Given a discrete conformal map with generic spectral data we can write down an explicit formula for it (up to Möbius equivalence) in terms of the Riemann theta function of Σ. For this we need the analogue of lemma 6 proved earlier. Recall we identify each $\Gamma(\mathcal{L}_{k,m})$ with $\Gamma(\mathcal{L}_{0,0})$ in the following way. To $T^\lambda_{k,m}$ and $\hat{T}^\lambda_{k,m}$ we have corresponding sections $\tau_{k,m}$ of $\mathcal{L}_{k,m}\otimes\mathcal{L}^{-1}_{k+1,m}$ and $\hat{\tau}_{k,m}$ of $\mathcal{L}_{k,m}\otimes\mathcal{L}^{-1}_{k,m+1}$. Since every section $\sigma\in\Gamma(\mathcal{L}_{k,m})$ is determined entirely by its restriction $\sigma|\infty=(\sigma|P_\infty,\sigma|\tilde{P}_\infty)$ we may define bijective linear maps

$$\begin{array}{ll} t_{k,m}:\Gamma(\mathcal{L}_{k+1,m})\to\Gamma(\mathcal{L}_{k,m}) & \text{where } t_{k,m}(\sigma)|\infty=(\sigma\otimes\tau_{k,m})|\infty; \\ \hat{t}_{k,m}:\Gamma(\mathcal{L}_{k,m+1})\to\Gamma(\mathcal{L}_{k,m}) & \text{where } \hat{t}_{k,m}(\sigma)|\infty=(\sigma\otimes\hat{\tau}_{k,m})|\infty. \end{array} \tag{13}$$

To write this is terms of global sections we need to introduce the following divisors. First, let $D_{k,m}$ be the unique positive divisor in the linear system of $\mathcal{L}_{k,m}(-O_{k,m})$. Now define the divisors $E_{k,m}$ by taking $E_{0,0}$ to be trivial and making

$$\begin{aligned} E_{k+1,m}-E_{k,m}&=S-O_{k,m}\\ E_{k,m+1}-E_{k,m}&=Q-O_{k,m}\end{aligned}$$

Lemma 9 *Up to Möbius equivalence the discrete conformal map with spectral data $(\Sigma,\lambda,\mathcal{L},O,S,Q)$ is given by the map $z:\mathbf{Z}^2\to\mathbf{P}^1$ for which $z_{k,m}=\psi_{k,m}(P_\infty)/\psi_{k,m}(\tilde{P}_\infty)$, where $\psi_{k,m}$ is the (unique up to scaling) rational function on Σ with divisor*

$$(\psi_{k,m})=D_{k,m}+E_{k,m}-D_{0,0}. \tag{14}$$

The proof is the same as for lemma 6.

To explicitly compute the $z_{k,m}$, we need to fix a basis $\{a_j,b_j\}_{j=1}^g$ for the first homology of Σ with the standard intersection pairing. We use the a-cycles to fix a dual basis $\{\omega_j\}_{j=1}^g$ of holomorphic one forms and thereby equate $Jac(\Sigma)$ with $\mathbf{C}^g/\Lambda$ where Λ is the lattice representing the first homology via integration of the vector $(\omega_1,\ldots,\omega_g)$ over each homology class. Given a base point B on Σ we have the Abel map

$$\mathcal{A}:\Sigma\to\mathbf{C}^g/\Lambda\ ,\quad P\mapsto\int_B^P(\omega_1,\ldots,\omega_g)\bmod\Lambda,$$

and, more generally, its extension to divisors by addition i.e. $\mathcal{A}(P+Q)=\mathcal{A}(P)+\mathcal{A}(Q)$. Further, let ω_{g+1} be the unique meromorphic differential satisfying: (1) ω_{g+1} is holomorphic except at P_∞ and $\tilde{P}_\infty$ where it has simple poles of residue $1/2\pi i$ and $-1/2\pi i$ respectively; (2) its integral around any a-cycle is zero.

Given all this, we define maps $\alpha_j : \mathbf{Z}^2 \to \mathbf{C}$ for $j = 1, \ldots, g+1$ by setting $\alpha_j(0,0) = 0$ and

$$\begin{aligned} \alpha_j(k+1,m) - \alpha_j(k,m) &= \textstyle\int_{O_{k,m}}^{S} \omega_j; \\ \alpha_j(k,m+1) - \alpha_j(k,m) &= \textstyle\int_{O_{k,m}}^{Q} \omega_j. \end{aligned}$$

In the formula to follow we will write $\alpha'_{k,m} : \mathbf{Z}^2 \to \mathbf{C}^{g+1}$ for the map whose j-th component is $\alpha_j(k,m)$, for $j = 1, \ldots, g+1$ and $\alpha_{k,m} : \mathbf{Z}^2 \to \mathbf{C}^g$ for its projection onto the first g components. We think of the former as lying over the generalised Jacobian J' of the curve Σ' obtained by identifying P_∞ with $\tilde{P}_\infty$ to obtain a node. The group J' may be analytically realised as $\mathbf{C}^{g+1}/\Lambda'$ where Λ' represents the first homology of $\Sigma - \{P_\infty, \tilde{P}_\infty\}$, the open variety of smooth points on Σ', via integration of the augmented vector $(\omega_1, \ldots, \omega_{g+1})$ (see e.g. [15] p101). This point of view is useful for computing the periodicity conditions.

Theorem 1 *The formula*

$$z_{k,m} = \exp[2\pi i \alpha_{g+1}(k,m)] \frac{\theta(\mathcal{A}(P_\infty) + \alpha_{k,m} - \mathcal{A}(D_{0,0}) - \kappa)}{\theta(\mathcal{A}(\tilde{P}_\infty) + \alpha_{k,m} - \mathcal{A}(D_{0,0}) - \kappa)}, \qquad (15)$$

where κ is the vector of Riemann constants, recovers the discrete conformal map with spectral data $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$ up to Möbius transform. This map is periodic (in k) with period n precisely when $\alpha'_{k+n,m} \equiv \alpha'_{k,m} \bmod \Lambda'$ for some (and hence all) k, m.

Proof. We begin by computing a function $\psi_{k,m}$ with divisor (14). For any $A \in \Sigma$ let $\eta^A_{k,m}$ be the unique meromorphic differential on Σ with zero a-periods and simple poles only at A and $O_{k,m}$, where it has residues $1/2\pi i$ and $-1/2\pi i$ respectively. On the universal cover of Σ we may define functions $\beta_{k,m}$ by setting $\beta_{0,0} \equiv 0$ and

$$\begin{aligned} \beta_{k+1,m}(P) - \beta_{k,m}(P) &= \textstyle\int_B^P \eta^S_{k,m}, \\ \beta_{k,m+1}(P) - \beta_{k,m}(P) &= \textstyle\int_B^P \eta^Q_{k,m}. \end{aligned}$$

Then define

$$\psi_{k,m} = \exp(2\pi i \beta_{k,m}(P)) \frac{\theta(\mathcal{A}(P) + \alpha_{k,m} - \mathcal{A}(D_{0,0}) - \kappa)}{\theta(\mathcal{A}(P) - \mathcal{A}(D_{0,0}) - \kappa)} \qquad (16)$$

where every integral from B to P is along the same path. By a standard reciprocity formula for differentials (see e.g. [7])

$$\oint_{b_j} \eta^A_{k,m} = \int_{O_{k,m}}^{A} \omega_j.$$

Using this it is a simple exercise to check that $\psi_{k,m}$ is well-defined on Σ. By construction, the denominator vanishes precisely on $D_{0,0}$ and the exponential term contributes the divisor $E_{k,m}$. Moreover, by definition $\alpha_{k,m} = \mathcal{A}(E_{k,m})$ and $\mathcal{A}(D_{k,m}) = \mathcal{A}(D_{0,0}) - \mathcal{A}(E_{k,m})$ so the numerator vanishes precisely on $D_{k,m}$. Hence $\psi_{k,m}$ has divisor (14). By lemma 9 the discrete conformal map is, up to Möbius transform, given by $\psi_{k,m}(P_\infty)/\psi_{k,m}(\tilde{P}_\infty)$. However, by another reciprocity formula

$$\int_{O_{k,m}}^{A} \omega_{g+1} = \int_{\tilde{P}_\infty}^{P_\infty} \eta^A_{k,m}$$

hence $\beta_{k,m}(P_\infty) - \beta_{k,m}(\tilde{P}_\infty) = \alpha_{g+1}(k,m)$. Hence, up to multiplication by a constant independent of k, m, we obtain (15).

For the periodicity, we will give a proof which works even when Σ is singular, since we will compute examples with such spectral curves shortly. First recall (from e.g. [15]) that the curve Σ' has its own Abel map:

$$\mathcal{A}' : \Sigma - \{P_\infty, \tilde{P}_\infty\} \to J', \qquad P \mapsto \int_B^P (\omega_1, \ldots, \omega_{g+1}) \bmod \Lambda'.$$

With this notation we have

$$\alpha'_{k+n,m} - \alpha'_{k,m} = \mathcal{A}'(E_{k+n,m} - E_{k,m}) = \begin{cases} \mathcal{A}'(nS - \frac{n}{2}O - \frac{n}{2}\tilde{O}), & n \text{ even;} \\ \mathcal{A}'(nS - nO), & n \text{ odd.} \end{cases}$$

In particular this is independent of k, m. Also recall that Abel's theorem holds for Σ' i.e. $\mathcal{A}'(D) \equiv 0$ if and only if D is the divisor of a rational function on Σ' (equally, D is the divisor of a rational function on Σ taking the same value at $P_\infty, \tilde{P}_\infty$). Now the rational function $\psi_{k+n,m}/\psi_{k,m}$ has divisor $D_{k+n,m} - D_{k,m} + E_{k+n,m} - E_{k,m}$ so if $\mathcal{A}'(E_{k+n,m} - E_{k,m}) \equiv 0$ then $D_{k+n,m} - D_{k,m}$ is itself the divisor of a rational function on Σ. But each $D_{k,m}$ is positive of degree g and non-special, therefore $D_{k+n,m} = D_{k,m}$. Thus $\psi_{k+n,m}/\psi_{k,m}$ must take the same value at $P_\infty, \tilde{P}_\infty$, whence $z_{k+n,m} = z_{k,m}$.

On the other hand, we have seen earlier that when the map is periodic the function $p+\lambda^{-1}\mu$ (representing the eigenvalues of the holonomy matrix) has divisor $E_{k+n,m}-E_{k,m}$. Since $H_0^\infty = I$ this function takes the same values at $P_\infty, \tilde{P}_\infty$. $\square$

Singularities and the θ-divisor. The formula (15) need not give a discrete conformal map for all k,m. Indeed we expect there to be singularities when both translates of the θ-function are zero: this will occur on some codimension two subvariety of $\mathbf{C}^g$. However, and perhaps less obviously, the map will also have singularities whenever $\mathcal{L}_{k,m}(-P_\infty-\tilde{P}_\infty)$ fails to be non-special i.e. on some translate of the θ-divisor. It is interesting to see what happens in this circumstance: typically the map fails to be a discrete immersion i.e. adjacent points fail to be distinct (see Figure 3).

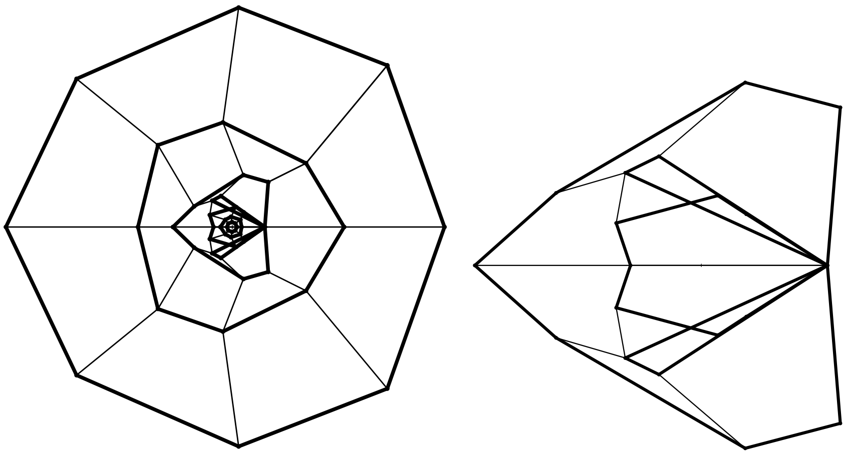

Figure 3: The collapsing of points at a singularity (with close up on the right). Points with the same m are joined by bolder lines.

Let us see why this occurs. Suppose that $\mathcal{L}(-P_\infty-\tilde{P}_\infty)$ is special Since all pairs $P+\tilde{P}$ are linearly equivalent this means the divisor class for $\mathcal{L}(-O-\tilde{O})$ contains a positive divisor (of degree $g-1$), E say, and we take $D=E+\tilde{O}$. Now let us define

$$f(P)=\exp(2\pi i\int_O^P\omega_{g+1})\frac{\theta(\mathcal{A}(P)-\mathcal{A}(D-P_\infty)-\kappa)}{\theta(\mathcal{A}(P)-\mathcal{A}(D-\tilde{P}_\infty)-\kappa)}$$

with the base point for the Abel map at O. In that case, unless $D-P_\infty, D-\tilde{P}_\infty$ are both special (which is the codimension two condition mentioned above), this is a well-defined rational function on Σ and a careful comparison with (15) shows that

$$z_{0,0} = f(O),\ z_{1,0} = f(S),\ z_{-1,0} = f(\tilde{S}),\ z_{0,1} = f(Q),\ z_{0,-1} = f(\tilde{Q}).$$

The divisor for f is $P_\infty - \tilde{P}_\infty + C - C'$ where C, C' are the unique positive divisors of degree g for which $C - O \sim D - P_\infty$ and $C' - O \sim D - \tilde{P}_\infty$. But $D = E + \tilde{O}$, $O + \tilde{O} \sim P_\infty + \tilde{P}_\infty$ and C, C' are unique, hence $C = E + \tilde{P}_\infty$ and $C' = E + P_\infty$. Therefore the poles and zeroes of f cancel i.e. f is constant, so the five points above are identical. In that case the cross-ratio condition breaks down locally.

Geometric interpretation. One of the reasons for choosing to write (15) in this form is to exhibit an elegant geometric interpretation which supports the claim (made in [11]) that the discrete conformal map equations are a discretization of the Schwarzian KdV (SKdV) equations:

$$z_t = S(z) z_x, \qquad S(z) = z_{xxx} z_x^{-1} - \frac{3}{2}(z_{xx} z_x^{-1})^2. \tag{17}$$

Here $S(z)$ is the Schwarzian derivative: it is well-known that $u(x,t) = 2S(g)$ satisfies the KdV equation. The 'finite gap' solutions of (17) are related to the formula (15) in the following way.

Let us assume that in the spectral data the point O is ramified (i.e. $O = \tilde{O}$). It is not hard to see that the projection $p : \mathbf{C}^{g+1} \to \mathbf{C}^g$ onto the first g coordinates projects the lattice Λ' onto Λ and therefore it induces a surjective homomorphism $p : J' \to Jac(\Sigma)$ whose kernel is $\Lambda'/\Lambda \simeq \mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^*$. Further, one can compute by means of multipliers that the pullback of the θ-line bundle over $Jac(\Sigma)$ to J' has its space of globally holomorphic sections spanned by the infinite collection

$$\{\theta_k(Z) = \exp(2\pi i k Z_{g+1})\theta(p(Z) + k\mathcal{A}(P_\infty - \tilde{P}_\infty)) :\ k \in \mathbf{Z},\ Z \in \mathbf{C}^{g+1}\}.$$

We see, therefore, that if we let U_P denote $\mathcal{A}'(P - O)$ then up to a scaling the formula (15) can be re-expressed as

$$z_{k,m} = \frac{\theta_1(kU_S + mU_Q + \tau)}{\theta_0(kU_S + mU_Q + \tau)}$$

for some constant $\tau \in \mathbf{C}^{g+1}$. Geometrically U_S and U_Q are the secants $\vec{OS}$ and $\vec{OQ}$ on $\mathcal{A}'(\Sigma_0)$ respectively.

On the other hand, if $U_1, U_3 \in \mathbf{C}^{g+1}$ represent, respectively, the tangent to $\mathcal{A}'(\Sigma_0)$ at O and its third derivative there (i.e. $U_1 = (\partial\mathcal{A}'/\partial\zeta)(0)$ and $U_3 = (\partial^3\mathcal{A}'/\partial\zeta^3)(0)$ for the local parameter $\zeta = \sqrt{\lambda}$), then

$$z(x,t) = \frac{\theta_1(xU_1 + tU_3 + \tau)}{\theta_0(xU_1 + tU_3 + \tau)}$$

satisfies the SKdV equation (17) (cf. the formula given at the end of [10]). Hence when viewed in the Jacobi variety (that is to say, after the equations have been linearised) the discretization is nothing other than the perturbation of a tangent into a secant on the spectral curve.

3.2 Examples.

Here we will present three examples, all based on taking Σ to be a rational nodal curve. The theory works equally well in this case and it is a good deal easier to calculate since the θ-functions are simply polynomials in exponentials. Also, the periodicity condition is easy to satisfy and we can obtain discrete maps with any period.

1. Σ is the Riemann sphere. Take Σ to be the Riemann sphere equipped with an affine coordinate ζ. In this case the θ-function is a constant which we may as well take to equal 1. We take the hyperelliptic involution to be $\zeta \mapsto -\zeta$ and prescribe the spectral data by $\zeta(O) = \epsilon$, $\zeta(S) = a$, $\zeta(Q) = b$ and $\zeta(P_\infty) = y$ where these are all distinct and different from $0, \infty$. It follows that

$$\lambda = \frac{(a^2 - y^2)}{(\zeta^2 - y^2)}\frac{(\zeta^2 - \epsilon^2)}{(a^2 - \epsilon^2)}.$$

Since $g = 0$ in this case we only need to compute ω_1 and its integrals. It is easy to see that $\omega_1 = \omega_y$ where

$$\omega_y = \frac{1}{2\pi i}\left(\frac{1}{\zeta - y} - \frac{1}{\zeta + y}\right)d\zeta \tag{18}$$

and therefore, following the procedure above, we see that $z_{k,m} = h_{k,m}(y)$ where

$$h_{k,m}(y) = \exp[2\pi i \alpha_1(k,m)] = \begin{cases} (\frac{a-y}{a+y})^k(\frac{b-y}{b+y})^m(\frac{\epsilon+y}{\epsilon-y}) & k+m \text{ odd}, \\ (\frac{a-y}{a+y})^k(\frac{b-y}{b+y})^m & k+m \text{ even}. \end{cases} \tag{19}$$

This map has cross-ratio $\lambda(Q)$ and period n whenever $(a+y)/(a-y)$ is an n-th root of unity (n must be even if $\epsilon \neq \infty$). Observe that this agrees with the periodicity condition given in theorem 1, for in this example Σ' is a one node curve and $J' = \mathbf{C}/\mathbf{Z}\langle \oint \omega_1 \rangle$, where the integral is around any cycle separating y from $-y$. It is not hard to see that parameter values can be chosen to give any cross-ratio with any period ≥ 4.

This is the general case of the zigzag: when $\epsilon = \infty$ we have a homomorphism $\mathbf{Z}^2 \to \mathbf{C}^*$. It is clear that this is the discrete exponential function $k, m \mapsto \exp(kA + mB)$ for constants A, B.

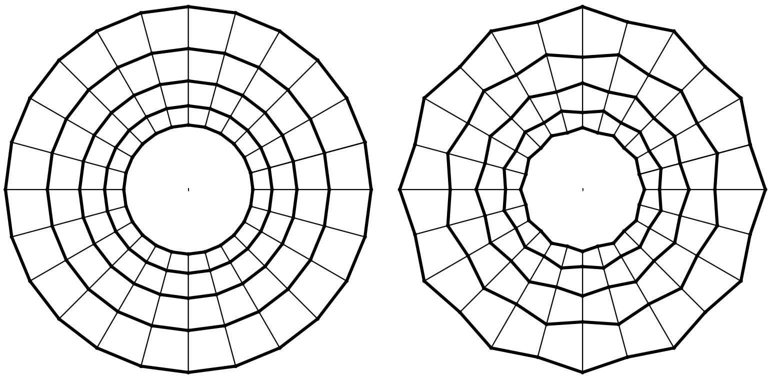

Figure 4: The discrete exponential (left) and a zigzag version (right).

2. Σ is a rational curve with one node. Here we take Σ to be $\mathbf{P}^1$ with the points $\pm x \neq 0, \infty$ identified together to obtain a node. In this case the θ-function is given by $\theta(Z) = \exp(2\pi i Z) - 1$. Here the space of regular one forms corresponds to the meromorphic forms on $\mathbf{P}^1$ with simple poles at $x, -x$ only (see [15] p68), hence the space is one dimensional and the (arithmetic) genus is $g = 1$. We keep the same notation as in the previous example and make the same choice for λ (so a, b, x, y, ϵ are all distinct). One easily checks that the appropriate choices for ω_1, ω_2 in this case are to take $\omega_1 = \omega_x$ and $\omega_2 = \omega_y$ using (18). By computing the integrals and applying the formula (15) we see that we can write

$$z_{k,m} = h_{k,m}(y) \left(\frac{\left(\frac{y-x}{y+x}\right) h_{k,m}(x) - e^{2\pi i c}}{\left(\frac{y+x}{y-x}\right) h_{k,m}(x) - e^{2\pi i c}} \right)$$

where $h_{k,m}(y)$ is given by (19) and the constant c represents the terms $\mathcal{A}(D_{0,0})+\kappa$ in (15). In this case Σ' is a two node curve with Jacobian $J'=\mathbf{C}/\mathbf{Z}\langle\oint\omega_1,\oint\omega_2\rangle$ where the integrals are around x and y respectively. According to theorem 1 the map is periodic when both $(a-y)/(a+y)$ and $(a-x)/(a+x)$ are n-th roots of unity (distinct, so that $x\neq\pm y$). In this case the periodicity problem can be solved for any period ≥ 5 for any value of the cross-ratio $\lambda(Q)$.

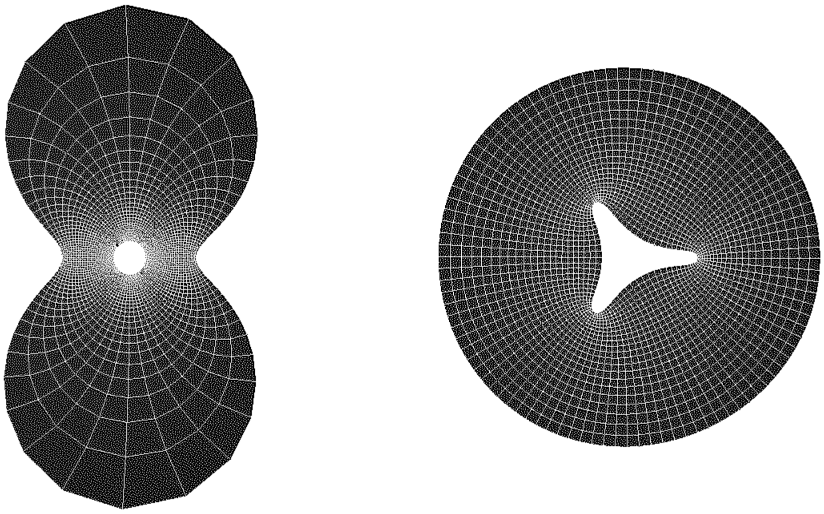

Figure 5: Two discrete 1-solitons: showing 2-fold and 3-fold symmetries.

Each of these maps behaves like a discrete 1-soliton in the sense that it has asymptotics in m like the discrete exponential (see Figure 5). Let $B=(b-y)/(b+y)$, then for $|\,B|<1$ we have

$$z_{k,m}/h_{k,m}(y)\to\begin{cases}1 \text{ as } m\to\infty;\\ (y-x)^2/(y+x)^2 \text{ as } m\to-\infty.\end{cases}$$

For $|B|>1$ the limits are interchanged.

3. Σ is a rational curve with two nodes. Take Σ to be $\mathbf{P}^1$ with $\pm x_1$, $\pm x_2$ identified in pairs. The θ-function here is given by

$$\theta(Z)=F(e^{2\pi iZ_1},e^{2\pi iZ_2}),\qquad F(X,Y)=\det\begin{pmatrix}X-1 & (Y+1)x_1\\ Y-1 & (X+1)x_2\end{pmatrix}.$$

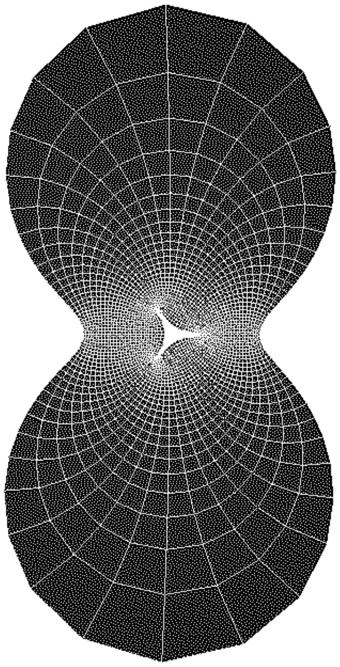

Figure 6: A discrete 2-soliton: superposition of 2-fold and 3-fold symmetries.

The arithmetic genus is $g = 2$ and $(\omega_1, \omega_2, \omega_3) = (\omega_{x_1}, \omega_{x_2}, \omega_y)$ using (18). Again, with λ chosen as above the appropriate computation yields

$$z_{k,m} = h_{k,m}(y)\frac{F((\frac{y-x_1}{y+x_1})h_{k,m}(x_1)e^{-2\pi ic_1}, (\frac{y-x_2}{y+x_2})h_{k,m}(x_2)e^{-2\pi ic_2})}{F((\frac{y+x_1}{y-x_1})h_{k,m}(x_1)e^{-2\pi ic_1}, (\frac{y+x_2}{y-x_2})h_{k,m}(x_2)e^{-2\pi ic_2})}$$

where c_1, c_2 are parameters corresponding to the initial point $\mathcal{A}(D_{0,0}) + \kappa$. The periodicity conditions can be solved for any period ≥ 7 and any cross-ratio $\lambda(Q)$. Each of these maps behaves like a 2-soliton in the sense that: (a) as $m \to \pm\infty$ it behaves like the discrete exponential, and; (b) by suitable choice of c_1, c_2 we observe that it behaves like two interacting 1-solitons (see Figure 6).

3.3 Spectral data which produces discrete conformal maps.

The aim of this section is to prove that any choice of spectral data satisfying the conditions described above yields a periodic discrete conformal map.

Theorem 2 *Let Σ be a compact hyperelliptic Riemann surface of genus g with: a degree two function λ unbranched at $\lambda = 1, \infty$; a degree $g+1$ line bundle $\mathcal{L}$ for which $\mathcal{L}(-P_\infty - \tilde{P}_\infty)$ is non-special; and non-singular points O, S, Q with $\lambda(O) = 0$, $\lambda(S) = 1$ and $\lambda(Q) \neq 0, 1, \infty$. Then the data $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$ is the spectral data for a discrete conformal map $z : \mathbf{Z}^2 \to \mathbf{P}^1$ of cross-ratio $\lambda(Q)$. This map is periodic with period n if and only if either: a) n is even and $\frac{n}{2}(2S - O - \tilde{O})$ is the divisor of a rational function on Σ taking the same value at each point over $\lambda = \infty$; or, b) n is odd and $n(S - O)$ is such a divisor.*

Remarks.(i) Strictly speaking it may be that $z_{k,m}$ has some singularities. These will occur only when $\mathcal{L}_{k,m}(-P_\infty - \tilde{P}_\infty)$ (where $\mathcal{L}_{k,m}$ is given by (11)) is special i.e. has a non-zero global section.
(ii) As we have seen in the examples above, the condition that Σ be smooth can be weakened to allow irreducible singular algebraic curves, provided the points $O, \tilde{O}, S, Q$ are all smooth. The proof we give works in this generality, in particular, we will not rely on the θ-function formula given earlier.

We can immediately deduce from this theorem a simple fact about smooth families of periodic discrete conformal maps.

Corollary 1 *Set $\Sigma^o = \Sigma - \lambda^{-1}(\{0, 1, \infty\})$. Up to Möbius equivalence, each discrete conformal map lies in a $g+1$-parameter family $z(Q, \mathcal{L})$ parameterised by $\Sigma^o \times Jac(\Sigma)$.*

Notice that if we swap Q with its involute $\tilde{Q}$ we get $z_{k,m}(\tilde{Q}, \mathcal{L}) = z_{k,-m}(Q, \mathcal{L})$, since $\mathcal{O}_\Sigma(\tilde{Q} - O) \simeq \mathcal{O}_\Sigma(O - Q)$.

Now let us turn to proving the theorem. We saw in the previous sections that the point $z_{k,m}$ is the image of the hyperplane $\Gamma(\mathcal{L}_{k,m}(-O_{k,m})) \subset \Gamma(\mathcal{L}_{k,m})$, where $O_{k,m}$ is O for $k + m$ even and $\tilde{O}$ otherwise, under an identification

$$\zeta_{k,m} : \mathbf{P}\Gamma(\mathcal{L}_{k,m})^* \to \mathbf{P}^1 \tag{20}$$

of $\mathbf{P}^1$ with the space $\mathbf{P}\Gamma(\mathcal{L}_{k,m})^*$ of hyperplanes in the two dimensional space $\Gamma(\mathcal{L}_{k,m})$. This identification is defined as follows. For each k, m there is, up to scaling, a unique section $\tau_{k,m}$ of $\mathcal{L}_{k,m}\otimes\mathcal{L}_{k+1,m}^{-1}$ with divisor $S-\tilde{O}_{k,m}$. Using (13) this fixes an isomorphism $t_{k,m}:\Gamma(\mathcal{L}_{k+1,m})\to\Gamma(\mathcal{L}_{k,m})$ which identifies sections by their behaviour over $\lambda=\infty$. Similarly using (13) we fix an isomorphism $\hat{t}_{k,m}:\Gamma(\mathcal{L}_{k,m+1})\to\Gamma(\mathcal{L}_{k,m})$ by choosing a section $\hat{\tau}_{k,m}$ of $\mathcal{L}_{k,m}\otimes\mathcal{L}_{k,m+1}^{-1}$ with divisor $Q-\tilde{O}_{k,m}$. Notice that $\hat{t}_{k,m}\circ t_{k,m+1}$ and $t_{k,m}\circ\hat{t}_{k+1,m}$ differ only by a scaling, since $\hat{\tau}_{k,m}\otimes\tau_{k,m+1}$ and $\tau_{k,m}\otimes\hat{\tau}_{k+1,m}$ differ only by a scaling (they are sections of the same line bundle and have the same divisor). Therefore we obtain, by composition, a well-defined isomorphism $\mathbf{P}\Gamma(\mathcal{L}_{k,m})\simeq\mathbf{P}\Gamma(\mathcal{L})$. Since the only freedom in the choice of $\tau_{k,m}$ and $\hat{\tau}_{k,m}$ is the scale, which is irrelvant when we pass to the projective spaces, this construction depends only on the spectral data. Finally, by fixing an identification of $\mathbf{P}\Gamma(\mathcal{L})^*$ with $\mathbf{P}^1$ we obtain the maps $\zeta_{k,m}$. Since the last step is not determined by the spectral data the discrete map is only determined up to Möbius equivalence.

At this point we can see why, under the conditions of the theorem, the map z should be periodic. It suffices to show that $\zeta_{k+n,m}=\zeta_{k,m}$. Consider the map $r_{k,m}=t_{k+n-1,m}\circ\ldots\circ t_{k,m}$: by (13) it is characterised by

$$r_{k,m}:\Gamma(\mathcal{L}_{k,m})\to\Gamma(\mathcal{L}_{k,m})\quad\text{where}\quad r_{k,m}(\sigma)|\infty=(\sigma\otimes h_{k,m})|\infty,$$

for $h_{k,m}=\tau_{k+n-1,m}\otimes\ldots\otimes\tau_{k,m}$. Notice that $h_{k,m}$ is a rational section of $\mathcal{L}_{k,m}\otimes\mathcal{L}_{k+n,m}^{-1}$ with divisor $\frac{n}{2}(2S-O-\tilde{O})$ for n even and $n(S-O)$ for n odd. Therefore, under the conditions of the theorem, $h_{k,m}$ is a rational function on Σ with $h_{k,m}(P_\infty)=h_{k,m}(\tilde{P}_\infty)$. It follows that $r_{k,m}$ is a scalar multiple of the identity whence $\zeta_{k+n,m}=\zeta_{k,m}$.

Now we must show that this recipe produces discrete conformal maps. First we observe the following convenient result.

Lemma 10 *The map $\zeta_{k,m}$ sends $\Gamma(\mathcal{L}_{k,m}(-S))$ and $\Gamma(\mathcal{L}_{k,m}(-Q))$ to $z_{k+1,m}$ and $z_{k,m+1}$ respectively.*

Proof. It suffices to show that the maps $t_{k,m}$ and $\hat{t}_{k,m}$ in (13) send, respectively, $\Gamma(\mathcal{L}_{k+1,m}(-O_{k+1,m}))$ to $\Gamma(\mathcal{L}_{k,m}(-S))$ and $\Gamma(\mathcal{L}_{k,m+1}(-O_{k,m+1}))$ to $\Gamma(\mathcal{L}_{k,m}(-Q))$. We will prove the former: the latter use the same proof with S replaced by Q. Let σ generate the line $\Gamma(\mathcal{L}_{k+1,m}(-O_{k+1,m}))$, then σ is a holomorphic section with a zero at $O_{k+1,m}=\tilde{O}_{k,m}$ and this is where $\tau_{k,m}$ has a simple pole. Hence $\sigma\otimes\tau_{k,m}$ is a holomorphic section of $\mathcal{L}_{k,m}$ with a zero at S i.e. it generates $\Gamma(\mathcal{L}_{k,m}(-S))$. □

To prove the theorem we will show that the action of tensoring sections with $\tau_{k,m}$ is represented by a matrix of the form (3) and deduce from this that the cross-ratios are constant. Since this matrix really acts on the eigenline bundle $\mathcal{E}_{k,m}$ dual to $\mathcal{L}_{k,m}$ we must take some care to describe the relationship between rational sections of $\mathcal{E}_{k,m}$ and global sections of $\mathcal{L}_{k,m}$.

Lemma 11 *Let $\mathcal{L}$ be a line bundle of degree $g+1$ over the genus g hyperelliptic curve Σ with $\Gamma(\mathcal{L}(-P-\tilde{P}))=0$ (for some $P \in \Sigma$) and let $\mathcal{E}$ denote the dual line bundle. Then $\Gamma(\mathcal{L})$ is canonically dual to $\Gamma(\mathcal{E}(R))$ where R is the ramification divisor of λ. This duality identifies the hyperplane $\Gamma(\mathcal{L}(-P))$ with the line $\Gamma(\mathcal{E}(R-\tilde{P}))$.*

Proof. Let $\mathcal{F}$ be the field of rational functions on X and let $\mathcal{K}$ denote the subfield of rational functions of λ. Clearly $\mathcal{F}$ is a two dimensional $\mathcal{K}$-space and we have a $\mathcal{K}$-linear map $\mathrm{Tr} : \mathcal{F} \to \mathcal{K}$ which gives to each element $f \in \mathcal{F}$ the trace of the matrix in $\mathfrak{gl}_2(\mathcal{K})$ representing the multiplication map $a \mapsto fa$ on $\mathcal{F}$ (since the trace is invariant this is independent of the $\mathcal{K}$-basis chosen for $\mathcal{F}$). It is easy to check that: (a) $\mathrm{Tr}(f)$ is globally holomorphic precisely when its divisor of poles is no worse than the ramification divisor R of λ; (b) for any such function f, $\mathrm{Tr}(f)=0$ if its divisor of zeroes includes $P+\tilde{P}$ (for some $P \in \Sigma$). We have a non-degenerate $\mathcal{K}$-bilinear form on $\mathcal{F}(\mathcal{E}) \times \mathcal{F}(\mathcal{L})$ by $(v,e) \mapsto \mathrm{Tr}(e(v))$ which, by the properties (a) and (b), pairs $\Gamma(\mathcal{L})$ non-degenerately with $\Gamma(\mathcal{E}(R))$. Further, if $e \in \Gamma(\mathcal{L}(-P))$ then $\mathrm{Tr}(e(v))=0$ if and only if $v \in \Gamma(\mathcal{E}(R-\tilde{P}))$.$\square$

Since $\mathcal{E}(R)$ also has the property $\Gamma(\mathcal{E}(R-P-\tilde{P}))=0$ it follows that $\mathcal{F}(\mathcal{E}))=\mathcal{K} \otimes_{\mathbf{C}} \Gamma(\mathcal{E}(R))$. As a result we may draw the following commuting diagram with which we define the map $T_{k,m}$:

$$\begin{array}{ccc} \mathcal{F}(\mathcal{E}_{k,m}) & \stackrel{\otimes\tau_{k,m}}{\to} & \mathcal{F}(\mathcal{E}_{k+1,m}) \\ \downarrow & & \downarrow \\ \mathcal{K}\otimes\Gamma(\mathcal{L}_{k,m})^* & \stackrel{T_{k,m}}{\to} & \mathcal{K}\otimes\Gamma(\mathcal{L}_{k+1,m})^* \end{array} \tag{21}$$

Now when we identify $\Gamma(\mathcal{L}_{k,m})^*$ with $\mathbf{C}^2$ using any lift of $\zeta_{k,m}$ we obtain a $\mathcal{K}$-valued matrix we will define to be $T^{\lambda}_{k,m}$. The theorem will follow from the next proposition.

Proposition 4 *The matrix $T^{\lambda}_{k,m}$ obtained from (21) is of the form $I - \lambda^{-1}A_{k,m}$ where: (i) $\ker(A_{k,m}) = z_{k,m}$; (ii) $\mathrm{im}(A_{k,m}) = z_{k+1,m} = \ker(I-A_{k,m})$; and (iii) $I-q^{-1}A_{k,m}$ maps $z_{k,m+1}$ to $z_{k+1,m+1}$ for $q=\lambda(Q)$. It follows that $[z_{k,m+1} : z_{k,m} : z_{k+1,m} : z_{k+1,m+1}] = q$ and therefore the map z is discrete conformal.*

Proof. To begin, observe that $T^{\infty}_{k,m}$ represents the map $\sigma \to \sigma \otimes \tau_{k,m}$ over $\lambda = \infty$. By (13) this is the identity. Further, since $\tau_{k,m}$ has degree one it follows that there is a constant matrix $A_{k,m}$ such that $T^{\lambda}_{k,m} = I - \lambda^{-1} A_{k,m}$. To prove (i), (ii) and (iii) we consider three different values of λ.

First, $A_{k,m} = (\lambda T^{\lambda}_{k,m})|_{\lambda=0}$ represents $\lambda\tau_{k,m}$ over $\lambda = 0$. But $\lambda\tau_{k,m}$ has a simple zero at $O_{k,m}$ and none at $\tilde{O}_{k,m}$, hence $(\sigma \otimes \lambda\tau_{k,m})|_{\lambda=0} = 0$ if and only if σ vanishes at $\tilde{O}_{k,m}$. By lemma 11 $z_{k,m}$ corresponds to $\Gamma(\mathcal{E}_{k,m}(R - \tilde{O}_{k,m}))$ and $z_{k+1,m}$ corresponds to $\Gamma(\mathcal{E}_{k+1,m}(R - O_{k,m}))$ (since $\tilde{O}_{k+1,m} = O_{k,m}$) so $\ker(A_{k,m}) = z_{k,m}$ while $\mathrm{im}(A_{k,m}) = z_{k+1,m}$.

Secondly, consider $I - A_{k,m}$, which represents $\tau_{k,m}$ over $\lambda = 1$. But $\tau_{k,m}$ has a zero at S so $(\sigma \otimes \tau_{k,m})|_{\lambda=1} = 0$ if and only if σ has a zero at $\tilde{S}$.Recall that $z_{k+1,m}$ corresponds to $\Gamma(\mathcal{E}_{k,m}(R - \tilde{S}))$ by lemmas 10 and 11 so $z_{k+1,m} = \ker(I - A_{k,m})$.

Thirdly, $I - q^{-1}A_{k,m}$ represents $\tau_{k,m}$ over $\lambda = q$. Here $\tau_{k,m}$ has neither zeroes nor poles so $\sigma \otimes \tau_{k,m}$ has a zero at $\tilde{Q}$ if and only if σ does. Since $z_{k,m+1}$ corresponds to $\Gamma(\mathcal{E}_{k,m}(R - \tilde{Q}))$ and $z_{k+1,m+1}$ corresponds to $\Gamma(\mathcal{E}_{k+1,m}(R - \tilde{Q}))$ (by lemmas 10 and 11) we see that $I - q^{-1}A_{k,m}$ maps $z_{k,m+1}$ to $z_{k+1,m+1}$.

Finally, it is an elementary computation (which we will leave to the reader) to establish that for $A_{k,m}$ to have these properties we are obliged to have $[z_{k,m+1} : z_{k,m} : z_{k+1,m} : z_{k+1,m+1}] = q$.$\square$

4 The Lax pair and a loop group action.

In [11] Nijhoff and Capel wrote down a Lax pair for the discrete conformal map equations. Using this we will show that all discrete conformal maps: (i) come in 1-parameter families obtained by deforming the cross-ratio (we have already seen this for periodic maps in corollary 1); (ii) admit non-trivial transformations by elements of an infinite dimensional Lie group (in fact, a loop group). To begin, let us define $\mathcal{Z}_q = \{z : \mathbf{Z}^2 \to \mathbf{C} |$ discrete conformal with cross-ratio q and $z_{0,0} = 0\}$. We will say maps of this type are based at 0. Since we are working with maps into the plane we may define

$$u_{k,m} = z_{k+1,m} - z_{k,m} \ , \qquad v_{k,m} = z_{k,m+1} - z_{k,m}. \tag{22}$$

Now fix a pair $\alpha, \beta \in \mathbf{C}$ for which $\beta^2/\alpha^2 = q$ and define

$$U_{k,m}(\lambda) = \begin{pmatrix} 1 & \alpha^2 u^{-1}_{k,m}\lambda^{-1} \\ u_{k,m} & 1 \end{pmatrix} \ , \qquad V_{k,m}(\lambda) = \begin{pmatrix} 1 & \beta^2 v^{-1}_{k,m}\lambda^{-1} \\ v_{k,m} & 1 \end{pmatrix}. \tag{23}$$

One readily computes that the pair U, V is a discrete Lax pair i.e.

$$U_{k,m}V_{k+1,m} = V_{k,m}U_{k,m+1}. \tag{24}$$

We consider these as maps $\mathbf{Z}^2 \to \mathcal{G}$ where $\mathcal{G}$ is the infinite dimensional Lie group of holomorphic maps from $\mathbf{P}^1_\lambda \backslash \{0, \alpha^2, \beta^2\}$ to GL_2. By the Lax equations (24) there exists a unique map $\Phi : \mathbf{Z}^2 \to \mathcal{G}$ satisfying:

$$\Phi_{k+1,m} = \Phi_{k,m}U_{k,m} \ ; \qquad \Phi_{k,m+1} = \Phi_{k,m}V_{k,m} \ ; \qquad \Phi_{0,0} = I. \tag{25}$$

We will call such maps *extended frames.* Each $z \in \mathcal{Z}_q$ has a unique extended frame Φ and z is recovered from the first column of $\Phi(\infty)$. To see this simply observe that Φ is constructed inductively from (25) and is uniquely specified by the initial condition $\Phi_{0,0} = I$. One readily checks that

$$\begin{pmatrix} 1 & 0 \\ z_{k,m} & 1 \end{pmatrix}$$

satisfies (25) for $\lambda = \infty$ so, by uniqueness, this must be $\Phi_{k,m}(\infty)$.
Example: the vacuum solution. The simplest example of a discrete conformal map is the tiling of the plane by a parallelogram of cross-ratio q i.e.

$$z_{k,m} = k\alpha + m\beta.$$

We will adopt a common terminology from soliton theory and call this solution the 'vacuum solution': it corresponds to having $u_{k,m} = \alpha, v_{k,m} = \beta$ for all k, m. In this case it is easy to check that for this map the extended frame is

$$\Phi_{k,m} = (I + \alpha\Lambda)^k (I + \beta\Lambda)^m \ , \quad \text{where } \Lambda = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}. \tag{26}$$

Now let us show that each $z \in \mathcal{Z}_q$ gives rise to a 1-parameter family of discrete conformal maps. Set $e_1 = (1,0)^t, e_2 = (0,1)^t$ and let $B_0 \subset GL_2$ denote the subgroup of upper triangular matrices: this is the stabilizer of the line generated by e_1. For each $\lambda \in \mathbf{P}^1_\lambda \backslash \{0, \alpha^2, \beta^2\}$ define $z(\lambda) : \mathbf{Z}^2 \to \mathbf{P}^1$ by taking $z_{k,m}(\lambda)$ to be the line generated by $F_{k,m}(\lambda) = \Phi_{k,m}(\lambda)e_1$. We think of $F(\lambda) : \mathbf{Z}^2 \to \mathbf{C}^2$ as a lift of $z(\lambda)$.

Proposition 5 *The map $z(\lambda)$ is discrete conformal with cross-ratio $\beta^2(1 - \lambda^{-1}\alpha^2)/\alpha^2(1 - \lambda^{-1}\beta^2)$. If $z, \hat{z} \in \mathcal{Z}_q$ are Möbius equivalent maps then so are $z(\lambda), \hat{z}(\lambda)$ for each λ.*

Proof. The cross-ratio of the map $z(\lambda)$ may be written using the lift $F(\lambda)$ as

$$\frac{\det(F_{k,m+1}(\lambda)\ F_{k,m}(\lambda))}{\det(F_{k,m}(\lambda)\ F_{k+1,m}(\lambda))}\frac{\det(F_{k+1,m}(\lambda)\ F_{k+1,m+1}(\lambda))}{\det(F_{k+1,m+1}(\lambda)\ F_{k,m+1}(\lambda))}.$$

where if $a, b \in \mathbf{C}^2$ then $(a\ b)$ denotes the matrix with these columns. But $F_{k+1,m} = \Phi_{k,m}U_{k,m}e_1$ and $F_{k,m+1} = \Phi_{k,m}V_{k,m}e_1$. Inserting these expressions and using (23), (24) this reduces to

$$\frac{\det\begin{pmatrix} 1 & 1 \\ v_{k,m} & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ 0 & u_{k,m} \end{pmatrix}}\frac{\det\begin{pmatrix} 1 & 1 \\ 0 & v_{k+1,m} \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ u_{k,m+1} & 0 \end{pmatrix}}\frac{\det U_{k,m}}{\det V_{k,m}} = \frac{\beta^2(1-\lambda^{-1}\alpha^2)}{\alpha^2(1-\lambda^{-1}\beta^2)},$$

Now suppose $z, \hat{z}$ are Möbius equivalent. Since they are both based at 0 it suffices to consider two types of Möbius transforms: 1) $\hat{z} = cz$; 2) $\hat{z} = z/(1-cz)$ for any non-zero constant c. We wish to show that in each case there exists $a : \mathbf{Z}^2 \to B_0$ for which

$$\hat{U}_{k,m} = a_{k,m}^{-1}U_{k,m}a_{k+1,m}, \quad \hat{V}_{k,m} = a_{k,m}^{-1}V_{k,m}a_{k+1,m}.$$

For then the extended frames $\Phi, \hat{\Phi}$ satisfy $\hat{\Phi}_{k,m} = a_{0,0}^{-1}\Phi_{k,m}a_{k,m}$, whence $\hat{z}(\lambda) = a_{0,0}^{-1} \circ z(\lambda)$ (Möbius action). We leave it to the reader to verify that for the two cases above we can take:

$$1)\ a_{k,m} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \qquad 2)\ a_{k,m} = \begin{pmatrix} z_{k,m} - c^{-1} & -1 \\ 0 & 1/(z_{k,m} - c^{-1}) \end{pmatrix}. \square$$

Remark. Notice that each $z(\lambda)$ will be based at 0, although we cannot guarantee that it maps into $\mathbf{C} \subset \mathbf{P}^1$. The parameterisation of this family $z(\lambda)$ depends upon the choice of α^2, β^2 but the family itself clearly does not, since a rescaling of λ will allow us to move through all pairs α, β such that $\beta^2/\alpha^2 = q$.

Corollary 2 *Any Lax pair of the form (23) produces a 1-parameter family of discrete conformal maps.*

This follows immediately from the first part of the previous proof.

4.1 The dressing action.

Let us now fix q and choose α, β so that $|\alpha|, |\beta| < 1$ (such a pair can clearly be found for each q). Then we may view U, V (and hence Φ) as taking values in the connected loop group $LGL_2 = \{g \in C^\omega(S^1, GL_2) : \det(g)$ has winding number zero$\}$. In fact U, V and Φ all take values in the subgroup $N = \{g \in LGL_2 : g$ extends holomorphically into the disc $|\lambda^{-1}| < 1$ with $g(\infty)$ lower unipotent $\}$ (we will say a matrix A is lower unipotent if $I - A$ is strictly lower triangular). The dressing action uses the fact that almost every $g \in LGL_2$ can be factorised into a product $g = g_N.g_B$ where $g_N \in N$ and g_B belongs to the subgroup $B = \{g \in LGL_2 : g$ extends holomorphically into $|\lambda| < 1$ with $g(0) \in B_0\}$. One knows (from e.g. [12]) that the space $N.B$ of all products is open dense in LGL_2. Given an extended frame Φ and $g \in B$ we might expect $g\Phi$ to map into $N.B$, in which case we could define $g \circ \Phi = (g\Phi)_N$ and therefore we have

$$g\Phi_{k,m} = (g \circ \Phi)_{k,m} \Psi_{k,m} \ , \quad \text{for some } \Psi : \mathbf{Z}^2 \to B. \tag{27}$$

Lemma 12 *For any $g \in B$ for which it exists, the map $g \circ \Phi : \mathbf{Z}^2 \to N$ is again the extended frame for a discrete conformal map $g \circ z$ of cross-ratio β^2/α^2.*

Proof. It suffices to show that the Lax pair for $g \circ \Phi$ is again of the form (23), since it is clear that $g \circ \Phi_{0,0} = g_N = I$. Observe that $g \circ \Phi = g\Phi\Psi^{-1}$ so that we can define

$$\begin{aligned} \hat{U}_{k,m} &= (g \circ \Phi)^{-1}_{k,m}(g \circ \Phi)_{k+1,m} = \Psi_{k,m} U_{k,m} \Psi^{-1}_{k+1,m} \ , \\ \hat{V}_{k,m} &= (g \circ \Phi)^{-1}_{k,m}(g \circ \Phi)_{k,m+1} = \Psi_{k,m} V_{k,m} \Psi^{-1}_{k,m+1} \ . \end{aligned}$$

Since $\Psi_{k,m} \in B$ we can expand it in Fourier series as

$$\Psi_{k,m} = \begin{pmatrix} a_{k,m} & b_{k,m} \\ 0 & c_{k,m} \end{pmatrix} + O(\lambda).$$

Consequently we compute

$$\Psi_{k,m} U_{k,m} \Psi^{-1}_{k+1,m} = \alpha^2 \lambda^{-1} \begin{pmatrix} 0 & e_{k,m} \\ 0 & 0 \end{pmatrix} + O(1),$$

for some expression $e_{k,m}$. But $g \circ \Phi$ takes values in N so the only other terms in this expression are constant in λ and lower unipotent i.e.

$$\hat{U}_{k,m} = \begin{pmatrix} 1 & \alpha^2\lambda^{-1}e_{k,m} \\ \hat{u}_{k,m} & 1 \end{pmatrix},$$

for some $\hat{u}_{k,m}$. Now we claim that $\det(\Psi_{k,m}) = \det(g)$ for all k, m, whence $\det(\hat{U}_{k,m}) = \det(U_{k,m}) = 1 - \alpha^2\lambda^{-1}$ so that $e_{k,m} = \hat{u}_{k,m}^{-1}$ and we are done. To see the claim, notice that

$$\det(\Phi)\det(g \circ \Phi)^{-1} = \det(g)^{-1}\det(\Psi).$$

The terms on the left extend holomophically into $|\lambda| < 1$ while on the right they extend holomorphically into $|\lambda^{-1}| < 1$. Therefore both sides are constant and by evaluating at $k, m = 0$ we see this constant is 1. A similar argument for $\hat{V}$ finishes the proof.□

Formally, at least, one can check that $\Phi \mapsto g \circ \Phi$ is a (left) group action and therefore we obtain an action of B on $\mathcal{Z}_q$. By analogy with the theory of the KdV equation we call this the dressing action (see e.g.[16]). The obstruction to making this more rigorous is that we cannot guarantee that $g \circ \Phi$ exists off $(0,0) \in \mathbf{Z}^2$ (compare with the KdV theory where one has at least local solutions guaranteed).

Remark: the effect of the scaling $\alpha, \beta \mapsto k\alpha, k\beta$**.** For fixed q we are only permitted to change α, β by the scaling $k\alpha, k\beta$, where we require $|k| < \min\{|\alpha|^{-1}, |\beta|^{-1}\}$. If $U(\lambda), V(\lambda)$ is the Lax pair for z, α, β then the Lax pair for $z, k\alpha, k\beta$ is $U(k^{-2}\lambda), V(k^{-2}\lambda)$. It is clear that our choice of working with loops on the unit circle limits us if the scaling is not unimodular. But this limitation is unnecessary: we could equally well work with loops on any circle and apply the dressing theory. It is not hard to see that by rescaling the unit circle the dressed map $g \circ z$ obtained using α, β can be obtained using $k\alpha, k\beta$ and $g(k^{-2}\lambda)$.

For the next result, we think of B_0 as the subgroup of constant loops in B.

Lemma 13 *The dressing action of the subgroup $B_0 \subset B$ corresponds precisely to the action of B_0 as the full group of base point preserving Möbius transformations.*

Proof. Let $z \in \mathcal{Z}_q$ and let Φ be its extended frame, so $F = \Phi(\infty)e_1$ lifts z into $\mathbf{C}^2$. Take $g \in B_0$, then $g\Phi$ extends holomorphically into $|\lambda^{-1}| < 1$. If

$g\Phi_{k,m}$ factorises then $g \circ z$ (i.e. the action of g by Möbius transforms) has lift

$$gF = (g\Phi)_N(\infty)(g\Phi)_B(\infty)e_1 = k(g \circ \Phi)(\infty)e_1, \quad \text{for some } k \in \mathbf{C},$$

since $(g\Phi)_B(\infty)$ takes value in B_0 which is the stabilizer of the line generated by e_1. Now we observe that $g\Phi_{k,m}$ factorises precisely when $(g\Phi_{k,m})(\infty)e_1 \neq e_2$, i.e. when $z_{k,m}$ belongs to the plane $\mathbf{C}$. $\square$

Now observe that B_0 is a normal subgroup of B and $B/B_0 \simeq LG_+ = \{g \in B : g(0) = I\}$. Since the quotient space $\mathcal{Z}_q/B_0$ can be identified with the space of all Möbius equivalence classes of discrete conformal maps we have an induced action of LG_+ on $\mathcal{Z}_q/B_0$. This is what we shall focus on from now on.

4.2 Dressing orbit of the vacuum solution.

Our main interest in the dressing action is to examine the dressing orbit of the vacuum solution, which we will denote by $z^{(0)}$. Throughout this section Φ will be the extended frame (26) for that solution. First we will describe this dressing orbit as a quotient space. It can be shown (by a tedious but straightforward argument using Fourier expansions) that $g \circ \Phi = \Phi$ if and only if $g \in \Gamma_+ = \{g \in B : g\Lambda = \Lambda g\}$. Therefore the dressing orbit of Φ is identifiable with B/Γ_+, so the dressing orbit of the vacuum solution in $\mathcal{Z}_q/B_0$ is identifiable with the double coset space $B_0\backslash B/\Gamma_+$. Since B_0 is a normal subgroup the group LG_+ acts transitively on the left so we may identify $B_0\backslash B/\Gamma_+$ with the quotient space LG_+/Γ_+. Here the action of Γ_+ on LG_+ is given by $\gamma \cdot g = \gamma(0)^{-1}g\gamma$.

The principal result in this section is that this dressing orbit contains all those periodic discrete conformal maps which are "small perturbations" of their corresponding SKdV solution in the following sense.

Theorem 3 *Let $z : \mathbf{Z}^2 \to \mathbf{P}^1$ be a periodic discrete conformal map (of cross-ratio q) with spectral data $(\Sigma, \lambda, O, S, Q)$ for which: (a) $\lambda = 0$ is a branch point (i.e. $O=\tilde{O}$); (b) the disc $|\lambda| \leq \max\{1, |q|\}$ contains no other branch points. Then z is in the dressing orbit of the vacuum solution.*

To prove this we will use the Grassmannian picture of the dressing orbit of the KdV vacuum solution developed by Segal and Wilson [14](see also [12]). Let $H = L^2(S^1, \mathbf{C})$ and recall that LGL_2 acts on this space by 'inter-leaving Fourier series' i.e. we use the isometric isomorphism between H and

$L^2(S^1, \mathbf{C}^2)$ given by $f(\zeta) \mapsto (f_0(\lambda), f_1(\lambda))^t$ where $f_0(\zeta^2) + \zeta^{-1} f_1(\zeta^2) = f(\zeta)$. The usual action of LGL_2 on $\mathbf{C}^2$-valued functions then passes to an action on H which we will denote by $g \cdot f$. In particular, the action of the commutative subgroup $\Gamma = \{g \in LGL_2 : g\Lambda = \Lambda g\}$ on H corresponds to the multiplication action of $C^\omega(S^1, \mathbf{C}^*)$. This follows from the observation that $\Lambda \cdot f(\zeta) = \zeta^{-1} f(\zeta)$. Therefore we have

$$\Phi_{k,m} \cdot f(\zeta) = (1 + \alpha\zeta^{-1})^k (1 + \beta\zeta^{-1})^m f(\zeta) \tag{28}$$

for the vacuum extended frame.

Next, recall that H has an orthogonal decomposition $H_- \oplus H_+$ where H_- (respectively, H_+) is the subspace of functions whose Fourier series possess only non-positive (respectively, positive) powers of ζ. [Readers familiar with [12, 14] should note that we have applied the involution $\lambda \mapsto \lambda^{-1}$ to the the picture described there.] We will define the Grassmannian to be $Gr = \{g \cdot H_- : g \in LGL_2\}$. We recall that the orbit of H_- under the subgroup LG_+ is open dense (the 'big cell') and that it is characterised as being the set of all $W \in Gr$ for which the orthogonal projection $pr_- : W \to H_-$ is invertible. Our assumption about the existence of $g \circ \Phi$ implies that $\Phi_{k,m}^{-1} \cdot W$ belongs to the big cell for all k, m, where $W = g^{-1} \cdot H_-$.

We will now associate to each $W \in Gr$ the discrete analogue of the Baker function used in [14]. First we observe that we may refine our current factorisation $N.B$ into $N.T.\tilde{N}$ where T denotes the subgroup of constant diagonal loops and $\tilde{N} = \{g \in B : g(0) \text{ is upper unipotent}\}$. Thus any $g \in N.B$ can be factorised into $g_N g_T g_{\tilde{N}}$. Let us now fix a $g \in LG_+$ and define

$$\psi = \Phi \Psi_{\tilde{N}}^{-1} \cdot 1 = g^{-1}(g \circ \Phi) \Psi_T \cdot 1,$$

using (27). Since both subgroups N and T preserve H_- we see that $\psi : \mathbf{Z}^2 \to W = g^{-1} \cdot H_-$.

Lemma 14 *Each $\psi_{k,m} \in W$ given in this way is uniquely determined by the property that*

$$pr_-((1 + \alpha\zeta^{-1})^{-k} (1 + \beta\zeta^{-1})^{-m} \psi_{k,m}) = 1.$$

Proof. Since we are assuming $\Phi^{-1} \cdot W$ is always in the big cell it suffices to show that $pr_-(\Phi_{k,m}^{-1} \cdot \psi_{k,m}) = 1$, given (28). But $\Phi^{-1} \cdot \psi = \Psi_{\tilde{N}}^{-1} \cdot 1$. It is easy to check that for any $n \in \tilde{N}$, $n \cdot 1 = 1 + O(\zeta)$.□

We will call this function the discrete Baker function for W. Now we want to reconstruct the dressed map $g \circ z^{(0)}$ from ψ. First we observe that

$\zeta^{-2}W \subset W$ and the quotient $W/\zeta^{-2}W$ is two dimensional (since $g \cdot \zeta^{-2} f = \zeta^{-2} g \cdot f$ and $H_-/\zeta^{-2}H_-$ is two dimensional).

Lemma 15 *Choose any linear identification of $W/\zeta^{-2}W$ with $\mathbf{C}^2$ and let $[\psi] : \mathbf{Z}^2 \to \mathbf{P}^1$ be the map given by $(k,m) \mapsto \psi_{k,m} + \zeta^{-2}W$. Then $[\psi]$ and $g \circ z^{(0)}$ are Möbius equivalent (so in particular this gives a discrete conformal map).*

Proof. Observe that $1+\zeta^{-2}H_-$ and $\zeta^{-1}+\zeta^{-2}H_-$ span $H_-/\zeta^{-2}H_-$. Therefore by setting $e_1 = g^{-1}\cdot 1$ and $e_2 = g^{-1}\cdot\zeta^{-1}$ we obtain a basis $e_1+\zeta^{-2}W, e_2+\zeta^{-2}W$ for $W/\zeta^{-2}W$. We can write $g \circ \Phi$ and Ψ_T in the form

$$g \circ \Phi = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + O(\lambda^{-1}), \quad \Psi_T = \begin{pmatrix} s & 0 \\ 0 & * \end{pmatrix}$$

so that the map $g \circ z^{(0)}$ has homogeneous coordinates $[a,b]$ and $\psi_T \cdot 1 = s$. Now observe that $(g \circ \Phi)\Psi_T \cdot 1 = sa + \zeta^{-1}sb + O(\zeta^{-2})$ and therefore

$$\begin{aligned} s(ae_1 + be_2) &\equiv g^{-1}(g\circ\Phi)\Psi_T \cdot 1 \bmod \zeta^{-2}W \\ &\equiv \psi \bmod \zeta^{-2}W. \end{aligned}$$

It follows that in this basis the map $[\psi]$ has homogeneous coordinates $[a,b]$.□

Now we are in a position to prove theorem 3. Given a periodic discrete conformal map z with spectral data satisfying the conditions of the theorem we will construct a $W \in Gr$ for which the map $\psi : \mathbf{Z}^2 \to W$ recovers z as $[\psi]$. By the previous lemma this will prove the theorem.

Given $(\Sigma, \lambda, \mathcal{L}, O, S, Q)$ we follow [14] and define W in the following manner. By assumption there exists $r > \max\{1, |q|\}$ for which the disc $|\lambda| \leq r$ contains no branch points. We set $\zeta = \sqrt{\lambda/r}$ so that the unit ζ-circle may be identified with the boundary of a disc Δ on Σ about O which contains no ramification points other than O i.e. this is a coordinate disc with coordinate ζ and contains the points O, S, Q. We define W to be the L^2-closure of the space W^0 of all $f \in H$ which extend meromorphically into $\Sigma - \Delta$ where they have divisor of poles no worse than D, where this is the unique positive divisor of degree g in the class $\mathcal{L}(-O)$. One knows that $W \in Gr$ (even the small Grassmannian we have chosen, since the whole construction is analytic). Now, with the definitions from previous sections, we define $\psi_{k,m}$ to be (the boundary of) a meromorphic function with divisor (14). Since the points O, S, Q are not in $\Sigma - \Delta$ this function belongs to W for all k, m. Moreover, if we set $\alpha = -\zeta(S)$ and $\beta = -\zeta(Q)$ we see that

$$(1+\alpha\zeta^{-1})^{-k}(1+\beta\zeta^{-1})^{-m}\psi_{k,m}$$

extends holomorphically into Δ and $\psi_{k,m}$ will be uniquely determined by requiring this expression to equal 1 when evaluated at $\zeta = 0$. By lemma 14 this must be the discrete Baker function for W and by lemma 9 the map $(k,m) \to \psi_{k,m}(P_\infty)/\psi_{k,m}(\tilde{P}_\infty)$ recovers $z : \mathbf{Z}^2 \to \mathbf{P}^1$ up to Möbius equivalence. But the map $W^0 \to \mathbf{P}^1$ given by $f \mapsto [f(P_\infty), f(\tilde{P}_\infty)]$ induces an isomorphism $\mathbf{P}(W^0/\zeta^{-2}W^0) \to \mathbf{P}^1$. Since W^0 is dense in W the spaces $W^0/\zeta^{-2}W^0$ and $W/\zeta^{-2}W$ are equal. Therefore, by the previous lemma the maps $[\psi]$ and z are equivalent, hence z is in the dressing orbit of the vacuum solution.

The discrete cubic. Lemma 15 allows us to compute examples which are difficult to obtain otherwise. For example, take $W = \mathbf{C}\langle\zeta\rangle + \zeta^{-1}H_-$, where $\mathbf{C}\langle\zeta\rangle$ is the vector space generated by ζ. One knows that $W \in Gr$ (see [14] §7): indeed W lies in the Grassmanian for rational loops in GL_2. It is elementary to show that in this case

$$\psi_{k,m} = (1+\alpha\zeta^{-1})^k(1+\beta\zeta^{-1})^m(1 - \frac{\zeta}{k\alpha + m\beta}).$$

Since $W/\zeta^{-2}W \simeq \mathbf{C}\langle\zeta, \zeta^{-2}\rangle$ we may take $\zeta + \zeta^{-2}W$ and $\zeta^{-2} + \zeta^{-2}W$ as a basis. A straightforward computation shows that

$$\psi_{k,m} \equiv \frac{1}{k\alpha + m\beta}\zeta - \frac{z_{k,m}}{6(k\alpha + m\beta)}\zeta^{-2} \bmod \zeta^{-2}W,$$

where

$$z_{k,m} = (k+1)k(k-1)\alpha^3 + 3k^2m\alpha^2\beta + 3km^2\alpha\beta^2 + (m+1)m(m-1)\beta^3.$$

It follows that this is a discrete conformal map with cross-ratio β^2/α^2. Indeed this is the discrete analogue of the cubic $z(x) = x^3$. In the smooth (binomial) limit $(1+\alpha\zeta^{-1})^k(1+\beta\zeta^{-1})^m \to \exp(x\zeta^{-1})$ we obtain the smooth Baker function $\psi(x) = \exp(x\zeta^{-1})(1 - \zeta/x)$ and

$$\psi(x) \equiv -\frac{1}{x}\zeta + \frac{2}{3}x^2\zeta^{-2} \bmod \zeta^{-2}W.$$

See [14] §7 for a description of how this corresponds to taking the spectral curve $\mu^2 = \lambda^{-3}$ and for the similar spaces corresponding to the spectral curves with equations of the form $\mu^2 = \lambda^{-2g-1}$. All of these should produce (discrete) rational functions but it is difficult to describe a priori which rational functions arise in this way.

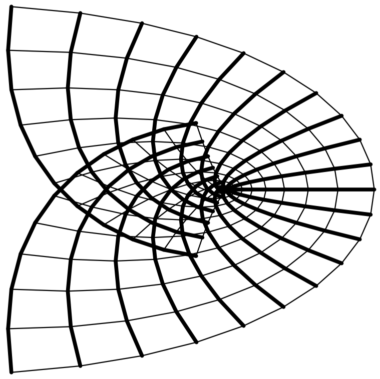

Figure 7: The discrete cubic.

4.3 Discrete conformal maps as Darboux transformations of the KdV hierarchy.

In this section we will prove rigourously a rather surprising relationship between discrete conformal maps and Darboux (Bäcklund) transforms of solutions to the KdV equation. We will work inside the dressing orbit of the vacuum solution of KdV. Recall from [14] that to every $W \in Gr$ there exists a solution $u_W(x,t)$ of the KdV equation, given by

$$(\partial^2 + u_W)\psi_W = \zeta^{-2}\psi_W \ , \qquad \partial = \partial/\partial x,$$

where $\psi_W : \mathbf{C}^2 \to W$ is the (smooth) Baker function for W i.e. the unique function with the property that $pr_-(\exp(-x\zeta^{-1} - t\zeta^{-3})\psi_W) = 1$ for almost all $x, t \in \mathbf{C}$. If $\Phi_{k,m}$ denotes the vacuum frame (26) then $W_{k,m} = \Phi_{k,m}^{-1} \cdot W$ defines a map $\mathbf{Z}^2 \to Gr$. We will show that $\{u_{W_{k,m}}\}$ is a $\mathbf{Z}^2$-family of Darboux transforms of u_W.

To begin, let us recall, from e.g. [6] the Darboux transform for the KdV hierarchy. Given any solution $u(x,t)$ of the KdV equation (or any other equation in the hierarchy) we can produce other solutions using the following procedure. Set $L = \partial^2 + u$. For any pair (ψ, c), consisting of a function $\psi(x,t)$ and a complex number c for which $L\psi = c\psi$, define $v = \psi_x \psi^{-1}$. One readily checks that $u = -v_x - v^2 + c$ and therefore

$$L - c = (\partial + v)(\partial - v).$$

The Darboux transform for the pair (ψ, c) maps u to $\tilde{u}$ where

$$\tilde{L} - c = (\partial - v)(\partial + v) \ , \quad \text{i.e. } \tilde{u} = v_x - v^2 + c.$$

One can think of this as a formal conjugation of $L - c$ into $\tilde{L} - c$ in the algebra of pseudo-differential operators and deduce that $\tilde{u}$ too satisfies the KdV equation (see [6]).

Proposition 6 *Whenever $0 < |\alpha| < 1$ the map $W \mapsto V = (1 + \alpha\zeta^{-1})W$ on Gr induces the Darboux transform on u_W determined by taking $\psi = \psi_W(x,t;-\alpha)$ and $c = \alpha^{-2}$.*

Proof. First, since W is a linear space we can write $V = (\zeta^{-1} + \alpha^{-1})W$. For convenience, set $\gamma(x,t) = \exp(x\zeta^{-1} + t\zeta^{-3})$, then

$$\psi_W = \gamma(1 + a_W\zeta + O(\zeta^2)),$$

for some function a_W of x, t. Comparing the Fourier series for $\partial\psi_W, \psi_W$ and $(\zeta^{-1} + \alpha^{-1})\psi_V$, all of which take values in W, we see that if we set $v = a_W - a_V - \alpha^{-1}$ then

$$\gamma^{-1}[(\partial - v)\psi_W - (\zeta^{-1} + \alpha^{-1})\psi_V] : \{(x,t)\} \to H_+,$$

i.e. it has only positive powers of ζ in its Fourier expansion. But $\gamma^{-1}W \cap H_+ = \{0\}$ for almost all x, t, hence this expression must be identically zero. Therefore

$$(\partial - v)\psi_W = (\zeta^{-1} + \alpha^{-1})\psi_V.$$

Further, $(\zeta^{-1} - \alpha^{-1})V = (\zeta^{-2} - \alpha^{-2})W = W$ so that a similar calculation gives

$$(\partial + v)\psi_V = (\zeta^{-1} - \alpha^{-1})\psi_W.$$

Therefore

$$\begin{aligned}(\partial + v)(\partial - v)\psi_W &= (\zeta^{-2} - \alpha^{-2})\psi_W, \\ (\partial - v)(\partial + v)\psi_V &= (\zeta^{-2} - \alpha^{-2})\psi_V,\end{aligned}$$

so in particular $u_W = -v_x - v^2 + \alpha^{-2}$ and $u_V = v_x - v^2 + \alpha^{-2}$. Hence if we take $\psi = \psi_W(x, t; -\alpha)$ and $c = \alpha^{-2}$ then $(\partial^2 + u_W)\psi = c\psi$ and $(\partial - v)\psi = 0$. Thus u_V is the Darboux transform of u_W for the pair (ψ, c).$\Box$.

Remark. In fact if one examines [6] p5, Theorem (ii), one sees that a Darboux transform of a certain type preserves the KdV spectral curve and shifts the line bundle by $\mathcal{L} \mapsto \mathcal{L}(Q - O)$ for some $Q \in \Sigma$ (recall that KdV solutions of finite type have spectral data $(\Sigma, \lambda, \mathcal{L}, O)$ with all the properties of the spectral data above). From this we see that this relationship between discrete conformal maps and Darboux transforms is true for all periodic maps, not just those in the dressing orbit of the vacuum.

Appendix: Non-singular spectral curves are generic.

Here we will prove our earlier claim that non-singular spectral curves exist (and are therefore generic) for periodic discrete curves with n points for any $n \geq 4$. We will use the notation of section 1 throughout.

Let $X_n \subset \mathbf{P}^1 \times \ldots \mathbf{P}^1$ be the space of periodic discrete curves of period n. It is clearly an irreducible affine open subvariety. Let Y_n be the space of data $(\Sigma, O, S, P_\infty, [y])$ where: Σ is a complete irreducible algebraic curve of arithmetic genus g (equal to $(n-4)/2$ for n even and $(n-3)/2$ for n odd) admitting a rational function λ of degree 2; O, S, P_∞ are smooth points on Σ with λ-values $0, 1, \infty$ respectively, at which λ is unramified (unless n is odd in which case O is a ramification point); y is a rational function on Σ with divisor of poles nS and $[y]$ denotes its image in the complete linear system $\mathbf{P}\Gamma(\mathcal{O}_\Sigma(nS))$. Since $n > 2g+2$ this linear system has dimension $n-g$. Notice that the map $(\Sigma, O, S, P_\infty, [y]) \mapsto (\Sigma, O, S, P_\infty)$ displays Y_n as a $\mathbf{P}^{n-g}$-bundle over the subvariety of $\mathbf{P}^{2g+2}$ corresponding to the possible configurations of branch divisors. Inside Y_n we consider two subvarieties: Y_n^s, wherein Σ is singular; Y_n^r, wherein Σ is rational with nodes only. In particular, Y_n^s is a hypersurface in Y while $Y_n^r \subset Y_n^s$ clearly has codimension g in Y_n.

To each $\Gamma \in X_n$ we assign the data $(\Sigma, O, S, P_\infty, [y])$ where Σ, O, S, P_∞ are given by the characteristic polynomial of M_0^λ and $y = \det(H_0^\lambda)$. This y has divisor

$$D_n = \begin{cases} n(O-S) & \text{for } n \text{ odd,} \\ \frac{n}{2}(O + \tilde{O} - 2S) & \text{for } n \text{ even,} \end{cases}$$

and satisfies $y(P_\infty) = y(\tilde{P}_\infty)$. Thus we have an algebraic map $F : X_n \to Y_n$ with image

$$V = \{(\Sigma, O, S, P_\infty, [y]) \in Y_n : (y) = D_n, y(P_\infty) = y(\tilde{P}_\infty)\}.$$

Since V is irreducible, either $V \subset Y_n^s$ or there exists a generic curve. Locally Y_n^s is given by a single equation, say $Z = 0$, in Y. We will show that the codimension of $V \cap Y_n^r$ in V is g hence Z cannot vanish identically on V since Y_n^r has only codimension $g-1$ in Y_n^s.

First let us describe $V \cap Y_n^r$. Each irreducible rational curve Σ with g nodes has normalisation $\mathbf{P}^1$. We choose a rational parameter t on $\mathbf{P}^1$ such that the hyperelliptic involution is $t \mapsto 1/t$ and $t(S) = \infty$. The preimage of the singularities under the normalisation will be g pairs of the form $a_j, 1/a_j$ $(j = 1, \ldots g)$ and all such curves Σ arise this way. For simplicity set $b = t(P_\infty)$

and $c = t(O)$. Then the parameters a_j, b, c determine (Σ, O, S, P_∞). If y has divisor D_n then, up to scaling,

$$y = \begin{cases} (t-c)^n & \text{for } n \text{ odd,} \\ (t-c)^m (t-c^{-1})^m & \text{for } n = 2m, \end{cases}$$

and $a_1, \ldots, a_g, b$ must all be roots of $y(t) = y(1/t)$. Certainly this many distinct roots exist, so $V \cap Y_n^r$ is non-empty and has dimension 1 for n even, since the only free parameter is c, while for n odd it has dimension zero, since $O = \tilde{O}$ forces $c = \pm 1$. Now let us compute the dimension of V. The fibre of the map $F : X_n \to Y_n$ over data involving any nodal rational curve is $PSL_2 \times Jac(\Sigma)$ since we have ignored Möbius invariance and the line bundle $\mathcal{L}$. Therefore V has dimension $n - (g+3)$ which equals $g+1$ for n even and g for n odd. Therefore $V \cap Y_n^r$ has codimension g in V (for every n) whence the result follows.

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