NEW CONSTANT MEAN CURVATURE TRINOIDS

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Abstract. I construct two families of constant mean curvature genus-zero surfaces with three ends via the DPW construction. One of these families is known and has three unduloid ends; the other is a new family with two unduloid and one nodoid end.

The classification of the almost-embedded CMC genus-zero surfaces with three ends — the trinoids with unduloid ends, or triunduloids — was announced by Große-Brauckmann, Kusner, and Sullivan [2]. In this note, I construct these examples explicitly (result 1) using a method due to Dorfmeister, Pedit and Wu [1], and present a new family of CMC trinoids (result 2) which enjoy two unduloid and one nodoid end.

Taking $H > 0$ to be a fixed mean curvature, the space of triunduloids may be described as follows.

Result 1. *Let n_1, n_2, n_3 be real numbers in the interval $(0, 1/(2H)]$ satisfying*

$$n_1 \le n_2 + n_3,\ n_2 \le n_1 + n_3,\ n_3 \le n_1 + n_2,\ n_1 + n_2 + n_3 \le 1/H.$$

Then there exist two potentials which, with appropriate initial conditions, produce trinoids with n_1, n_2, n_3 as neck radii, except when any of the above four inequalities is an equality; in this case there is exactly one such trinoid.

We extend this result to trinoids with two unduloid and and one nodoid end (that is, ends asymptotic to Delaunay unduloids and nodoids respectively).

Result 2. *Let n_1, n_2, n_3 be real numbers in the intervals $(0, 1/(2H)]$, $(0, 1/(2H)]$, $(-\infty, 0)$ respectively satisfying*

$$n_1 \le n_2 - n_3, \quad n_2 \le n_1 - n_3, \quad n_1 + n_2 - n_3 \le 1/H.$$

Suppose further that the end weights w_1, w_2, w_3 corresponding to the neck radii n_1, n_2, n_3 satisfy $w_1 + w_2 + w_3 \ge 0$. Then there exist two potentials which, with appropriate initial conditions, produce trinoids

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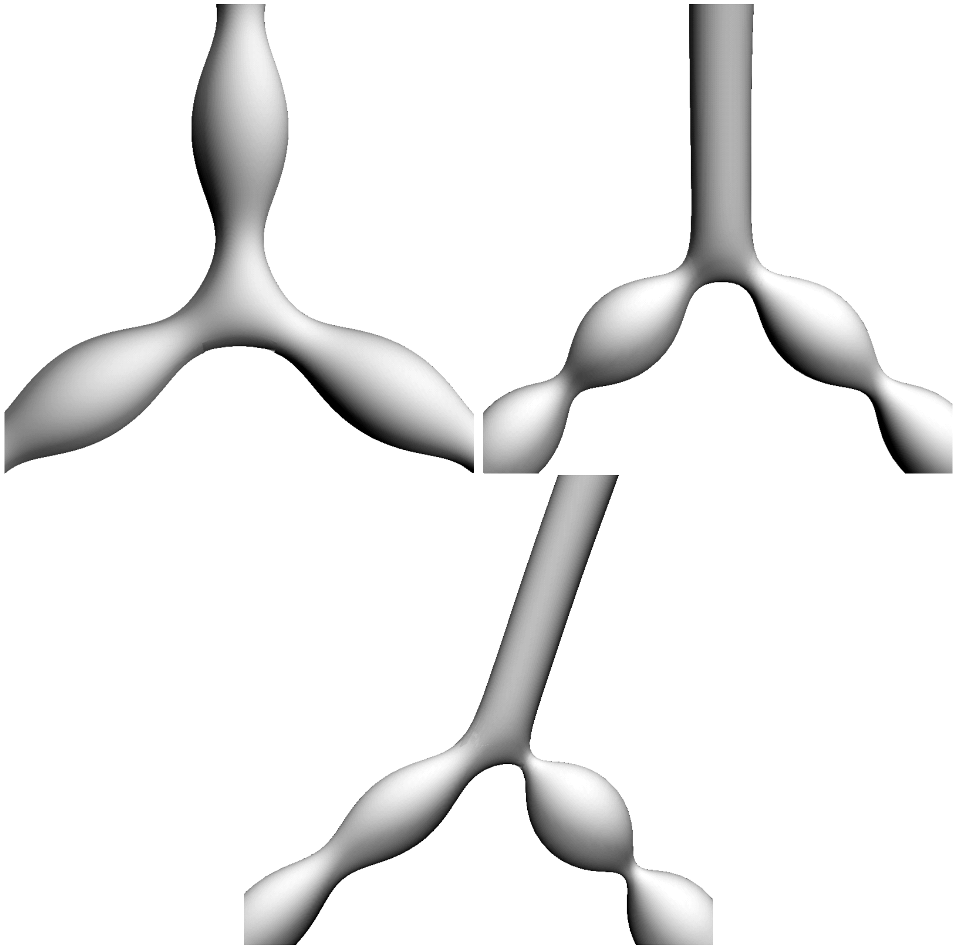

FIGURE 1. These CMC triunduloids lie in the one-parameter family whose neck radius sum is maximal. Each of the isosceles and scalene examples has a cylindrical end.

with n_1, n_2, n_3 *as neck radii, except when any of the above three inequalities for* n_1, n_2, n_3 *is an equality; in this case there is one such trinoid.*

These surfaces were produced with two tools I developed at the Center for Geometry, Analysis, Numerics and Graphics (GANG). The algorithm for the main software, `DPWLab`, which computes CMC surfaces from DPW potentials, is discussed in [1]. The second tool, `killerB`, discussed in more detail below, uses an unsophisticated but very stable algorithm to compute the initial condition for which a surface simultaneously closes at its ends.

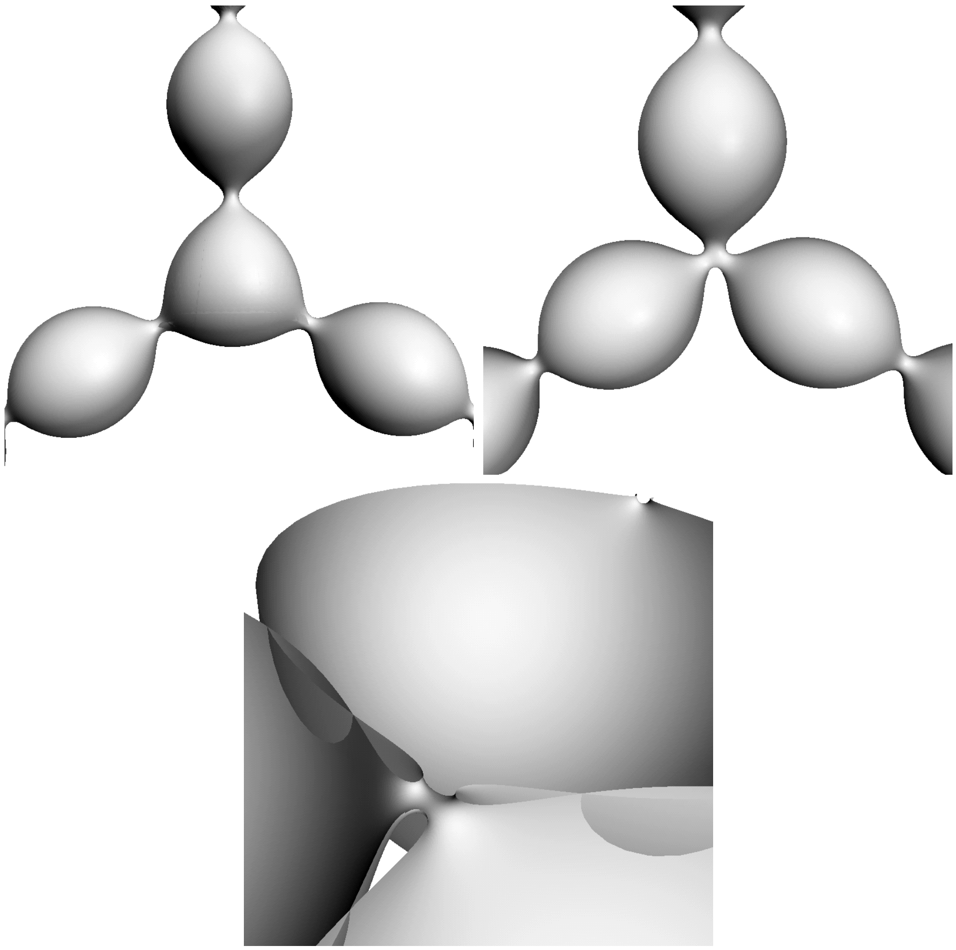

FIGURE 2. The top images show equilateral trinoids with equal neck radii but opposite phase; the one joined at a central bulge, the other meeting in a thin neck. The lower image is a cutaway view of an equilateral trinoid whose neck radii are so small that it self-intersects.

Neck Radii and End Weights. The minimum and maximum radii a, b of a Delaunay unduloid satisfy $0 < a < 1/(2H) < b < 1/H$, with constant sum $a + b = 1/H$. It is natural to extend this notion of neck and bulge radii to the round cylinder ($a = b = 1/(2H)$), the round sphere ($a = 0$, $b = 1/H$), and to the nodoid, where $a < 0 < 1/H < b$. In all cases, the sum is $a + b = 1/H$.

The *end weight*, defined as $w = ab$, is at most $1/(4H)$ (for the round cylinder) and like the neck radius, is positive for unduloids, zero for the round sphere, and negative for nodoids.

The Family of Trinoid Potentials. The family of DPW potentials generating the trinoids of results 1 and 2 are meromorphic on the thrice-punctured Riemann sphere and are asymptotic to the DPW potential for a Delaunay surface at each puncture. For $a,\, b \in \mathbb{C}$ and $c \in \mathbb{R}$ I denote this potential by

$$\mathrm{Del}_e(a,\, b,\, c) = \begin{pmatrix} c & a\lambda^{-1} + \bar{b}\lambda \\ b\lambda^{-1} + \bar{a}\lambda & -c \end{pmatrix} \frac{dz}{z - e} \tag{1}$$

The Delaunay surface produced by this potential is closed if and only if $a,\, b,\, c$ satisfy

$$|a + \bar{b}| + c^2 = 1/4. \tag{2}$$

Fix distinct points $\{e_1,\, e_2,\, e_3\} \in \mathbb{C}\mathrm{P}^1$. The space of trinoid potentials is parameterized by

$$U = \big\{(a_1,\, a_2,\, a_3,\, v) \in \mathbb{R}^4 \mid (a_1 + a_2 + a_3)^2 + v^2 = 1/4, \\ a_1a_2 + a_1a_3 + a_2a_3 \geq 0\big\}$$

as follows. Fix $(a_1,\, a_2,\, a_3,\, v) \in U$ and let $s = a_1 + a_2 + a_3$. Let $d \in \mathbb{C}\mathrm{P}^1$ satisfy $\gamma(e_1,\, d,\, e_2,\, e_3) = -a_2/a_3$. Here, γ denotes the cross ratio

$$\gamma(z_1,\, z_2,\, z_3,\, z_4) = (z_1 - z_3)(z_2 - z_4)/((z_1 - z_4)(z_2 - z_3)).$$

Then the trinoid potential $\xi \in \Lambda^\sigma \mathrm{SL}(2, \mathbb{C}) \otimes \Omega^{(1,0)}\mathbb{C}\mathrm{P}^1$ is the unique form with divisor $-[e_1] - [e_1] - [e_3] - [d]$ and residues

$$\begin{aligned} \mathrm{res}_{e_1}\xi &= \mathrm{Del}_{e_1}(-s + a_1,\, -a_1,\, -v) \\ \mathrm{res}_{e_2}\xi &= \mathrm{Del}_{e_2}(a_2,\, s - a_2,\, v) \\ \mathrm{res}_{e_3}\xi &= \mathrm{Del}_{e_3}(a_3,\, s - a_3,\, v) \\ \mathrm{res}_{d}\xi &= \mathrm{Del}_{d}(0,\, -s,\, -v). \end{aligned}$$

ξ is the potential for a (closed) constant mean curvature immersion $f_\xi : \mathbb{C}\mathrm{P}^1 \setminus \{e_1,\, e_2,\, e_3\} \mapsto \mathbb{R}^3$ under the DPW procedure for initial condition found numerically. The resulting trinoid will be denoted T_ξ.

The Parameter Space Conditions. Of the two conditions defining U,

$$(a_1 + a_2 + a_3)^2 + v^2 = 1/4 \tag{3}$$

$$a_1a_2 + a_1a_3 + a_2a_3 \geq 0 \tag{4}$$

(3) insures that at each of its poles $e_1,\, e_2,\, e_3$, the Delaunay potential (1) to which ξ is asymptotic satisfies the closing condition (2).

The remaining condition (4) defining U arises from a balancing formula [3] according to which the sum of the forces exerted by each end is zero. It follows that the end weights must satisfy the inequality $|w_i| \leq |w_j| + |w_k|$ for each permutation $i,\, j,\, k$ of 0, 1, 2.

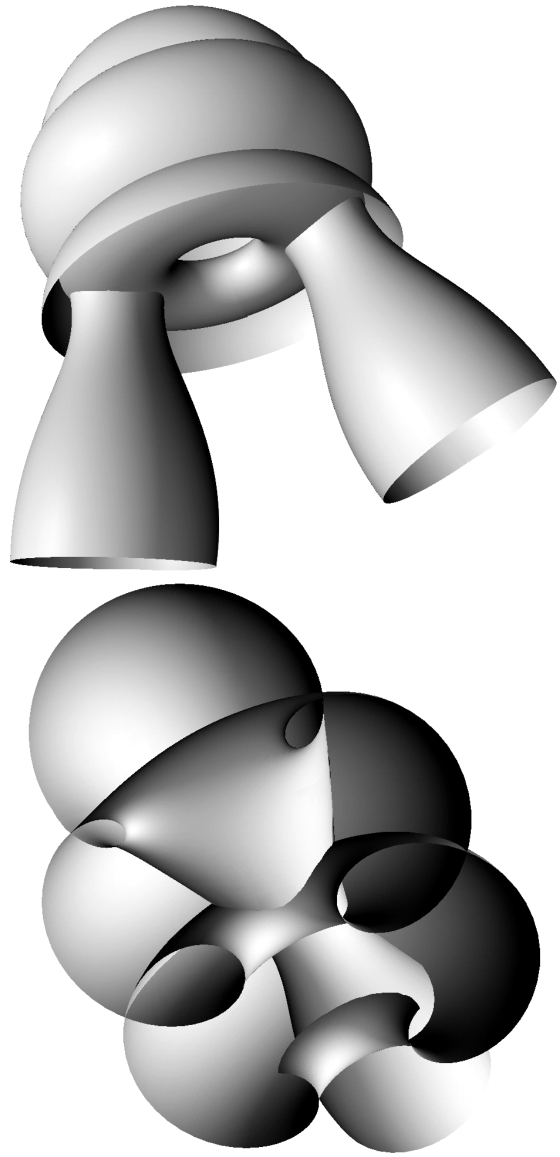

FIGURE 3. Two views of an isosceles CMC trinoid with one nodoid end; all three ends extend infinitely downward. In contrast to the triunduloids, the two umbilic points lie not symmetrically about the mirror plane but on it.

Each trinoid T_ξ is represented several times in U because (a) permutations of the parameters (a_1, a_2, a_3, v) produce the same trinoid with ends permuted, and (b) (a_1, a_2, a_3, v) and $(-a_1, -a_2, -a_3, -v)$ trivially produce the same surface.

On the other hand, there are two trinoids for each choice of neck radii unless $v = 0$. The two values $\pm v$ give two triunduloids with identical neck radii but opposite phase: one has a bulge in the middle, and the other a neck. This phenomenon is illustrated in figures 1 and 5.

End Types. The family of trinoids parametrized by ξ must have either (a) three unduloid ends, or (b) two unduloid and one nodoid end. Trinoids with more than one nodoid end cannot arise in this family. To see this, note that the ends e_1, e_2, e_3 are unduloidal or nodoidal according as their weights $4a_1(a_2+a_3)$, $4a_2(a_1+a_3)$, $4a_3(a_1+a_2)$ are positive or negative respectively. Because (a_1, a_2, a_3) and $(-a_1, -a_2, -a_3)$ give the same trinoid, as do permutations of (a_1, a_2, a_3), without loss of generality there are two cases: (a) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and (b) $a_1 > 0$, $a_2 > 0$, $a_3 < 0$. Case (a) yields a trinoid with three unduloid ends. In case (b), $a_3(a_1 + a_2) < 0$, making e_3 nodoidal. The condition $a_1a_2 + a_1a_3 + a_2a_3 > 0$ means that $a_1(a_2 + a_3) > -a_2a_3$ and $a_2(a_1 + a_3) > -a_1a_3$. But $-a_2a_3 > 0$, so $a_1(a_2 + a_3) > -a_2a_3 > 0$, and likewise $-a_1a_3 > 0$, so $a_2(a_1 + a_3) > -a_1a_3 > 0$. Hence each of the remaining two ends are unduloidal.

Asymptotics. At each of its poles e_1, e_2, e_3, the z-leading-order term of ξ is the DPW potential for a closed Delaunay surface D_i, and the trinoid T_ξ converges to D_i near e_i.

Near e_1, ξ is a is a special perturbation of a Delaunay surface in which the leading-order z-term of ξ commutes with its holomorphic part. This means that there is a linear space of initial conditions for which the CMC immersion closes (has trivial holonomy) at this end.

Cone Points. The point $d \in \mathbb{C}\mathrm{P}^1$, while a simple pole of ξ, is a regular point for the resulting immersion. Writing

$$\xi = \begin{pmatrix} \gamma & \alpha\lambda^{-1} + \beta\lambda \\ \beta\lambda^{-1} + \alpha\lambda & \gamma \end{pmatrix}$$

the cross-ratio condition for d guarantees that $\mathrm{ord}_d\alpha \geq 1$, so the Hopf form has no pole at d.

Symmetry. If $\tau : \mathbb{C}\mathrm{P}^1 \longrightarrow \mathbb{C}\mathrm{P}^1$ is the anticonformal diffeomorphism fixing e_1, e_2, e_3, then $\tau^* f_\xi = T f_\xi$ for a reflection R of $\mathbb{R}^3$. Hence each trinoid ξ has a plane of reflective symmetry.

Umbilics. Each trinoid T_ξ has two umbilic points, whose preimages u_1, $u_2 \in \mathbb{C}\mathrm{P}^1$ are the zeros of the coefficient of λ^{-1} in the lower-right entry of ξ. For triunduloids u_1, u_2 are interchanged by the involution τ and hence by the reflection T_ξ of $\mathbb{R}^3$. For trinoids with one nodoid end, u_1, u_2 are individually fixed by τ and lie in the plane of reflective symmetry of the trinoid.

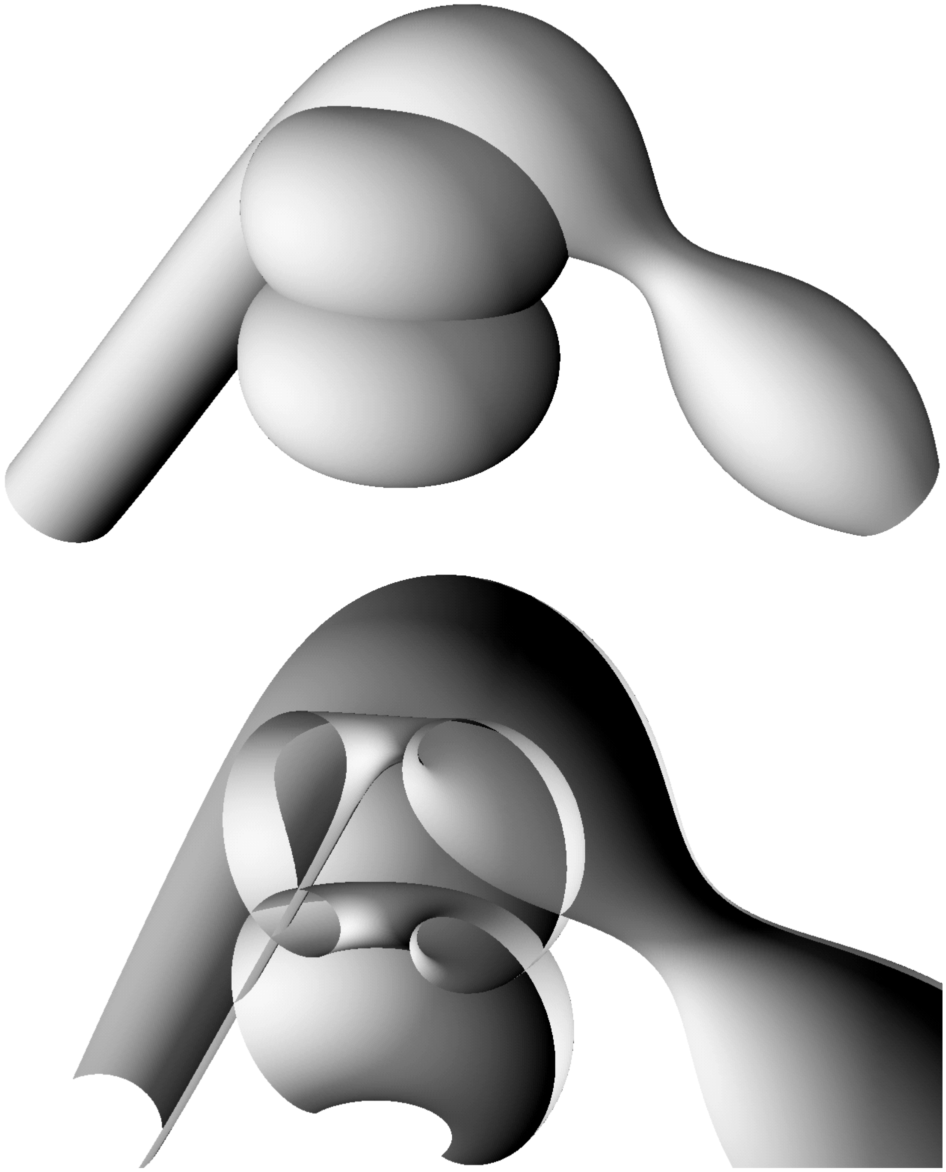

FIGURE 4. This CMC trinoid has three ends which are asymptotically a cylinder, a nodoid, and an unduloid. The three downward ends have zero-sum force because the nodoid end's force is upward.

Period Closing. As the DPW potentials for these surfaces became clear, I implemented period-closing software `killerB` to solve the holonomy problem numerically and verify that the potentials indeed produce trinoids. Since the desired initial condition is not a constant but

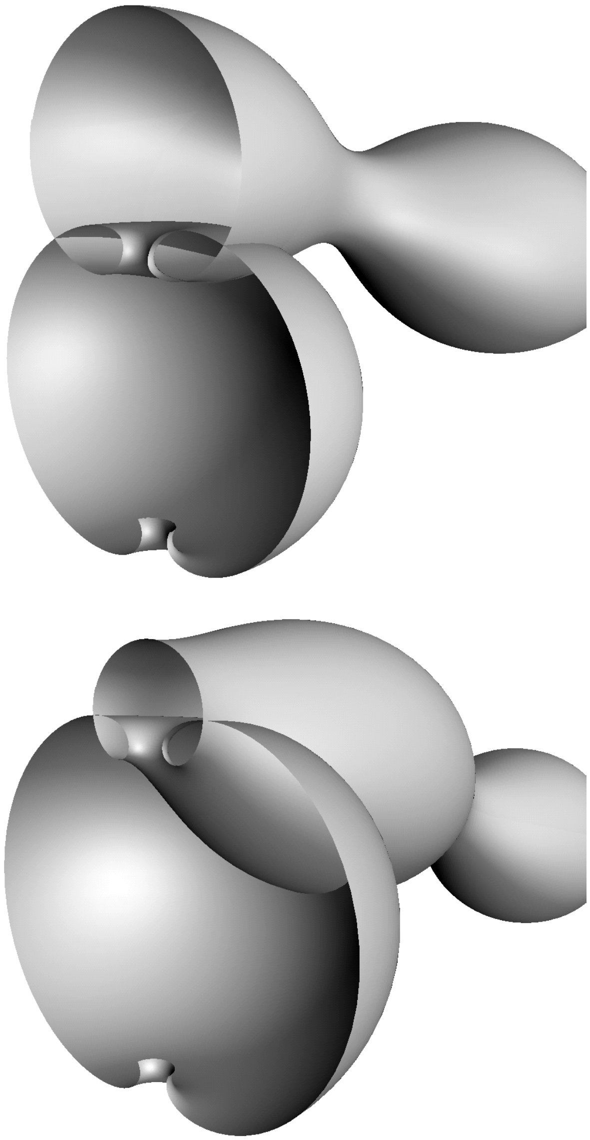

Figure 5. These two isosceles trinoids with one nodoid end are sliced through a mirror plane. As with the triunduloids in figure 2 they have correspondingly equal neck radii and opposite phase.

a loop group element, the period closer's search space is an infinite dimensional space of Fourier series, so for the purposes of computation it is necessary to restrict to a finite dimensional subspace by truncating the series.

The period-closing software uses a brute-force algorithm. Starting with an arbitrary initial guess, it cycles through the dimensions of the search space (the loop group coefficients), varying one parameter at a time while holding the others fixed. In this way it minimizes, one dimension at a time, an error measure of the holonomy by quadratic interpolation. By stair-stepping through the parameters, the algorithm slowly converges to a stable relative minimum which in practice is the global minimum. It is interesting to note that the periods of ends with small neck radii converge much more quickly than do those near the maximal limit.

As noted above, due to the special structure of the potential, at one end a linear space of initial conditions can be found for which the trinoid closes. Within this space, the period-closing software must be used to close a second end. Because the holonomy around the third end coincides with the composition of holonomies around the other two, the closing condition on the third end is vacuous.

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