Curved Flats and Isothermic Surfaces

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Abstract

We show how pairs of isothermic surfaces are given by curved flats in a pseudo Riemannian symmetric space and vice versa. Calapso's fourth order partial differential equation is derived and, using a solution of this equation, a Möbius invariant frame for an isothermic surface is built.

1 Introduction

These notes grew out of a series of discussions on a recent paper by J. Cieśliński, P. Goldstein and A. Sym [3]: these authors give a characterization of isothermic surfaces as "soliton surfaces" by introducing a spectral parameter. In trying to understand the geometric meaning of this spectral parameter, we observed some analogies with the theory of conformally flat hypersurfaces in a four-dimensional space form: Guichard's nets may be understood as analogues of isothermic parametrizations of Riemannian surfaces (cf.[6, no.3.4.1]), and so it seems natural to look for relations between the theory of isothermic surfaces in three-dimensional space forms and the theory of conformally flat hypersurfaces in four-dimensional space forms. Here we would like to present some results we found — especially the possibility of constructing isothermic surfaces using

2 Curved Flats

A curved flat is the natural generalization of a developable surface in Euclidean space: it is an (intrinsically) flat submanifold of a symmetric space for which the curvature tensor of the ambient space vanishes on each tangent space and whose dimension is equal to the rank of the symmetric space — this is the maximal dimension possible. Thus, a curved flat may be thought of as the enveloping submanifold of a congruence of flats — totally

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geodesic submanifolds — of the symmetric space[1]. Taking a regular parametrization $\gamma : M^r \to G/K$ of a curved flat, r being the rank of G/K, and a framing $F : M \to G$ of this parametrization, the Maurer-Cartan form $\Phi = F^{-1}dF$ of the framing has a natural decomposition $\Phi = \Phi_k + \Phi_p$ according to the Cartan decomposition $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{p}$ of the Lie algebra $\boldsymbol{g}$. Now the condition for γ to parametrize a curved flat may be formulated as[2]

$$[\Phi_p \wedge \Phi_p] \equiv 0 \ . \tag{1}$$

In the case of a pseudo-Riemannian symmetric space this should be taken as the **Definition** of a curved flat: An immersion $\gamma : M^r \to G/K$ is said to parametrize a CURVED FLAT, if the $\boldsymbol{p}$-part in the Cartan decomposition[3] of the Maurer-Cartan form $F^{-1}dF = \Phi = \Phi_k + \Phi_p$ of a framing $F : M \to G$ of γ defines a congruence $p \mapsto \Phi_p|_p(T_pM)$ of maximal abelian subalgebras[4] of $\boldsymbol{g}$.

At this point we should remark that curved flats naturally arise in one parameter families [5]: setting

$$\Phi_\lambda := \Phi_k + \lambda\Phi_p \tag{2}$$

the Maurer-Cartan equation $d\Phi_\lambda + \frac{1}{2}[\Phi_\lambda \wedge \Phi_\lambda] = 0$ for the loop $\lambda \mapsto \Phi_\lambda$ of forms splits into the three equations

$$\begin{array}{l} 0 = d\Phi_k + \frac{1}{2}[\Phi_k \wedge \Phi_k] \\ 0 = d\Phi_p + [\Phi_k \wedge \Phi_p] \\ 0 = [\Phi_p \wedge \Phi_p] \ , \end{array} \tag{3}$$

and hence the integrability of the loop $\lambda \mapsto \Phi_\lambda$ is equivalent to the forms Φ_λ being the Maurer-Cartan forms for some framings $F_\lambda : M^r \to G$ of curved flats $\gamma_\lambda : M^r \to G/K$. Thus integrable systems theory may be applied to produce examples.

Now we will consider the case leading to the theory of isothermic surfaces: let

$$G := O_1(5) \quad \text{and} \quad K := O(3) \times O_1(2) \ . \tag{4}$$

The coset space $G_+(5,3) = G/K$ of space-like 3-planes in the Minkowski space $I\!R^5_1$ becomes a six dimensional semi-Riemannian symmetric space of signature $(3,3)$ when endowed with the pseudo Riemannian metric induced by the Killing form. The rank of this

[1] To get developable surfaces in Euclidean space as a special case of curved flats, we have to omit the dimension assumption.

[2] The product

$$[\Phi \wedge \Psi](v, w) := [\Phi(v), \Psi(w)] - [\Phi(w), \Psi(v)]$$

defines a symmetric product on the space of Lie algebra valued 1-forms with values in the space of Lie algebra valued 2-forms.

[3] The subgroup K is not necessarily compact — nevertheless the decomposition $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{p}$ is called "Cartan decomposition", if the characteristic conditions

$$[\boldsymbol{k}, \boldsymbol{k}] \subset \boldsymbol{k}, [\boldsymbol{k}, \boldsymbol{p}] \subset \boldsymbol{p}, [\boldsymbol{p}, \boldsymbol{p}] \subset \boldsymbol{k}$$

are satisfied.

[4] As a consequence, γ is curvature isotropic in the sense [5] and hence intrinsically and extrinsically flat. It is not clear, however, whether curvature isotropic surfaces are necessarily curved flats.

symmetric space being 2, we will consider two-dimensional curved flats

$$\gamma : M^2 \to G_+(5,3)\ . \tag{5}$$

Fixing a pseudo orthonormal basis $(e_1, \ldots, e_5)$ of the Minkowski space $I\!R^5_1$ with

$$(\langle e_i, e_j \rangle)_{ij} = E_5 := \begin{pmatrix} I_3 & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{6}$$

we get the matrix representations

$$\begin{array}{l} O_1(5) = \{A \in Gl(5, I\!R) | A^t E_5 A = E_5\} \\ \boldsymbol{o}_1(5) = \{X \in gl(5, I\!R) | (E_5 X) + (E_5 X)^t = 0\}\ . \end{array} \tag{7}$$

The subalgebra $\boldsymbol{k}$ and its complementary linear subspace $\boldsymbol{p}$ in the Cartan decomposition of $\boldsymbol{o}_1(5)$ are given by the $+1$- resp. -1-eigenspaces of the involutive automorphism $\mathrm{Ad}(Q) : \boldsymbol{o}_1(5) \to \boldsymbol{o}_1(5)$ with $Q = \left(\begin{smallmatrix} -I_3 & 0 \\ 0 & I_2 \end{smallmatrix}\right)$. Writing down the Maurer-Cartan form of a framing $F : M^2 \to O_1(5)$ of our curved flat $\gamma : M^2 \to G_+(5,3)$ with this notation we obtain

$$\begin{array}{lll} F^{-1}dF = \Phi = \Phi_k + \Phi_p & \text{with} & \\ \Phi_k = \begin{pmatrix} \Omega & 0 \\ 0 & \boldsymbol{\nu} \end{pmatrix} & : & TM \to \boldsymbol{o}(3) \times \boldsymbol{o}_1(2) \\ \Phi_p = \begin{pmatrix} 0 & \eta \\ -E_2\eta & 0 \end{pmatrix} & : & TM \to \boldsymbol{p}\ . \end{array} \tag{8}$$

Since Φ_p is abelian, we can put η into the standard form

$$\eta = \begin{pmatrix} \omega_1 & -a\omega_1 \\ \omega_2 & a\omega_2 \\ 0 & 0 \end{pmatrix} \quad (a \in C^\infty(M)) \tag{9}$$

by taking an $O(3)$-gauge $\left(\begin{smallmatrix} H & 0 \\ 0 & I_2 \end{smallmatrix}\right) : M \to O(3) \times O_1(2)$. Henceforth, we make the regularity assumption that a is never zero[5]. We may then take $a \equiv 1$ by applying an $O_1(2)$-gauge.

Calculating the Maurer-Cartan equation using the ansatz

$$\Omega = \begin{pmatrix} 0 & \omega & -\psi_1 \\ -\omega & 0 & -\psi_2 \\ \psi_1 & \psi_2 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nu} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \tag{10}$$

together with η given by (9), we see that

$$d\omega_1 = d\omega_2 = 0\ . \tag{11}$$

[5]This is equivalent to the assumption that the metric induced by the immersion γ (by pulling back the Killing form) is nondegenerate.

So we are given canonical coordinates $(x, y) : M \to I\!R^2$ by integrating[6] the forms ω_1 and ω_2. Moreover, since we also have $d\nu = 0$, we may set $\nu = -du$ for a suitable function $u \in C^\infty(M)$ — this gives us $\omega = u_y dx - u_x dy$, where u_x and u_y denote the partial derivatives of u in x- resp. y-directions. Finally, the equations $\psi_1 \wedge \omega_1 = 0$ and $\psi_2 \wedge \omega_2 = 0$ show that $\psi_1 = e^u k_1 dx$ and $\psi_2 = e^u k_2 dy$ for two functions $k_i \in C^\infty(M)$.

We now perform a final $O_1(2)$-gauge $\begin{pmatrix} I_3 & 0 \\ 0 & \begin{smallmatrix} e^u & 0 \\ 0 & e^{-u} \end{smallmatrix} \end{pmatrix} : M \to O(3) \times O_1(2)$ and insert the spectral parameter λ to obtain the Maurer-Cartan form discussed in (cf.[3]):

$$\Phi_\lambda = \begin{pmatrix} 0 & u_y dx - u_x dy & -e^u k_1 dx & \lambda e^u dx & -\lambda e^{-u} dx \\ -u_y dx + u_x dy & 0 & -e^u k_2 dy & \lambda e^u dy & \lambda e^{-u} dy \\ e^u k_1 dx & e^u k_2 dy & 0 & 0 & 0 \\ \lambda e^{-u} dx & -\lambda e^{-u} dy & 0 & 0 & 0 \\ -\lambda e^u dx & -\lambda e^u dy & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

We are now lead directly to the theory of

3 Isothermic Surfaces

In the context of Möbius geometry the three sphere S^3 is viewed as the projective light-cone $I\!PL^4$ in $I\!R^5_1$ while the Lorentzian sphere $\{v \in I\!R^5_1 | \langle v, v\rangle = 1\}$ should be interpreted as the space of (oriented) spheres in the three sphere[7] (cf.[1]). Now, denoting by

$$\begin{aligned} n &:= Fe_3 : M \to S^5_1 = \{v \in I\!R^5_1 | \langle v, v\rangle = 1\} \\ f &:= Fe_4 : M \to L^4 = \{v \in I\!R^5_1 | \langle v, v\rangle = 0\} \\ \hat{f} &:= Fe_5 : M \to L^4 \end{aligned} \tag{13}$$

one of the sphere congruences resp. the two immersions given by our frame F, we see that

Theorem: *The sphere congruence n given by our curved flat is a Ribeaucour sphere congruence[8], which is enveloped by two isothermic immersions f and $\hat{f}$* (cf.[1, S.362]):

Since

$$\begin{aligned} \langle f, n\rangle = 0 \quad &\text{and} \quad \langle df, n\rangle \equiv 0, \\ \langle \hat{f}, n\rangle = 0 \quad &\text{and} \quad \langle d\hat{f}, n\rangle \equiv 0\ , \end{aligned} \tag{14}$$

the immersions f and $\hat{f}$ do envelop the sphere congruence n and, since the bilinear forms

$$\begin{aligned} \langle df, dn\rangle &= \lambda e^{2u}(k_1 dx^2 + k_2 dy^2), \\ \langle d\hat{f}, dn\rangle &= \lambda(-k_1 dx^2 + k_2 dy^2) \end{aligned} \tag{15}$$

[6] Since our theory is local, all closed forms may be assumed to be exact.

[7] Or, equivalently, it may be interpreted as the space of (oriented) spheres and planes in Euclidean three space $I\!R^3$: the polar hyperplane to a vector v of the Lorentz sphere intersects the three sphere — thought of as the absolute quadric in projective four space — in a two sphere. Stereographic projection yields a sphere in $I\!R^3$ or, if the projection center lies on the sphere, a plane.

[8] The curvature lines on the two enveloping immersions correspond.

are diagonal with respect to the induced metrics

$$
\begin{aligned}
\langle df, df\rangle &= \lambda^2 e^{2u}(dx^2+dy^2),\\
\langle d\hat f, d\hat f\rangle &= \lambda^2 e^{-2u}(dx^2+dy^2)\ ,
\end{aligned}
\tag{16}
$$

the two immersions f and $\hat f$ are isothermic[9].

It is quite difficult to calculate the first and second fundamental forms of these isothermic immersions, when they are projected to S^3 resp. $I\!R^3$, but applying a (constant) conformal change (constant $O_1(2)$-gauge)

$$
\begin{array}{llll}
f \rightsquigarrow \frac{1}{\lambda} f & \text{and} & \hat f \rightsquigarrow \lambda \hat f & \text{or}\\
f \rightsquigarrow \lambda f & \text{and} & \hat f \rightsquigarrow \frac{1}{\lambda}\hat f &
\end{array}
\tag{17}
$$

and sending $\lambda \to 0$, $\hat f$ resp. f becomes a constant vector — $\Phi_{\lambda=0}e_5$ resp. $\Phi_{\lambda=0}e_4$ vanishes. This constant light-like vector may be interpreted as the point at infinity and we therefore obtain an isothermic immersion $f: M \to I\!R^3$ with first and second fundamental forms

$$
\begin{aligned}
I &= e^{2u}(dx^2+dy^2)\\
II &= e^{2u}(k_1dx^2+k_2dy^2)
\end{aligned}
\tag{18}
$$

resp. its Euclidean dual surface $\hat f: M\to I\!R^3$ with first and second fundamental forms

$$
\begin{aligned}
I &= e^{-2u}(dx^2+dy^2)\\
II &= -k_1dx^2+k_2dy^2\ .
\end{aligned}
\tag{19}
$$

We now recognise the remaining three equations from the Maurer-Cartan equation for Φ_λ

$$
\begin{aligned}
0 &= \Delta u + e^{2u}k_1k_2\\
0 &= k_{1y} + (k_1-k_2)u_y\\
0 &= k_{2x} - (k_1-k_2)u_x
\end{aligned}
\tag{20}
$$

as the Gauß and Codazzi equations of the Euclidean immersion f resp. its dual $\hat f$ [10] [2], [3]. As a consequence, we can invert our construction and build a curved flat from an

[9] The bundle defined by span$\{n, f, \hat f\}$ over M is flat (cf.(12)) and so the map $p \mapsto d_pf(T_pM)$ defines a normal congruence of circles [4]: for each $p\in M$

$$
t \mapsto f_t(p) := \frac{1}{\sqrt{2}}\sin t\cdot n(p) + \frac{1}{2}(1+\cos t)\cdot f(p) - \frac{1}{2}(1-\cos t)\cdot \hat f(p)
$$

parametrizes the circle $(d_pf(T_pM))^\perp$ meeting the sphere $n(p)$ in $f(p)$ and $\hat f(p)$ orthogonal. Since n, f and $\hat f$ are parallel sections in this bundle, the maps $p\mapsto f_t(p)$ (which generically are not degenerate) parametrize the surfaces orthogonal to this congruence of circles.

In general the immersions f and $\hat f = f_\pi$ will be the only isothermic surfaces among the surfaces.

[10] When the normal congruence of circles mentioned in footnote 9 (p.5) is projected to Euclidean three space $I\!R^3$, we see that, in the limit $\lambda\to 0$, the circles become straight lines — circles meeting the collapsed surface $\hat f$ resp. f in the point at infinity — while the Ribeaucour sphere congruence enveloped by the two surfaces f and $\hat f$ becomes the congruence of tangent planes of f resp. $\hat f$.

isothermic surface:
Theorem. *Given an isothermic surface $f : M^2 \to I\!R^3$ and its Euclidean dual surface $\hat{f} : M \to I\!R^3$ we get a curved flat $\gamma : M \to G_+(5,3)$ integrating the Maurer-Cartan form (12), which we are able to write down knowing the first and second fundamental forms of the immersions f and $\hat{f}$* [11].

Another way to obtain these two Euclidean immersions is presented in [3]. Applying Sym's formula to the associated family of frames $F = F(\lambda)$, we obtain a map

$$(\tfrac{\partial}{\partial\lambda}F)F^{-1}|_{\lambda=0} : M \to \boldsymbol{p}\ ; \tag{21}$$

interpreting $\boldsymbol{p}$ as two copies of Euclidean three space[12] $I\!R^3$ this map gives us the immersion f, and in the other copy of $I\!R^3$, its dual $\hat{f}$: this can be seen by looking at the differential

$$\begin{aligned} d(\tfrac{\partial}{\partial\lambda}F)F^{-1}|_{\lambda=0} &= F_0\Phi_p F_0^{-1} \\ &\cong H_3 \begin{pmatrix} e^u dx & -e^{-u}dx \\ e^u dy & e^{-u}dy \\ 0 & 0 \end{pmatrix} . \end{aligned} \tag{22}$$

Here $F_0 = \begin{pmatrix} H_3 & 0 \\ 0 & I_2 \end{pmatrix}$ solves the equation $F_0^{-1}dF_0 = \Phi_k$ and thus we may view $H_3 : M \to O(3)$ as an Euclidean framing of f resp. $\hat{f}$.

There is another possibility for producing isothermal surfaces in Euclidean space $I\!R^3$ (or S^3): that is, by using a solution of

4 Calapso's equation

To understand this, we write down the Maurer-Cartan form of a frame $F : M \to O_1(5)$, which is Möbius-invariantly connected to a given immersion: taking $f = Fe_4$ the (unique) isometric lift of the isothermic immersion and $n = Fe_3$ the central sphere congruence (conformal Gauß map) of the immersion, the frame is determined by the assumption of being an adapted frame (i.e. $Fe_1 = f_x$ and $Fe_2 = f_y$). The associated Maurer-Cartan form will be

$$\Phi = \begin{pmatrix} 0 & 0 & kdx & dx & \chi_1 \\ 0 & 0 & -kdy & dy & \chi_2 \\ -kdx & kdy & 0 & 0 & \tau \\ -\chi_1 & -\chi_2 & -\tau & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix} , \tag{23}$$

k^2 being the conformal factor relating the metric induced by the central sphere congruence to the isometric one, and the 1-forms χ_1, χ_2 and τ to be determined. From the Maurer-

[11]Since this construction depends on the conformal rather than the Euclidean geometry of the ambient space, we generally get a whole three parameter family of "loops" of curved flats from one isothermic surface: when viewing our given isothermic surface as a surface in the three sphere S^3, we may choose the point at infinity arbitrarily.

[12]Here the Euclidean metric is induced by the quadratic form $\frac{1}{2}\mathrm{tr}\Phi_p^t\Phi_p$ instead of the Killing form.

Cartan equation for this form we learn that

$$
\begin{aligned}
\tau &= k_x dx - k_y dy \\
\chi_1 &= (\tfrac{1}{2}k^2 - u)dx - \tfrac{k_{xy}}{k}dy \\
\chi_2 &= -\tfrac{k_{xy}}{k}dx + (\tfrac{1}{2}k^2 + u)dy \ ,
\end{aligned} \tag{24}
$$

where $u \in C^\infty(M)$ is a function satisfying the differential equation

$$du = -((\tfrac{k_{xy}}{k})_y + (k^2)_x)dx + ((\tfrac{k_{xy}}{k})_x + (k^2)_y)dy \tag{25}$$

— the integrability condition of this equation is a fourth order partial differential equation closely related to Calapso's original equation [2]:

$$0 = \Delta(\tfrac{k_{xy}}{k}) + 2(k^2)_{xy} \tag{26}$$

This shows, that
Theorem: *Any isothermic surface gives rise to a solution of Calapso's equation.*

Conversely, from a solution $k \in C^\infty(M)$ of Calapso's equation we can construct a Möbius invariant frame of an isothermic surface by integrating the Maurer-Cartan form (23), where the function u is a solution of (25).

Now, applying a conformal change $f \rightsquigarrow \frac{1}{k}f$ while fixing the central sphere congruence $n \rightsquigarrow n$, the Maurer-Cartan form of the associated frame becomes

$$\Phi = \begin{pmatrix} 0 & \omega & kdx & \frac{1}{k}dx & \chi_1 \\ -\omega & 0 & -kdy & \frac{1}{k}dy & \chi_2 \\ -kdx & kdy & 0 & 0 & 0 \\ -\chi_1 & -\chi_2 & 0 & 0 & 0 \\ -\frac{1}{k}dx & -\frac{1}{k}dy & 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

where

$$
\begin{aligned}
\omega &= -\tfrac{k_y}{k}dx + \tfrac{k_x}{k}dy \\
\chi_1 &= k(\tfrac{k_{xx}}{k} - \tfrac{k_x^2+k_y^2}{2k^2} + \tfrac{1}{2}k^2 - u)dx \\
\chi_2 &= k(\tfrac{k_{yy}}{k} - \tfrac{k_x^2+k_y^2}{2k^2} + \tfrac{1}{2}k^2 + u)dy
\end{aligned} \ . \tag{28}
$$

Here we see that the central sphere congruence of an isothermic surface is a Ribeaucour sphere congruence, which actually is a characterisation of isothermic surfaces (cf.[1, S.374]), and hence it has flat normal bundle as a codimension two surface in the Lorentz sphere S^4_1.

In general, the second enveloping surface of the central sphere congruence of an isothermic surface will not be an isothermic surface and it seems to be difficult to built a curved flat starting with it. But in a quite simple case this is possible:

5 Example

Starting with a surface of revolution

$$f(x,y) = (r(x)\cos y, r(x)\sin y, z(x)) \ , \tag{29}$$

the functions r and z satisfying the differential equation

$$r^2 = r'^2 + z'^2 \ , \tag{30}$$

i.e. the curve (r, z) being parametrized by arc length (thought of as a curve in the Poincaré half plane), we may write down the loop of Maurer-Cartan forms

$$\Phi_\lambda = \begin{pmatrix} 0 & -\frac{r'}{r}dy & -\frac{r'z''-r''z'}{r^2}dx & \lambda r\,dx & -\frac{\lambda}{r}dx \\ \frac{r'}{r}dy & 0 & -\frac{z'}{r}dy & \lambda r\,dy & \frac{\lambda}{r}dy \\ \frac{r'z''-r''z'}{r^2}dx & \frac{z'}{r}dy & 0 & 0 & 0 \\ \frac{\lambda}{r}dx & -\frac{\lambda}{r}dy & 0 & 0 & 0 \\ -\lambda r\,dx & -\lambda r\,dy & 0 & 0 & 0 \end{pmatrix} , \tag{31}$$

which gives us the immersion f and its dual $\hat{f}$ in the limit $\lambda \to 0$.

On the other hand, denoting by $H = \frac{1}{2}(\frac{z'}{r^2} + \frac{r'z''-r''z'}{r^3})$ the mean curvature of our surface of rotation, the central sphere congruence of f is $n + Hf$. The metric it induces has conformal factor k^2 (relative to the metric induced by f) given by

$$k = \tfrac{1}{2r^2}(rz' - r'z'' + r''z') \ . \tag{32}$$

Since $k_y \equiv 0$, this is obviously a solution of Calapso's equation and a function u solving (25) is $u = \lambda^2 - k^2$. So the Maurer-Cartan form (23) becomes

$$\Phi_\lambda = \begin{pmatrix} 0 & 0 & k\,dx & dx & (\frac{3}{2}k^2 - \lambda^2)dx \\ 0 & 0 & -k\,dy & dy & (-\frac{1}{2}k^2 + \lambda^2)dy \\ -k\,dx & k\,dy & 0 & 0 & k_x\,dx \\ -(\frac{3}{2}k^2 - \lambda^2)dx & (\frac{1}{2}k^2 - \lambda^2)dy & -k_x\,dx & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix} . \tag{33}$$

A change $n \rightsquigarrow n + kf$ of the sphere congruence, enveloped by f, followed by an $O_1(2)$-gauge $f \rightsquigarrow \lambda f$ and $\hat{f} \rightsquigarrow \lambda^{-1}\hat{f}$ gives us the Maurer-Cartan form

$$\Phi_\lambda = \begin{pmatrix} 0 & 0 & 2k\,dx & \lambda dx & -\lambda dx \\ 0 & 0 & 0 & \lambda dy & \lambda dy \\ -2k\,dx & 0 & 0 & 0 & 0 \\ \lambda dx & -\lambda dy & 0 & 0 & 0 \\ -\lambda dx & -\lambda dy & 0 & 0 & 0 \end{pmatrix} . \tag{34}$$

of a curved flat, quite different from that coming from (31).

To understand the geometry of the two enveloping immersions $f = Fe_4$ and $\hat{f} = Fe_5$, we remark that the sphere congruence $n = Fe_3$ depends only on one variable and hence the two immersions parametrize a channel surface; moreover all spheres of the congruence are perpendicular to the fixed circle[13] given by $\text{span}\{Fe_2, F(e_4 + e_5)\}$, which may be thought as an axis of rotation: the immersions f and $\hat{f}$ parametrize pieces of a surface of revolution[14], f and $\hat{f}$ being axisymmetric[15]. Taking now the limit $\lambda \to 0$, we obtain a cylinder resp. its dual, which is an (axial) reflection of the original cylinder.

[13] We have $\Phi e_2 = -(e_4 + e_5)dy$ and $\Phi(e_4 + e_5) = 2e_2 dy$.

[14] The meridian curve is given by $\frac{1}{\sqrt{2}}(f - \hat{f})$ — which only depends on one variable — thought as a curve in the Poincaré half plane; its tangent field is given by Fe_1 and its unit normal field by $n = Fe_3$.

[15] The circles $\{F(p)e_1, F(p)e_2\}^\perp$ intersecting the sphere $n(p)$ orthogonally in $f(p)$ and $\hat{f}(p)$ all meet the axis (cf. footnote 9, page 5).

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