CROSSING NUMBERS OF TIGHT KNOTS

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Sir — Applications of knots to the study of polymers have emphasized geometric measures on curves, like "energy"[1,2,3,4] and "ropelength"[5,6,7], which (when minimized over different configurations of a knot) give numerically computable knot invariants related to physical quantities[8]. The electrophoretic mobility of DNA knots seems correlated to the average crossing number of their ropelength-minimizing configurations[9], and a linear relation between crossing number and ropelength for these configurations was observed[10]. This linear relation cannot hold in general: we construct infinite families of knots whose ropelength grows as the $\frac{3}{4}$ power of crossing number[11]; no smaller power is possible[12,13,14,15].

One measure of geometric complexity for a space curve γ is $\mathcal{A}(\gamma)$, the average number of crossings in planar projections of γ. Another scale-invariant measure is *ropelength* $\mathcal{L}(\gamma) = L/T$, the quotient of arclength by *thickness* (the diameter of the largest uniform tube centered on γ). If we fix a knot type K (or a link type K: our methods work for curves with one or many components) the infima of $\mathcal{A}$ and $\mathcal{L}$ over $\gamma \in K$ are the crossing number $\mathcal{C}(K)$ and ropelength $\mathcal{L}(K)$ of K. A curve γ achieving the infimum $\mathcal{L}(K)$ is called *tight*. Experiments[9] with knotted DNA and simulations[10] of thermal averages of such knots lead us to seek mathematical relations between the invariants $\mathcal{C}$ and $\mathcal{L}$, and between $\mathcal{A}$ and $\mathcal{L}$ for tight curves.

For any fixed power $\frac{3}{4} \le p \le 1$ we construct infinite families of knots and of links which have $\mathcal{L} \sim \mathcal{C}^p$, meaning the ratio $\mathcal{L}/\mathcal{C}^p$ is bounded above and below across each family. No $p < \frac{3}{4}$ can work, since $\mathcal{C} \le \mathcal{A}$ and there is a universal lower bound[12] for $\mathcal{L}/\mathcal{A}^{3/4}$. Furthermore, ropelength minimizers in a family with $\mathcal{L} \sim \mathcal{C}^{3/4}$ must satisfy $\mathcal{C} \sim \mathcal{A}$; thus these tight curves have $\mathcal{L} \sim \mathcal{A}^{3/4}$.

Any knot or link arises from some *braid*, a collection of N ascending arcs in a cylinder Z, joining N points on the bottom of Z to the same N points on the top: bend Z into a solid torus S, identify the top and bottom disks, and join the arcs to give a link in S. For a braid with bounds on its slope, curvature and horizontal strand separation, this bending fixes ropelength within a uniform factor (depending on the shape of Z and S).

For a family of N-component Hopf links with $\mathcal{L} \sim \mathcal{C}^{3/4}$, take a cylinder Z of height h and radius r. Distribute N points on the bottom disk of Z, and rotate these one full turn as they ascend (see figure). The resulting braid has N helical strands of bounded slope and curvature provided $h \sim r$; to separate strands, we need $r \sim N^{1/2}$. Thus the braid (and the corresponding link) has ropelength $\mathcal{L} \sim hN \sim N^{3/2}$. Since each component links every other component exactly once, $\mathcal{C} \ge N(N-1)$; the standard projection has $N(N-1)$ crossings, so $\mathcal{C} = N(N-1) \sim N^2$.

For $(N, N-1)$-torus knots in S with $\mathcal{L} \sim \mathcal{C}^{3/4}$, repeat this construction in the lower half of Z; to define the rest of the braid, choose a circuit joining the N points, as in the figure, and slide each point along this circuit to the next while ascending through the upper half of Z. The horizontal distance a point travels is comparable to at most r. So as before, we need $h \sim r \sim N^{1/2}$ and ropelength $\mathcal{L} \sim N^{3/2}$. The minimum crossing number of this knot[16] is $\mathcal{C} = N(N-2) \sim N^2$. (Lattice knots with the same growth of crossing number were found independently[17].) Many further families of links can be constructed similarly[11].

If we plot $\mathcal{C}$ versus $\mathcal{L}$ (on a log–log scale, as in the figure) then links in any family with $\mathcal{L} \sim \mathcal{C}^p$ approach the ray of slope p. We have seen examples with $p = \frac{3}{4}$ (sublinear growth).

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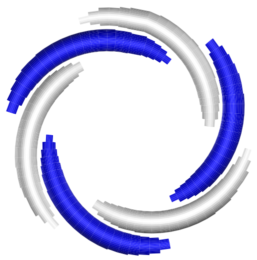

FIGURE 1. Hopf links with ropelength $\mathcal{L}$ growing as the $\frac{3}{4}$ power of crossing number $\mathcal{C}$ are constructed from braids (left). To get torus knots with the same property, modify these braids by following a circuit (center). The graph (right) illustrates the general relation between crossings and ropelength when viewed at large scale: for any family of links, the growth rate must lie between powers $\frac{3}{4}$ and 2, and all powers below 1 are realized.

To get $p = 1$ (linear growth), consider simply linked chains of N components, for which[11] $\mathcal{L} = 2\pi N + 2(N-2)$, while $\mathcal{C} = 2(N-1)$. Combining these examples yields families of links with any $\frac{3}{4} \leq p \leq 1$. It is unknown whether any family has superlinear growth. However (excluding links with split, unknotted components) an argument based on embedding planar projections[11] proves no family has growth $p > 2$.

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