ON THE MEROMORPHIC POTENTIAL FOR A HARMONIC SURFACE IN A k-SYMMETRIC SPACE.

JOSEF DORFMEISTER, IAN MCINTOSH, FRANZ PEDIT, AND HONGYOU WU

1. Introduction

Over the past few years there has been significant progress in the study of harmonic maps of compact Riemann surfaces into compact symmetric spaces. Whereas the case of harmonic 2-spheres in various symmetric spaces and Lie groups can be completely understood in terms of certain holomorphic curves into associated twistor spaces, the case of harmonic 2-tori is considerably more involved and is closely tied to certain integrable systems on loop spaces. Besides being motivated by field theories of mathematical physics, considerable motivation for studying this case was provided by the question of the existence of constant mean curvature tori in 3-space (posed by H. Hopf and resolved much later, to the affirmative, by H. Wente). By now it is fair to say we understand how 'the generic' harmonic 2-tori (in the sense of [5]) in compact symmetric spaces arise: they are obtained by solving a hierarchy of completely integrable ODE in Lax form on certain loop algebras, whose solutions can be expressed in terms of theta functions on certain algebraic curves, the *spectral curves* of the flows [6].

A comparable theory for harmonic maps of higher genus (≥ 2) surfaces is far from comprehended. Recently an attempt has been made to provide such a theory which resulted in the description of *all* harmonic maps on a simply connected domain into a compact symmetric space G/K in terms of $\dim(G/K)$ many meromorphic 1-forms, the so-called *meromorphic potential* [8]. This is reminiscent of the Weierstrass-representation for minimal surfaces in $\mathbb{R}^3$. Of course, to make headway in this approach one has to gain a solid understanding of how the meromorphic potential encodes the geometry of the harmonic map it describes. This provokes the question addressed in the present note: given a harmonic map from a simply connected domain how can one calculate its meromorphic potential? We will give a formula for the meromorphic potentials η in the dressing orbit of a given harmonic map. Since all (semisimple) finite type harmonic maps $\psi : \mathbb{R}^2 \to G/K$ are dressing equivalent [7], [9] to $exp(zA + \bar{z}\bar{A}) \cdot K$ for some appropriate semi-simple $A \in \mathfrak{g}^{\mathbb{C}}$ with $[A, \bar{A}] = 0$, this yields an explicit formula for the meromorphic potentials of finite type harmonic maps and thus of 'most' harmonic 2-tori (and indeed all non-conformal harmonic 2-tori in compact symmetric spaces of rank 1 [5]).

In the case which is of primary interest in classical geometry, namely non-conformal harmonic maps into the Riemann sphere $\mathbb{C}P^1$ (the Gauss maps of constant mean curvature surfaces in $\mathbb{R}^3$), the harmonic map equations reduce (away from conformal points) to the sinh-Gordon equation and

Partially supported by NSF grant DMS-9205293.
Supported by SERC grant GR/J33043 and the SFB 288 at Technische Universität Berlin.
Partially supported by NSF grant DMS-9205293, the SFB 288 at Technische Universität Berlin and the Graduierten Kolleg at Humboldt Universität Berlin.
Partially supported by NSF grant DMS-9205293.

we obtain the following. If $\psi : \mathbb{R}^2 \to \mathbb{C}P^1$ is the finite type harmonic map corresponding to the solution $u : \mathbb{R}^2 \to \mathbb{R}$ of the sinh-Gordon equation

$$u_{z\bar{z}} + 2\sinh(2u) = 0$$

then the corresponding meromorphic potential is given by

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & e^{-v(0,0)+2v(z,0)} \\ e^{v(0,0)-2v(z,0)} & 0 \end{pmatrix} dz$$

where $v : \mathbb{C}^2 \to \mathbb{C}$ is the complex solution to the complex sinh-Gordon equation

$$v_{zw} + 2\sinh(2v) = 0$$

which is the (unique) extension of $u(z,\bar{z})$ off $w = \bar{z}$. That such an extension exists in the finite type case follows from the explicit formulas given in [2]. From there one also sees that $e^{2v(z,0)}$ is indeed meromorphic since it is given as a ratio of theta functions. As a further application we derive the meromorphic potentials of the generalized Smyth surfaces (see [8]).

2. Loop group definitions and factorisations.

We begin by fixing the k-symmetric space G/K: here G is a compact semisimple Lie group and K is the fixed-point subgroup of a periodic automorphism σ of order $k \geq 2$. This automorphism extends uniquely to the complexification $G^{\mathbb{C}}$. The construction of meromorphic potentials [8] involves the choice of an appropriate loop group. In this paper the loop group in which everything will take place is a group of smooth maps from a pair of circles in $\mathbb{C}$ to $G^{\mathbb{C}}$ which are equivariant with respect to σ. To define this we fix $0 < \epsilon < 1$ and let C_1, C_2 denote the circles of radius ϵ, ϵ^{-1} respectively. Let C denote their union, then we define

$$\Lambda_C G^{\mathbb{C}}_\sigma = \{\text{smooth } g : C \to G^{\mathbb{C}} \mid g(\omega\lambda) = \sigma g(\lambda), \text{ for } \lambda \in C\}$$

where $\omega = e^{2\pi i/k}$ is a primitive k-th root of unity. It will often be convenient to think of an element of this group as a pair $g = (g_1, g_2)$, where $g_i = g \mid C_i$.

If we think of C as a contour on the λ-sphere $\mathbb{P}_\lambda$ it is the boundary for three regions: the annulus E and two discs I_0 and I_∞, about $\lambda = 0$ and ∞ respectively, whose union we will call I. With respect to these we may define the following subgroups of $\Lambda_C G^{\mathbb{C}}_\sigma$:

$$\begin{aligned} \Lambda_E G^{\mathbb{C}}_\sigma = &\ \{g \in \Lambda_C G^{\mathbb{C}}_\sigma : \text{boundaries of holomorphic maps } E \to G^{\mathbb{C}}\}, \\ \Lambda_I G^{\mathbb{C}}_\sigma = &\ \{g \in \Lambda_C G^{\mathbb{C}}_\sigma : \text{boundaries of holomorphic maps } I \to G^{\mathbb{C}} \text{ with } g(\infty) = e\}. \end{aligned}$$

The reason for introducing these subgroups lies in the following result (cf. [10]).

Theorem 1. *The multiplication map*

$$\Lambda_E G^{\mathbb{C}}_\sigma \times \Lambda_I G^{\mathbb{C}}_\sigma \longrightarrow \Lambda_C G^{\mathbb{C}}_\sigma$$

is a diffeomorphism onto an open dense subset $\mathcal{U} \subset \Lambda_C G^{\mathbb{C}}_\sigma$. For each $g \in \mathcal{U}$ we write $g = g^E g^I$ to denote the unique factorisation which results.

The proof relies on the Birkhoff factorisation [12] of elements of each loop group

$$L_i G^{\mathbb{C}}_\sigma = \{\text{smooth } \sigma\text{-equivariant } g : C_i \to G^{\mathbb{C}}\}.$$

Since we will use the factorisation in theorem 1 explicitly let us recall it now. In $L_1 G^{\mathbb{C}}_\sigma$ we may write every element as a product

$$g = g_- w_1 g_+ \quad , \quad g_-(\infty) = e,$$

where g_+ extends holomorphically into I_0, g_- extends holomorphically into $E \cup I_\infty$ and w_1 represents an element of the affine Weyl group. Likewise, in $L_2 G_\sigma^{\mathbb{C}}$ we may write

$$h = h_+ w_2 h_- \quad , \quad h_-(\infty) = e,$$

with h_+ and h_- extending holomorphically into $E \cup I_0$ and I_∞ respectively. Now one checks that we may write

$$(g, h) = (g_- p_+^{-1}, h_+ p_-)(p_+ g_+, p_-^{-1} h_-) = (g, h)^E (g, h)^I \tag{1}$$

precisely when $p = w_2^{-1} h_+^{-1} g_- w_1$, which is defined throughout E, factorises into $p_- p_+$ over C_1 (observe that these factors extend to all of E).

Later we will need to compare this factorisation with the global Iwasawa-type decomposition which exists on a particular real subgroup of $\Lambda_C G_\sigma^{\mathbb{C}}$. Let

$$\Lambda_C G_\sigma = \{g \in \Lambda_C G_\sigma^{\mathbb{C}} : g_2 = \bar{g}_1\}$$

where $(\bar{g})(\lambda)$ denotes $\overline{g(\bar{\lambda}^{-1})}$ (with conjugation in $G^{\mathbb{C}}$ taken with respect to the real form G). To recall the global decomposition of this group we first fix an Iwasawa decomposition $K^{\mathbb{C}} = KB$ on the reductive subgroup $K^{\mathbb{C}}$, in which B denotes the solvable factor. Now we define the following three subgroups of $\Lambda_C G_\sigma$:

$$\begin{array}{lcl} \Lambda_E G_\sigma & = & \Lambda_C G_\sigma \cap \Lambda_E G_\sigma^{\mathbb{C}} \\ \Lambda_I G_\sigma & = & \{g \in \Lambda_C G_\sigma : \ g \text{ extends holomorphically into } I_0 \text{ with } g(0) = e\} \\ B & = & \{(b, \bar{b}) : b \in B\} \end{array}$$

For convenience we also denote the product $B.\Lambda_I G_\sigma$ by $\Lambda_{I,B} G_\sigma$. Then the global decomposition proven in [11] may be generalised [8] to give:

Theorem 2. *The multiplication map $\Lambda_E G_\sigma \times \Lambda_{I,B} G_\sigma \to \Lambda_C G_\sigma$ is a diffeomorphism. For each g we write $g = g_E g_I$ to denote the resulting factorisation.*

We will refer to this as the Iwasawa decomposition for $\Lambda_C G_\sigma$. A comparison between the two factorisations gives:

Lemma 1. *The subgroup $\Lambda_C G_\sigma \subset \Lambda_C G_\sigma^{\mathbb{C}}$ lies entirely in $\mathcal{U}$.*

Proof. For $g \in \Lambda_C G_\sigma$ write $g_I = (b, \bar{b})(n, \bar{n})$ for the factors arising from $\Lambda_{I,B} G_\sigma = B.\Lambda_I G_\sigma$. Then

$$g = g_E \bar{b} (\bar{b}^{-1} b n, \bar{n}) \tag{2}$$

and $g_E \bar{b}$ belongs to $\Lambda_E G_\sigma^{\mathbb{C}}$ while $(\bar{b}^{-1} b n, \bar{n})$ belongs to $\Lambda_I G_\sigma^{\mathbb{C}}$. Thus $g \in \mathcal{U}$ and its unique factors g^E, g^I are given by (2). □

Finally, we will require a result concerning the Birkhoff factorisation of holomorphic families of loops.

Theorem 3 ([8]). *Let $\Delta \subset \mathbb{C}^n$ be an open domain and $g : \Delta \to L_i G_\sigma^{\mathbb{C}}$ be holomorphic with $g(z_0) = e$ for some $z_0 \in \Delta$. Then $g = g_- g_+$ off an analytic subvariety S of Δ and the factors g_-, g_+ are holomorphic on $\Delta \backslash S$. Moreover these factors extend meromorphically to all of Δ.*

We deduce that for a holomorphic map $g : \Delta \to \Lambda_C G_\sigma^{\mathbb{C}}$ with $g(z_0) = e$ we have $g = g^E g^I$ almost everywhere and the factors g^E and g^I are meromorphic throughout Δ.

3. Harmonic maps and holomorphic potentials.

Recall that G/K is a reductive homogeneous space: the automorphism σ equips the Lie algebra $\mathfrak{g}$ of G with reductive summands $\mathfrak{k}$ and $\mathfrak{m}$, where

$$\mathfrak{m}^{\mathbb{C}} = \sum_{j\in\mathbb{Z}_k\backslash\{0\}} \mathfrak{g}_j^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$$

in which $\mathfrak{g}_j$ denotes the ω^j-eigenspace for σ (recall that $\omega = e^{2\pi i/k}$). As is standard (see e.g. [7]) we identify $T^{\mathbb{C}}(G/K)$ with the subbundle $[\mathfrak{m}^{\mathbb{C}}] = G\times_K \mathfrak{m}^{\mathbb{C}} \subset G/K\times\mathfrak{g}^{\mathbb{C}}$. Since K acts adjointly on each subspace $\mathfrak{g}_j$ we may decompose $[\mathfrak{m}^{\mathbb{C}}]$ into the direct sum of subbundles $[\mathfrak{g}_j] = G\times_K\mathfrak{g}_j$ for $j\in\mathbb{Z}_k\backslash\{0\}$.

In this article we will deal exclusively with harmonic maps into G/K from a non-compact, simply connected and connected Riemann surface Ω and therefore we may as well take Ω to be either the complex plane or the Poincaré disc. Following [4] we say a map $\psi:\Omega\to G/K$ is *primitive* whenever $\partial\psi: T^{1,0}\Omega\to[\mathfrak{m}^{\mathbb{C}}]$ has image in $[\mathfrak{g}_{-1}]$ (observe that this is always the case for $k=2$). One knows from [1, 4] that for $k>2$ primitive maps are harmonic for suitable choices of G-invariant metrics on G/K. Therefore (following [7]) we define ψ to be *primitive harmonic* if either $k=2$ and ψ is harmonic or $k>2$ and ψ is primitive.

We know (from [8] and its obvious generalisation to k-symmetric spaces) that every primitive harmonic map $\psi:\Omega\to G/K$ arises from a *holomorphic potential* in the following way: let $L_i\mathfrak{g}_\sigma^{\mathbb{C}}$ denote the Lie algebra of $L_iG_\sigma^{\mathbb{C}}$ and define

$$\Lambda_{-1,\infty} = \{X\in L_1\mathfrak{g}_\sigma^{\mathbb{C}}:\ \lambda X \text{ extends holomorphically into } I_0\}.$$

We will write the Fourier series on C_1 of $X\in\Lambda_{-1,\infty}$ as $X=\lambda^{-1}X_{-1}+\ldots$. Following [8] we define a holomorphic potential to be a holomorphic section ξ of $T^*_{1,0}\Omega\otimes\Lambda_{-1,\infty}$. Such a potential satisfies the Maurer-Cartan equations so, since Ω is simply connected, we may integrate to obtain a unique solution $\Phi^\xi:\Omega\to L_1G_\sigma^{\mathbb{C}}$ to

$$(\Phi^\xi)^{-1}d\Phi^\xi = \xi \qquad , \qquad \Phi^\xi(0)=e.$$

Let us use $\Phi(\xi)$ to denote the map $(\Phi^\xi(z),\overline{\Phi^\xi(z)}):\Omega\to\Lambda_C G_\sigma$. Let Φ_λ denote $(\Phi(\xi))_E:\Omega\to\Lambda_E G_\sigma$, which is defined for λ throughout E. Now from [8] we deduce:

Lemma 2. *For every $|\lambda|=1$ the map Φ_λ frames a primitive harmonic map $\psi:\Omega\to G/K$ and every primitive harmonic map arises from some holomorphic potential ξ.*

We call Φ_λ an extended frame for ψ. It is elementary to show that every primitive map possesses an extended frame: one then solves a $\bar\partial$-problem over Ω to show there is always a holomorphic potential.

Now let $\mathcal{P}$ denote the vector space of all holomorphic potentials and use $\mathcal{H}$ to denote the set of all primitive harmonic maps $\psi:\Omega\to G/K$ based by $\psi(0)=eK$. The previous lemma provides us with a surjective map $\phi:\mathcal{P}\to\mathcal{H}$ which assigns to each potential ξ the based primitive map $\phi(\xi)$ with frame Φ_1. Further, this map intertwines a $\Lambda_{I,B}G_\sigma$-action on $\mathcal{P}$ and $\mathcal{H}$. The group $\Lambda_{I,B}G_\sigma$ acts adjointly on the vector space $\Lambda_{-1,\infty}$ (set $g_1 = g|C_1$ for any element g of $\Lambda_{I,B}G_\sigma$, then $\mathrm{Ad}g_1.X$ belongs to $\Lambda_{-1,\infty}$ whenever X does) and this we transfer to an action on $\mathcal{P}$. On $\mathcal{H}$ we define an $\Lambda_{I,B}G_\sigma$-action, called the *dressing action*, by taking $g\#\phi(\xi)$ to be the primitive map with extended frame $g\#\Phi_\lambda = (g\Phi(\xi))_E$. This is shown to be an action in [7] with the property $g\#\phi(\xi)=\phi(\mathrm{Ad}g.\xi)$, so that ϕ is $\Lambda_{I,B}G_\sigma$-equivariant for this pair of actions. We will denote the dressing orbit of $\phi(\xi)$ in $\mathcal{H}$ by O_ξ.

Let α denote the pullback of the (left) Maurer-Cartan form under $(g\#\Phi_\lambda)_{\lambda=1} : \Omega \to G$ and let $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ denote its splitting with respect to the decomposition of $\mathfrak{g}$. A useful observation about maps in a fixed dressing orbit O_ξ is:

Lemma 3 (cf. [8])**.** *Fix $\xi \in \mathcal{P}$ and for $g \in \Lambda_{I,B}G_\sigma$ let α denote the Maurer-Cartan form for $(g\#\Phi_\lambda)_{\lambda=1}$. Then ξ_{-1} and $\alpha'_{\mathfrak{m}}$ are $C^\infty(\Omega, B)$-conjugate sections of $T^*_{1,0}\Omega \otimes \mathfrak{g}_{-1}$.*

Proof. Since $g \in \Lambda_{I,B}G_\sigma$ we see that $\Phi(Adg.\xi)_E = [g\Phi(\xi)]_E$ and therefore we may write the Iwasawa decomposition of $g\Phi(\xi)$ as $(g\#\Phi_\lambda)(s,\bar{s})(n,\bar{n})$ where $s : \Omega \to B$. Therefore we find that

$$(g\#\Phi_\lambda)^{-1}\partial(g\#\Phi_\lambda) = \begin{cases} \mathrm{Ad}sn.\xi + \mathrm{Ad}s.(n\partial n^{-1}) + s\partial s^{-1} & \text{on } C_1, \\ \mathrm{Ad}\bar{s}.(\bar{n}\partial\bar{n}^{-1}) + \bar{s}\partial\bar{s}^{-1} & \text{on } C_2. \end{cases} \tag{3}$$

A comparison shows that this must be holomorphic on $\mathbb{P}_\lambda$ except for a simple pole at $\lambda = 0$, and therefore that

$$(g\#\Phi_\lambda)^{-1}\partial(g\#\Phi_\lambda) = \lambda^{-1}\mathrm{Ad}s.\xi_{-1} + \bar{s}\partial\bar{s}^{-1}. \tag{4}$$

This shows that $\alpha'_{\mathfrak{m}} = \mathrm{Ad}s.\xi_{-1}$ and we are done. □

3.1. **Complexification of the 'field equations'.** The commutativity of the ∂ and $\bar{\partial}$ derivatives of $g\#\Phi_\lambda$ above (i.e. (4) and its conjugate) imply the equations

$$\begin{aligned} [\partial + \bar{s}\partial\bar{s}^{-1}, \bar{\partial} + s\bar{\partial}s^{-1}] &= -[\mathrm{Ad}s.\xi_{-1}, \mathrm{Ad}\bar{s}.\bar{\xi}_{-1}], \\ [\bar{\partial} + s\bar{\partial}s^{-1}, \mathrm{Ad}s.\xi_{-1}] &= 0, \end{aligned} \tag{5}$$

a version of the Yang-Mills-Higgs equations (cf.[6]). The first equation may also be viewed as a (very broad) generalisation of the elliptic Toda field equations for the field $s : \Omega \to B$ — when σ is the Coxeter automorphism and $\xi_{-1}(\partial/\partial z)$ is constant semisimple we obtain the (abelian) Toda field equations associated to $\mathfrak{g}^{\mathbb{C}}$ (see, for example, [3, 6]).

In fact these equations occur as the real form of a system in two complex variables z and w. Let $a : \Omega^2 \to K^{\mathbb{C}}$ and consider the system

$$[\partial_z, \partial_w + a\partial_w a^{-1}] = -[\mathrm{Ad}a.\xi_{-1}(z), \overline{\xi_{-1}(\bar{w})}], \tag{6}$$

(we have $\bar{\Omega} = \Omega$ since Ω is either the disc or the plane). It is easy to check that we obtain solutions of (6) along $w = \bar{z}$ from solutions of (5) by setting $a = \bar{s}^{-1}s$ (this corresponds to a gauge transformation of the first equation in (5)). Conversely, if a solves (6) and is Hermitian positive definite along $w = \bar{z}$ (i.e. the $\mathbb{R}$-bilinear form $(X,Y) = \langle \mathrm{Ad}a.X, \bar{Y}\rangle$ on $\mathfrak{g}^{\mathbb{C}}$ obtained from the Killing form $\langle\cdot,\cdot\rangle$ is Hermitian positive definite) then $a = \bar{s}^{-1}s$ for a (unique) $s : \Omega \to B$ and this satisfies (5). Therefore we will think of (6) as the complexification of (5).

Although there may well be other complexifications this is the natural one with respect to the case of the Toda fields mentioned earlier. In that case one knows (from e.g. [10]) that the system (6) arises from complexifying the map $z \mapsto (\Phi_\xi(z), \overline{\Phi_\xi(z)})$ and then using the factorisation available in $\mathcal{U} \subset \Lambda_C G^{\mathbb{C}}_\sigma$. This can be done for any holomorphic potential in the following way.

For fixed $\xi \in \mathcal{P}$ and $g \in \Lambda_{I,B}G_\sigma$ let us define $\chi : \Omega^2 \subset \mathbb{C}^2 \to \Lambda_C G^{\mathbb{C}}_\sigma$ to be the holomorphic map given by

$$\chi(z,w) = g(\Phi^\xi(z), \overline{\Phi^\xi(\bar{w})}). \tag{7}$$

Since $\chi(0,0)=g$ belongs to $\mathcal{U}$ (by lemma 1) we deduce from theorem 3 that off an analytic subvariety $S\subset\Omega^2$ this maps into $\mathcal{U}$ and we write it as $\chi=\chi^E\chi^I$ there. With a slight abuse of notation we will write $\chi^I=(\chi_1,\chi_2)$. Now we observe

$$(\chi^E)^{-1}\partial_z\chi^E=\begin{cases}\mathrm{Ad}\chi_1.\xi(z)+\chi_1\partial_z\chi_1^{-1} & \text{on } C_1,\\ \chi_2\partial_z\chi_2^{-1} & \text{on } C_2.\end{cases}$$

It follows that this must be meromorphic on $\mathbb{P}_\lambda$ with a simple pole only at 0 and a simple zero only at ∞. Therefore

$$(\chi^E)^{-1}\partial_z\chi^E=\lambda^{-1}\mathrm{Ad}a(z,w).\xi_{-1}(z) \tag{8}$$

where $a=\chi_1|_{\lambda=0}$. Similarly we compute the w-derivative to obtain

$$(\chi^E)^{-1}\partial_w\chi^E=a\partial_w a^{-1}+\lambda\overline{\xi_{-1}(\bar{w})}. \tag{9}$$

The compatibility conditions between (8) and (9) are clearly (6). Moreover, we have:

Lemma 4. *Along $w=\bar{z}$ we have $a=\bar{s}^{-1}s$ and $g\#\Phi_\lambda=\chi^E\bar{s}^{-1}$.*

Proof. The map χ equals $g\Phi(\xi)$ along $w=\bar{z}$ and we can write the Iwasawa decomposition for the latter as $(g\#\Phi_\lambda)(s,\bar{s})(n,\bar{n})$. Using (2) it follows that $\chi^E=(g\#\Phi_\lambda)\bar{s}$ and $\chi^I|_{\lambda=0}=(\bar{s}^{-1}s,e)$. □

4. Meromorphic potentials.

One of the principal observations of [8] (stated there for symmetric spaces but equally applicable to the present case) is that every map in $\mathcal{H}$ can also be obtained from a *meromorphic* section of $T^*_{1,0}\Omega\otimes\mathfrak{g}_{-1}$ using the procedure described above for holomorphic potentials. Our aim is to show that these meromorphic potentials may be expressed purely in terms of the leading order term $\xi_{-1}(z)$ and the field $a(z,w)$ evaluated along $w=0$. So first we recall how to obtain the meromorphic potential for $\phi(\xi)$.

Since $\Phi^\xi:\Omega\to L_1G^{\mathbb{C}}_\sigma$ is holomorphic with $\Phi^\xi(0)=e$ it has a Birkhoff factorisation $\Phi^\xi=F_-F_+$ off a discrete subset $D\subset\Omega$ by theorem 3. We compute

$$\begin{aligned}F_-^{-1}\partial F_- &= \mathrm{Ad}F_+.\xi+F_+\partial F_+^{-1}\\ &= \lambda^{-1}\mathrm{Ad}f.\xi_{-1}\end{aligned} \tag{10}$$

where $f=F_+|_{\lambda=0}$ is a meromorphic $K^{\mathbb{C}}$-valued function on Ω. Therefore $\eta=\mathrm{Ad}f.\xi_{-1}$ is a meromorphic differential on Ω with values in $\mathfrak{g}_{-1}$: we call this the *meromorphic potential* for $\phi(\xi)$. We may recover the map $\phi(\xi)$ by combining the observation that F_- above is the unique solution to (10) normalised at $z=0$ with the result:

Theorem 4 ([8])**.** *Let $F=(F_-,\bar{F}_-):\Omega\to\Lambda_CG_\sigma$ and define $\tilde{\Phi}=F_E$. Then $\tilde{\Phi}_1$ frames the harmonic map $\phi(\xi)$.*

Moreover, from the proof of this theorem in [8] we learn that the meromorphic potential is independent of the frame. Now, keeping the notations of the previous section, our principal observation is:

Theorem 5. *Let $g\in\Lambda_{I,B}G_\sigma$ then the meromorphic potential $\eta(z)$ for $\phi(\mathrm{Ad}g.\xi)$ is given by*

$$\eta(z)=\mathrm{Ad}\bar{b}a(z,0).\xi_{-1}(z), \tag{11}$$

where $(b,\bar{b})$ is the B-factor of g.

Observe that η is determined by ξ_{-1} and $a(z,0)$ only, since b is uniquely determined by the equation $\bar{b}^{-1}b = a(0,0)$.

Proof. Write $\mu = \mathrm{Ad}bn.\xi$, where $g = (b,\bar{b})(n,\bar{n})$. Then $\mu_{-1} = \mathrm{Ad}b.\xi_{-1}$ so that

$$\eta = \mathrm{Ad}(F_+|_{\lambda=0}b).\xi_{-1}$$

where $\Phi^\mu = F_-F_+$ on $\Omega\backslash D$ for some discrete subset D. Observe that $\Phi^\mu = \mathrm{Ad}bn.\Phi^\xi$ so that $P = bn\Phi^\xi$ has factorisation $P = P_-P_+$ on $\Omega\backslash D$ with $P_+(bn)^{-1} = F_+$ (since $(bn)_+ = bn$). On the other hand, the definition (7) gives $\chi(z,0) = (P(z), \bar{b}\bar{n})$ and using the formula (1) we compute

$$\chi_1(z,0) = [(\bar{b}\bar{n})_+^{-1}P_-]_+P_+ = \bar{b}^{-1}P_+$$

Therefore, $F_+|_{\lambda=0}$ is given by

$$[\bar{b}\chi_1(z,0)(bn)^{-1}]_{\lambda=0} = \bar{b}a(z,0)b^{-1}$$

from which we obtain (11). □

As a corollary we deduce a formula relating the meromorphic potential for $\phi(\mathrm{Ad}g.\xi)$ to that for $\phi(\xi)$. For clarity, let η_g denote the meromorphic potential for the former and let $a_g(z,w)$ denote the solution of (6) corresponding to the point $\mathrm{Ad}g.\xi$ in $\mathcal{P}$. Then

$$\eta_g(z) = \mathrm{Ad}\,\bar{b}a_g(z,0)a_e(z,0)^{-1}.\eta_e(z). \tag{12}$$

4.1. **Example: Abelian Toda fields.** To interpret theorem 5 more geometrically we look at the case of primitive harmonic maps into the full flag manifolds of simple Lie groups, i.e., the abelian Toda field equations (see [3]). The simplest example, when the group is SU_2, is of classical interest in surface theory since it describes harmonic maps into $\mathbb{C}P^1$, the Gauss normal maps of constant mean curvature surfaces. The corresponding field equation is the sinh-Gordon equation.

The setup is as follows: G is a simple compact Lie group and σ is the Coxeter automorphism so that $K = T$ is a maximal torus and every element of $\mathfrak{g}_{-1}$ is either nilpotent or semisimple. Recall that in this case $\mathfrak{g}_1$ $(= \overline{\mathfrak{g}_{-1}})$ is the sum of root spaces for the roots $\alpha_0, \dots, \alpha_n$, where $\alpha_1, \dots, \alpha_n$ are simple roots and $-\alpha_0$ is the maximal root. Therefore there are positive integers n_i for which $\sum n_i\alpha_i = 0$. We can always find root vectors e_i for these roots for which the numbers

$$\alpha_j([e_i, -\bar{e}_i]) = c_{ji}$$

give the entries of the extended Cartan matrix for $\mathfrak{g}^{\mathbb{C}}$. If we set

$$\xi_{-1} = -\sum_{i=0}^{n} \sqrt{n_i}\bar{e}_i dz \tag{13}$$

then the equations (5) for $s = \exp(u)$, $u : \Omega \to \mathfrak{a}$, where $\mathfrak{a} = i\mathfrak{t}$, amount to the affine Toda field equations

$$u_{z\bar{z}} - \sum_{i=0}^{n} n_i e^{-2\alpha_i(u)}[\bar{e}_i, e_i] = 0. \tag{14}$$

In the traditional (but linearly dependent) coordinates $u_j = \alpha_j(u)$ these look like

$$\frac{\partial^2 u_j}{\partial z \partial \bar{z}} - \sum_{i=0}^{n} n_i e^{-2u_i} c_{ji} = 0.$$

The system (14) has the 'vacuum' solution $u \equiv 0$, for which the corresponding holomorphic potential is simply given by $\xi = \lambda^{-1}\xi_{-1}$ from (13). One knows from [7] that any constant holomorphic potential with this same leading term ξ_{-1} (in particular, all finite type solutions and thus all doubly periodic solutions) are contained in the same dressing orbit O_ξ. Now let $g \in \Lambda_{I,B}G_\sigma$ and let $a : \Omega^2 \to T^{\mathbb{C}}$ be given by $\chi_1|_{\lambda=1}$ as in the previous section. Then we can write $a = e^{2v}$ and, by lemma 4, the restriction $u = v(z,\bar{z})$ is a solution to (14). The corresponding meromorphic potential will be

$$\eta(z) = -\sum_{i=0}^{n} \sqrt{n_i} e^{v_i(0,0) - 2v_i(z,0)} \bar{e_i} \tag{15}$$

since $\bar{b}^{-1}b = e^{2v(0,0)}$ and $\bar{b}^{-1} = b$.

In the case of $G = SU_2$ the equations (14) reduce to the sinh-Gordon equation

$$u_{z\bar{z}} + 2\sinh(2u) = 0$$

and the meromorphic potential is given by

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & e^{-v(0,0)+2v(z,0)} \\ e^{v(0,0)-2v(z,0)} & 0 \end{pmatrix} dz\,.$$

To obtain the meromorphic potentials for constant mean curvature surfaces in the dressing orbit through the generalised Smyth surfaces (see [8]) we take as the vacuum solution the holomorphic potential

$$\mu = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix} dz$$

where $Q : \Omega \to \mathbb{C}$ is holomorphic, the Hopf differential of the corresponding constant mean curvature surface. If Q has zeros (the umbilics of the surface) then the dressing orbit through μ is different from the one through ξ (where $Q \equiv 1$) and the meromorphic potential for any surface in this orbit through μ is given by

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & e^{-v(0,0)+2v(z,0)} \\ Qe^{v(0,0)-2v(z,0)} & 0 \end{pmatrix} dz\,.$$

Here $v : \Omega^2 \to \mathbb{C}$ is the complex solution of the generalised sinh-Gordon equation

$$v_{zw} + (e^{2v} - Q(z)\overline{Q(\bar{w})}e^{-2v}) = 0$$

which gives the real solution u when evaluated along $w = \bar{z}$.

References

[1] M Black, *Harmonic maps into homogeneous spaces*, Pitman Research Notes in Math. 255, Longman, Harlow, 1991.

[2] A I Bobenko, *All constant mean curvature tori in* $\mathbf{R}^3$, S^3, H^3 *in terms of theta-functions*, Math. Ann. **290** (1991), 209-245.

[3] J Bolton, F Pedit & L M Woodward, *Minimal surfaces and the affine Toda field model*, J. reine angew. Math. 459 (1995), 119-150.

[4] F E Burstall, *Harmonic tori in spheres and complex projective spaces*, University of Bath Math. Preprint no. 93/01 (1993).

[5] F E Burstall, D Ferus, F Pedit & U Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. 138 (1993), 173-212.

[6] F E Burstall & F Pedit, *Harmonic maps via Adler-Kostant-Symes theory*, in *Harmonic maps and integrable systems* ed: A P Fordy & J C Wood, Aspects of Mathematics E23, Vieweg, Wiesbaden, 1994.

[7] F E Burstall & F Pedit, *Dressing orbits of harmonic maps*, GANG preprint IV.4, University of Massachusetts at Amherst (1994).
[8] J Dorfmeister, F Pedit & Hongyou Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, GANG preprint III.25, University of Massachusetts at Amherst (1994).
[9] J Dorfmeister & H Wu, *Constant mean curvature surfaces and loop groups*, J. Reine Angew. Math., 440 (1993), 43-76.
[10] I McIntosh,*Global solutions of the elliptic 2D periodic Toda lattice*, Nonlinearity 7 (1994), 85-108.
[11] I McIntosh, *Infinite dimensional Lie groups and the two-dimensional Toda lattice* in *Harmonic maps and integrable systems*, ed: A P Fordy & J C Wood, Aspects of Mathematics E23, Vieweg, Wiesbaden, 1994.
[12] A Pressley & G Segal, *Loop groups*, Oxford Mathematical Monographs, OUP, Oxford, 1986.

Department of Mathematics, University of Kansas, Lawrence, KS 66045

E-mail address, Josef Dorfmeister: `dorfmeister@math.ukans.edu`

School of Mathematical Sciences, University of Bath, Bath, BA7 2AY

E-mail address, Ian McIntosh: `i.r.mcintosh@maths.bath.ac.uk`

Department of Mathematics, University of Massachusetts, Amherst, MA 01003

E-mail address, Franz Pedit: `franz@gang.umass.edu`

Department of Mathematics, Northern Illinois University, DeKalb, IL 60115

E-mail address, Hongyou Wu: `wu@math.niu.edu`