MINIMAL SURFACES VIA LOOP GROUPS

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1. Introduction

The study of minimal surfaces in $\mathbb{R}^3$ has a long and rather fruitful history [10]. In this note we describe how the methods developed in [7] to construct harmonic maps of Riemann surfaces into symmetric spaces apply to minimal surfaces. The link is provided by the Gauss map which is harmonic for surfaces of constant mean curvature H. In fact, the Gauss map is holomorphic for minimal surfaces $H = 0$. As is well known, e.g. [14, 4], the infinte dimensional loop group constructions reduce to their finite dimensional analogs in this particular case. This note recovers these general observations for the special case at hand and thereby derives the classical constructions and formulas used in (local) minimal surface theory. To see this well known example worked through in somewhat more detail from the loop group point of view might assist Readers not aquainted with these techniques to become more familiar with them.

In the first section we briefly overview the so called DPW-method [7] for the special case of harmonic maps into the symmetric space S^2. We then specialize in the next section to holomorphic maps into S^2. We describe all Gauss maps of minimal surfaces (of a domain) in terms of meromorphic potentials and derive the first and second fundamental forms of the minimal surfaces from the meromorphic potential. We also describe all minimal surfaces to a given Gauss map. In the third section we derive the classical Weierstrass representation for minimal surfaces from the meromorphic potential, thus identifying the DPW-data with the classical Weierstrass data. Section 4 studies the dressing action on minimal surfaces, or to be more precise, on their Gauss maps. As to be expected this action is nothing but the finite dimensional dressing action induced by the $\mathbf{SL}(2, \mathbb{C})$ action on $S^2 = \mathbb{C}P^1$. Section 5 deals with symmetries of minimal surfaces using the approach of [6]. The note finishes with a couple of standard examples illustrating the various constructions described.

2. The DPW–Method Revisited

We briefly review the DPW–construction [7, 14] for harmonic maps $\varphi\colon D \to S^2$ from a contractible domain $D \subset \mathbb{C}$ into the 2–sphere S^2. In this case the harmonic map equation for φ reads

$$\Delta\varphi + |d\varphi|^2\varphi = 0 \tag{2.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian on $\mathbb{C} = \mathbb{R}^2$. To investigate (2.1) we consider the canonical projection $\pi\colon \boldsymbol{SU}(2) \to S^2$ and lift φ to a map $F\colon D \to \boldsymbol{SU}(2)$ with $\pi \circ F = \varphi$. Any such lift (which always exists, since D is contractible) we call a *framing* or a *frame* of φ. Since $F\colon D \to \boldsymbol{SU}(2)$ is determined (up to left multiplication by a constant element of $\boldsymbol{SU}(2)$) by its

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Maurer–Cartan form $\alpha = F^{-1}\,dF$, (2.1) can be expressed as an equation on α. To see this we have to decompose the Lie algebra of $\boldsymbol{SU}(2)$

$$\mathfrak{su}(2) = \mathfrak{k} \oplus \mathfrak{p} \tag{2.2}$$

into diagonal and off–diagonal matrices. This is the Cartan decomposition of the symmetric space $S^2 = \boldsymbol{SU}(2)/S^1$ under the inner involution $\sigma = Ad\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. Thus we can decompose the $\mathfrak{su}(2)$–valued 1–form $\alpha\colon TD \to \mathfrak{su}(2)$ into $\mathfrak{k}$ and $\mathfrak{p}$ parts

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}. \tag{2.3}$$

The harmonic map equation (2.1) then becomes

$$d\alpha_{\mathfrak{k}} = -[\alpha_{\mathfrak{p}}^{'} \wedge \alpha_{\mathfrak{p}}^{''}] \tag{2.4a}$$

$$d\alpha_{\mathfrak{p}}' + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{p}}'] = 0 \tag{2.4b}$$

where, for an $\mathfrak{su}(2)$–valued 1–form $\beta = \beta_z\,dz + \beta_{\bar z}\,d\bar z$ on D, we denote by $\beta' = \beta_z\,dz$ resp. $\beta'' = \beta_{\bar z}\,d\bar z$ its $(1,0)$–part resp. its $(0,1)$–part. Notice that the forms β', β'' take values in the complexification $\mathfrak{sl}(2,\mathbb{C})$ of $\mathfrak{su}(2)$ and, since β takes values in $\mathfrak{su}(2)$, $\beta'' = \overline{\beta'}$ where conjugation is always understood with respect to the compact real form $\mathfrak{su}(2)$, i.e. $\bar\xi = -\xi^*$, $\xi \in \mathfrak{sl}(2,\mathbb{C})$. The importance of this reformulation lies in the fact that equations (2.4) can be expressed as a *single* Maurer–Cartan equation involving an additional (spectral) parameter: consider, for $\lambda \in \mathbb{C}^*$, the $\mathfrak{sl}(2,\mathbb{C})$–valued 1–form

$$A = \lambda^{-1}\alpha_{\mathfrak{p}}' + \alpha_{\mathfrak{k}} + \lambda\alpha_{\mathfrak{p}}''. \tag{2.5}$$

Then one easily checks the following [7, 14]:

i. $A_{\lambda=1} = \alpha$,
ii. A is $\mathfrak{su}(2)$–valued if and only if $\lambda \in S^1 \subset \mathbb{C}$, i.e. $\overline{A_\lambda} = A_{1/\bar\lambda}$,
iii. $dA + \frac{1}{2}[A \wedge A] = 0$ if and only if equations (2.4) hold.

Thus we arrive at the following recipe to construct harmonic maps $\varphi\colon D \to S^2$: find a $\mathbb{C}^*$–family of $\mathfrak{sl}(2,\mathbb{C})$–valued 1–forms $A\colon TD \times \mathbb{C}^* \to \mathfrak{sl}(2,\mathbb{C})$, whose λ dependence is given by (2.5), satisfying the Maurer–Cartan equation $dA + \frac{1}{2}[A \wedge A] = 0$. Then the $\mathfrak{su}(2)$–valued 1–form $\alpha\colon = A_{\lambda=1}\colon TD \to \mathfrak{su}(2)$ integrates to a framing $F\colon D \to \boldsymbol{SU}(2)$, $F^{-1}\,dF = \alpha$, of the harmonic map $\varphi\colon = \pi \circ F\colon D \to S^2$. In fact, instead of evaluating A at $\lambda = 1$, we could have evaluated at any $\lambda \in S^1$, thereby obtaining an S^1–family of framings $F_\lambda\colon D \to \boldsymbol{SU}(2)$, $F_\lambda^{-1}\,dF_\lambda = A_\lambda$, and corresponding harmonic maps $\varphi_\lambda\colon = \pi \circ F_\lambda\colon D \to S^2$. Clearly *every* harmonic map $\varphi\colon D \to S^2$ comes from this construction. The S^1–family of harmonic maps φ_λ to any given harmonic map φ we call the *associated family* of harmonic maps.

To deal more effectively with the λ–dependence it turns out to be useful to study maps into loop spaces rather than to consider families of maps. Consider the infinite dimensional loop Lie group

$$\Lambda\boldsymbol{SL}(2,\mathbb{C}) = \{g\colon S^1 \to \boldsymbol{SL}(2,\mathbb{C})\,;\, g(-\lambda) = \sigma g(\lambda)\}$$

with the corresponding loop Lie algebra

$$\Lambda\mathfrak{sl}(2,\mathbb{C}) = \{\xi\colon S^1 \to \mathfrak{sl}(2,\mathbb{C})\,;\, \xi(-\lambda) = \sigma\xi(\lambda)\}.$$

There are various topologies one can put on these spaces but the choice turns out to be rather irrelevant for our purposes as long as certain factorization theorems hold. To simplify arguments (e.g. local existence of solutions to ODE's) we assume a topology which makes our loop spaces Banach spaces, e.g. any Sobolev H^s–topology with $s > \frac{1}{2}$ will do. If we express $\xi \in \Lambda\,\mathfrak{sl}(2,\mathbb{C})$ as a

Fourier series $\xi = \sum_{k\in\mathbb{Z}} \lambda^k \xi_k$ then the twisting condition $\xi(-\lambda) = \sigma\xi(\lambda)$ simply says that $\xi_{odd} \in \mathfrak{k}^{\mathbb{C}}$ and $\xi_{even} \in \mathfrak{p}^{\mathbb{C}}$. This is necessary if we wish to interpret the 1–form A of (2.5) as having values in $\Lambda\,\mathfrak{sl}(2,\mathbb{C})$. The following subgroups and corresponding sub Lie algebras will be important: the real loop group

$$\Lambda \boldsymbol{SU}(2) = \{g : S^1 \to \boldsymbol{SU}(2)\,;\, g(-\lambda) = \sigma g(\lambda)\} \subset \Lambda\, \boldsymbol{SL}(2,\mathbb{C})$$

with Lie algebra

$$\Lambda\mathfrak{su}(2) = \{\xi \colon S^1 \to \mathfrak{su}(2)\,;\, \xi(-\lambda) = \sigma\xi(\lambda)\} \subset \Lambda\,\mathfrak{sl}(2,\mathbb{C})$$

and the groups

$$\Lambda^+_H \boldsymbol{SL}(2,\mathbb{C}) = \{g \colon S^1 \to \boldsymbol{SL}(2,\mathbb{C})\,;\, g \text{ extends holomorphically to } |\lambda| < 1 \text{ and } g(0) \in H\}\,,$$

where $H \subset \boldsymbol{SL}(2,\mathbb{C})$ is a subgroup. If $H = \boldsymbol{SL}(2,\mathbb{C})$ we will omit the subscript at all and use $*$ if $H = \{1\}$. The Lie algebra of $\Lambda^+_H \boldsymbol{SL}(2,\mathbb{C})$ is

$$\Lambda^+_{\mathfrak{h}}\mathfrak{sl}(2,\mathbb{C}) = \{\xi \colon S^1 \to \mathfrak{sl}(2,\mathbb{C})\,;\, \xi \text{ extends holomorphically to } |\lambda| < 1 \text{ and } \xi(0) \in \mathfrak{h}\}\,,$$

where $\mathfrak{h} \subset \mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra of H. Similarly we define $\Lambda^-_H \boldsymbol{SL}(2,\mathbb{C})$ as those loops $g \colon S^1 \to \boldsymbol{SL}(2,\mathbb{C})$ which extend holomorphically to $|\lambda| > 1$ and for which $g(\infty) \in H$ and its Lie algebra is defined correspondingly.

With these loop groups and loop algebras in place we can reformulate the harmonic map problem as follows: find an $\Lambda\mathfrak{su}(2)$–valued 1–form $A \colon TD \to \Lambda\mathfrak{su}(2)$ such that

i. $dA + \frac{1}{2}[A \wedge A] = 0$.
ii. λA is $\Lambda^+\mathfrak{sl}(2,\mathbb{C})$–valued and $(\lambda A)_{\lambda=0} \in \Omega^{(1,0)}(D)$ is a $(1,0)$–form on D.

By integrating A an equivalent formulation is: find a map $F \colon D \to \Lambda\boldsymbol{SU}(2)$ such that

$$F^{-1}dF = \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 \tag{2.6}$$

with a $(1,0)$–form α_{-1}. Such a map F we will call an *extended framing* (of the harmonic map $\varphi = \pi \circ F_{\lambda=1}$). To construct all extended framings on the domain D from meromorphic data we need two factorization theorems for our loop groups, the Birkhoff and Iwasawa decompositions [12, 7]. The complexified maximal torus $(S^1)^{\mathbb{C}} = \mathbb{C}^* \subset \boldsymbol{SL}(2,\mathbb{C})$ has the decomposition $\mathbb{C}^* = S^1\mathbb{R}^+$, where $\mathbb{R}^+ \subset \boldsymbol{SL}(2,\mathbb{C})$ consists of the matrices $\left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$ with $r > 0$. Denote this subgroup by B.

Theorem 2.1. i. $\Lambda\boldsymbol{SL}(2,\mathbb{C}) = \bigcup_{w\in W} \Lambda^-_* \boldsymbol{SL}(2,\mathbb{C})\, w\, \Lambda^+\boldsymbol{SL}(2,\mathbb{C})$ *where W is the Weyl group of* $\Lambda\boldsymbol{SL}(2,\mathbb{C})$.
ii. *Multiplication* $\Lambda^-_*\boldsymbol{SL}(2,\mathbb{C}) \times \Lambda^+\boldsymbol{SL}(2,\mathbb{C}) \to \Lambda\boldsymbol{SL}(2,\mathbb{C})$ *is a diffeomorphism onto the open and dense subset* $\Lambda^-_*\boldsymbol{SL}(2,\mathbb{C})\,\Lambda^+\boldsymbol{SL}(2,\mathbb{C}) \subset \Lambda\boldsymbol{SL}(2,\mathbb{C})$ *called the* big cell.

Thus every g in the big cell has a unique decomposition

$$g = g_- g_+$$

with $g_- \in \Lambda^-_*\boldsymbol{SL}(2,\mathbb{C})$ *and* $g_+ \in \Lambda^+\boldsymbol{SL}(2,\mathbb{C})$.

Theorem 2.2. *Multiplication* $\Lambda\boldsymbol{SU}(2) \times \Lambda^+_B\boldsymbol{SL}(2,\mathbb{C}) \to \Lambda\boldsymbol{SL}(2,\mathbb{C})$ *is a diffeomorphism. Thus every* $g \in \Lambda\boldsymbol{SL}(2,\mathbb{C})$ *has unique decomposition*

$$g = g_u g_+$$

with $g_u \in \Lambda\boldsymbol{SU}(2)$ *and* $g_+ \in \Lambda^+_B\boldsymbol{SL}(2,\mathbb{C})$.

We can now state the DPW method [7] for the construction of all extended framings $F\colon D \to \Lambda \boldsymbol{SU}(2)$ (and hence all harmonic maps $\varphi\colon D \to S^2$): away from the set $S \subset D$ where F fails to take values in the big cell we can decompose

$$F = F_- F_+$$

with $F_-\colon D \setminus S \to \Lambda_*^- \boldsymbol{SL}(2,\mathbb{C})$ and $F_+\colon D \setminus S \to \Lambda^+ \boldsymbol{SL}(2,\mathbb{C})$. One easily checks

$$F_-\, dF_- = \lambda^{-1} \begin{pmatrix} 0 & p_1 \\ p_2 & 0 \end{pmatrix} dz \tag{2.7}$$

with $p_i\colon D \setminus S \to \mathbb{C}$ holomorphic.

Theorem 2.3. *The set $S \subset D$ is discrete and F_- extends meromorphically to D.*

We call meromorphic maps $F_-\colon D \to \Lambda_*^- \boldsymbol{SL}(2,\mathbb{C})$ *complex extended frames.* Notice that those frames are determined by two meromorphic functions $p_i\colon D \to \mathbb{C}$ as in (2.7), which assemble to the meromorphic $\mathfrak{p}^{\mathbb{C}}$–valued 1–form $\eta = \begin{pmatrix} 0 & p_1 \\ p_2 & 0 \end{pmatrix} dz$. We call the Maurer–Cartan form (2.7) of a complex extended frame a *meromorphic potential.* How does one recover the extended frame (and thus the harmonic map) from the meromorphic potential? Given a meromorphic potential $\xi = \lambda^{-1}\eta$ where η is a $\mathfrak{p}^{\mathbb{C}}$–valued meromorphic 1–form on D we first integrate $F_-\, dF_- = \xi$ to a complex extended frame $F_-\colon D \to \Lambda_*^- \boldsymbol{SL}(2,\mathbb{C})$. By Theorem 2.2 we can split $F_- = Fb$ where $F\colon D \to \Lambda \boldsymbol{SU}(2)$ and $b\colon D \to \Lambda_B^+ \boldsymbol{SL}(2,\mathbb{C})$.

Theorem 2.4. *The map $F\colon D \to \Lambda \boldsymbol{SU}(2)$ is an extended framing (with possible singularities along the pole divisor of F_-).*

Thus we have shown that every harmonic map $\varphi\colon D \to S^2$ is obtained from some meromorphic potential ξ. For a more detailed study of conditions on ξ yielding smooth harmonic maps see [5].

In order to obtain an essentially unique correspondence between harmonic maps and their potentials we have to introduce base points into our discussion. Choose a point $z_0 \in D$ and assume that all harmonic maps $\varphi : D \to S^2$ satisfy $\varphi(z_0) \in o = S^1$, and their (extended) framings F satisfy $F(z_0) = 1$. Then we have the following

Theorem 2.5. *Let $\varphi, \tilde{\varphi}\colon D \to S^2$ be harmonic maps with corresponding meromorphic potentials $\xi, \tilde{\xi}\colon TD \to \Lambda_*^- \mathfrak{sl}(2,\mathbb{C})$. Then φ and $\tilde{\varphi}$ differ by an isometry of S^2, i.e. $\tilde{\varphi} = \gamma\varphi$ for some $\gamma \in S^1$, if and only if $\tilde{\xi} = Ad\gamma(\xi)$.*

Proof. From our discussion above it follows that the extended framings $F, \tilde{F}\colon D \to \Lambda \boldsymbol{SU}(2)$ of $\varphi, \tilde{\varphi}$ are related by

$$\tilde{F} = \gamma F k\,,$$

where $k\colon D \to S^1$ has $k(z_0) = \gamma^{-1}$. Since $\gamma F_- F_+ k = (\gamma F_- \gamma^{-1})(\gamma F_+ k)$ and $\gamma F_- \gamma^{-1}$ takes values in $\Lambda_*^- \boldsymbol{SL}(2,\mathbb{C})$ whereas $\gamma F_+ k$ takes values in $\Lambda^+ \boldsymbol{SL}(2,\mathbb{C})$, the uniqueness of the Birkhoff decomposition (Theorem 2.1) implies $\tilde{F}_- = Ad\gamma(F_-)$ and thus

$$\tilde{\xi} = \tilde{F}_-^{-1}\, d\tilde{F} = Ad\gamma(F_-^{-1}\, dF_-) = Ad\gamma(\xi).$$

$\square$

3. Minimal Surfaces via DPW

The main geometric reason to study harmonic maps into S^2 is provided by the fact that surfaces of *constant mean curvature* H have harmonic Gauss maps [13]. The two cases $H = 0$ and $H \neq 0$ are rather different, the former being simpler and well–studied. The case $H \neq 0$ has only recently [11, 8, 3] become more tractable and the DPW–approach has been strongly motivated by its application to constant mean curvature surfaces. The aim of these notes is to reinterpret the constructions of DPW in terms of classical *minimal surface* ($H = 0$) theory. First note that the case $H = 0$ is characterized by the fact that the Gauss map $\varphi\colon D \to S^2$ of the minimal surface $f\colon D \to \mathbb{R}^3$ is holomorphic. Since we view $S^2 = \boldsymbol{SU}(2)/S^1$ its complex structure is described (at the base point $o = S^1$) by the splitting

$$(T_oS^2) = \mathfrak{p}^{\mathbb{C}} = \mathbb{C}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \oplus \mathbb{C}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = T_o^{(1,0)}S^2 \oplus T_o^{(0,1)}S^2 \,.$$

Hence a map $\varphi\colon D \to S^2$ is holomorphic if and only if one (and hence any) lift $F\colon D \to \boldsymbol{SU}(2)$ has $F^{-1}\,dF = \alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \alpha''_{\mathfrak{p}}$ with

$$\alpha'_{\mathfrak{p}} = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} dz \,, \tag{3.1}$$

where $p\colon D \to \mathbb{C}$ We summarize the discussion so far in

Theorem 3.1. *Let $f\colon D \to \mathbb{R}^3$ be an immersion and $\varphi\colon D \to S^2$ its Gauss map. Then the following are equivalent:*

i. *f is minimal, i.e., $H = 0$.*
ii. *φ is holomorphic.*
iii. *the meromorphic potential ξ of φ has the form*

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} dz$$

for some meromorphic $p\colon D \to \mathbb{C}$.

Proof. The equivalence of (i) and (ii) is classical. To see the equivalence of (ii) and (iii) note that our extended frames F resp. F_- of φ are related via the Birkhoff decomposition (Theorem 2.1) by

$$F = F_- F_+ \,.$$

Hence the meromorphic potential

$$\xi = F_-^{-1}\,dF_- = AdF_+(F^{-1}\,dF) - dF_+\,F_+^{-1} = \lambda^{-1}AdF_{+,0}(\alpha'_{\mathfrak{p}}),$$

where $F_{+,0} = F_{+|\lambda=0}\colon D \to B \cong \mathbb{R}^+$ and by (2.5) $F^{-1}\,dF = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}}$. The claim now follows from (3.1). □

Together with Theorem 2.5 we obtain the well known fact that the associated family of holomorphic maps $\varphi_\lambda\colon D \to S^2$ consists of congruent maps.

Corollary 3.1. *Let $\varphi\colon D \to S^2$ be holomorphic and $\varphi_\lambda\colon D \to S^2$ its associated family. Then*

$$\varphi_\lambda = \lambda^{-1/2}\varphi \,,$$

where we interpret $\lambda \in S^1 \subset \boldsymbol{SU}(2)$.

This follows immediately from Theorem 2.5 and the fact that for $\gamma = \mathrm{diag}(c, c^{-1})$, $c \in S^1$

$$Ad\gamma(\xi) = c^2\xi$$

if ξ is a meromorphic potential for a holomorphic $\varphi\colon D \to S^2$.

Of course, Corollary 3.1 just expresses the classical fact that the associated minimal surfaces of a given minimal surface all have the same Gauss map up to a rotation.

We now apply our construction to obtain all Gauss maps $\varphi\colon D \to \mathbb{C}$ of minimal surfaces $f\colon D \to \mathbb{R}^3$: we start with a meromorphic potential $\xi = \lambda^{-1}\eta$, $\eta = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right) dz$ and $p\colon D \to \mathbb{C}$ meromorphic. Then we can explicitly solve

$$\partial_z g = g\xi\,, \qquad g(z_0) = I$$

and obtain the complex extended frame

$$g(z) = I + \lambda^{-1}\int_{z_0}^{z} \eta\,. \tag{3.2}$$

(We assume that p has no residues at its poles and that z_0 is not a pole of p). The Iwasawa decomposition (Theorem 2.2) can be carried out explicitly and yields

$$g = FF_+$$

with

$$F = \frac{1}{\sqrt{1+|q|^2}}\begin{pmatrix} 1 & -\lambda\bar{q} \\ \lambda^{-1}q & 1 \end{pmatrix}, \tag{3.3}$$

where $q(z) = \int_{z_0}^{z} p$ and

$$F_+ = \frac{1}{\sqrt{1+|q|^2}}\begin{pmatrix} 1+|q|^2 & \lambda\bar{q} \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

To obtain an explicit formula for the holomorphic map $\varphi = \pi \circ F_{\lambda=1}\colon D \to S^2$ it is easiest to view $S^2 \subset \mathfrak{su}(2) = \mathbb{R}^3$ as the adjoint orbit of $e_3 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, i.e., $\varphi = AdF_{\lambda=1}\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. We leave this calculation to the reader.

Finally, we want to determine the fundamental forms of the corresponding minimal surface $f\colon D \to \mathbb{R}^3$. We recall that given an immersion $f\colon D \to \mathbb{R}^3$ we may assume that $z = x + iy$ are conformal coordinates for f, i.e. $|df|^2 = e^u\,|dz|^2$ for some function $u\colon D \to \mathbb{R}$. If $\varphi\colon D \to S^2$ is the Gauss map of f then we call

$$F = (e^{-\frac{u}{2}} f_x,\, e^{-\frac{u}{2}} f_y,\, \varphi)\colon D \to \mathbf{SO}(3) \tag{3.5}$$

the coordinate frame of f. Lifting F to the universal cover $\boldsymbol{SU}(2) \to \mathbf{SO}(3)$ yields an extended framing $F\colon D \to \Lambda\boldsymbol{SU}(2)$, which we also call a coordinate frame (of course, F also frames φ in the usual sense, i.e. $\pi \circ F_{\lambda=1} = \varphi$). Then a calculation yields

$$F^{-1}F_z = \begin{pmatrix} u_z/4 & \lambda^{-1}\frac{1}{2}He^{u/2} \\ \lambda^{-1}Qe^{-u/2} & -u_z/4 \end{pmatrix} = V\,, \qquad F^{-1}F_{\bar{z}} = -V^* \tag{3.6}$$

where $Q = (\varphi, f_{zz})$ is the Hopf differential. For a minimal surface $H = 0$ and Q is holomorphic. Computing $F^{-1}F_z$ for the frame in (3.3) and comparing it to (3.6) yields

$$u = 2\,ln(1+|q|^2) \tag{3.7a}$$

$$Q = q_z = p\,. \tag{3.7b}$$

Unlike in the case of surfaces of constant mean curvature $H \neq 0$, the Gauss map $\varphi : D \to S^2$ does not determine a minimal surface. If $\tilde{f} : D \to \mathbb{R}^3$ is a minmal surface with the same Gauss map $\varphi\colon D \to S^2$ as $f\colon D \to \mathbb{R}^3$, then its coordinate frame $\tilde{F}$ (3.5) must satisfy

$$\tilde{F} = Fk$$

where $k = \mathrm{diag}(e^{i\alpha},\, e^{-i\alpha})\colon D \to S^1 \subset \boldsymbol{SU}(2)$. Thus

$$\tilde F^{-1}\tilde F_z = Adk^{-1}(\bar F' F_z) + k^{-1}k_z$$

which unravels to

$$\tilde u_z = u_z + 4i\alpha_z \tag{3.8a}$$

$$\tilde Q = e^{2i\alpha}\, Q\, e^{\frac12(\tilde u - u)}\,. \tag{3.8b}$$

Integrating (3.8a) yields

$$\tilde u - u = 4i\alpha + h \tag{3.9}$$

for some antiholomorphic $h\colon D \to \mathbb{C}$. Since $\tilde u - u$ is real valued, (3.9) says that $\tilde u - u\colon D \to \mathbb{R}$ is harmonic. One also checks at once, using (3.9) and (3.8b), that Q is holomorphic, (which it has to be if $\tilde f$ is a minimal surface). Thus we have described all minimal surfaces and all holomorphic Gauss maps starting from a meromorphic potential. Keeping the above notation we may summarize this discussion in the following

Theorem 3.2. *Let $\xi = \lambda^{-1}\left(\begin{smallmatrix}0 & 0\\ p & 0\end{smallmatrix}\right) dz$ be a meromorphic potential on D and let $q\colon D \to \mathbb{C}$ be given by $q(z) = \int_{z_0}^{z} p(w)\,dw$. Then*

i. $F = \frac{1}{\sqrt{1+|q|^2}}\begin{pmatrix}1 & -\lambda\bar q\\ \lambda^{-1}q & 1\end{pmatrix}$ *is a coordinate frame of a minimal surface $f\colon D \to \mathbb{R}^3$ with induced metric $|df|^2 = (\frac{1}{1+|q|^2})^2|dz|^2$ and Hopf differential $Q = p\,dz^2$.*

ii. *Every other minimal surface $\tilde f\colon D \to \mathbb{R}^3$ having the same Gauss map $\varphi\colon D \to S^2$ as f is obtained as follows: choose an antiholomorphic map $h\colon D \to \mathbb{C}$ and put $\tilde u = u + Re\,h$ and $\alpha = Im\,h$, where $u = 2\,ln(1+|p|^2)$. Then $\tilde F = F\,diag(e^{i\alpha},\, e^{-i\alpha})$ is the coordinate frame of the minimal surface $\tilde f$ with induced metric*

$$|d\tilde f|^2 = e^{\tilde u}|dz|^2$$

and Hopf differential

$$\tilde Q = Qe^{h/2}\,.$$

4. The classical Weierstrass representation

From the extended frame equations (3.6) it is straightfoward to derive a formula for the minimal immersion $f\colon D \to \mathbb{R}^3$. Let $J : \mathbb{R}^3 \to \mathfrak{su}(2)$ be the map $J(x) = -\frac{i}{2}\sum x_k\sigma_k$ where $\sigma_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right)$ and $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ are the Pauli matrices. Extending J complex linearly to $J : \mathbb{C}^3 \to \mathfrak{sl}(2,\mathbb{C})$ we obtain from (3.5)

$$Jf_z = -\frac{i}{2}e^{\frac{u}{2}}\,Ad\,F\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right) = -\frac{i}{2}e^{\frac{u}{2}}\begin{pmatrix}a\bar b & a^2\\ -\bar b^2 & -\bar b a\end{pmatrix} \tag{4.1}$$

for the extended frame $F = \begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix}$. Equations (3.6) read

$$\begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix}_z = \begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix}\begin{pmatrix}u_z/4 & 0\\ \lambda^{-1}Q\,e^{\frac{-u}{2}} & -u_z/4\end{pmatrix}, \tag{4.2}$$

from which we deduce

$$a = e^{-\frac{u}{4}}s \qquad b = e^{-\frac{u}{4}}\bar r \tag{4.3}$$

with s, $r\colon D \to \mathbb{C}$ holomorphic. Inserting this into (4.1) gives

$$f_z = -\frac{i}{2} J^{-1} \left(\begin{smallmatrix} rs & s^2 \\ -r^2 & -rs \end{smallmatrix} \right) = \left(\tfrac{1}{2}(s^2 - r^2),\, \tfrac{i}{2}(s^2 + r^2),\, rs\right)\colon D \to \mathbb{C}^3 , \tag{4.4}$$

which is (a version of) the classical Weierstrass representation [10] of the minimal surface

$$f = Re \int_{z_0}^{z} f_z \, dz \colon D \to \mathbb{R}^3 .$$

The advantage of this version over the usual formula

$$f_z = \left(\tfrac{1}{2}(1 - \nu^2),\, \tfrac{i}{2}(1 + \nu^2),\, \nu\right)\mu ,$$

where $\mu = s^2$ and $\nu = r/s$ is the stereographically projected Gauss map, is that no artificial poles are introduced. Moreover, formula (4.4) can naturally be globalized by viewing r and s as two holomorphic spinor fields over a Riemann surface (rather than as functions on D) [9]. Since

$$1 = |a|^2 + |b|^2 = e^{-\frac{u}{4}}(|s|^2 + |r|^2) ,$$

we obtain as the induced metric

$$|df|^2 = (|s|^2 + |r|^2)^2 \, |dz|^2 .$$

Moreover, from the remaining equations in (4.2) we derive for the Hopf differential of $f\colon D \to \mathbb{R}^3$

$$Q = r s_z - s r_z$$

the Wronskian of r and s. Finally one can check that the λ–dependence of r and s is given by

$$r = \lambda^{-\frac{1}{2}} r_0 \quad s = \lambda^{-\frac{1}{2}} s_0$$

where r_0, s_0 are λ–dependent. This simply restates the classical fact that the associated family of minimal surfaces is given by

$$f_\lambda = Re\, \lambda^{-1} \int_{z_0}^{z} f_z \, dz , \qquad \lambda \in S^1 ,$$

where f is the minimal surface obatined from the Weierstrass data for $\lambda = 1$.

As an example we specialize to the frame F obtained by the DPW construction from the meromorphic potential

$$\xi = \lambda^{-1} \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix} \right) dz .$$

Comparing (3.3) and (3.7) with (4.3) we obtain for the classical Weierstrass data of the minimal surface f described by $F_{\lambda=1}$

$$s = 1 \qquad r = \nu = -\int_{z_0}^{z} p .$$

Thus

$$|df|^2 = (1 + |\nu|^2)^2 \, |dz|^2, \quad Q = -\nu_z = p ,$$

where ν is the Gauss map (stereographically projected from the south pole) of

$$f = Re \int_{z_0}^{z} \left(\tfrac{1}{2}(1 - \nu^2),\, \tfrac{i}{2}(1 + \nu^2), \nu\right) dz .$$

5. The dressing action on minimal surfaces

In the previous sections we discussed how one can describe minimal surfaces $f\colon D \to \mathbb{R}^3$ in terms of their meromorphic potentials (Theorem 3.2) and how this approach relates to the classical Weierstrass representation (Section 4). An important feature of our description in terms of loop groups is that we can deform a minimal surface by the dressing action. Given a meromorphic potential

$$\xi = \lambda^{-1}\eta \, dz\,, \qquad \eta = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right)\,, \quad p : D \to \mathbb{C} \tag{5.1}$$

with corresponding complex extended frame $g = 1 + \lambda^{-1}\int \eta\colon D \to \Lambda_*^- \boldsymbol{SL}(2,\mathbb{C})$, we can dress g by an element $h \in \Lambda^+\boldsymbol{SL}(2,\mathbb{C})$ using Theorem 2.1:

$$hg = \tilde{g}b\,, \tag{5.2}$$

where $\tilde{g} : D \smallsetminus S \to \Lambda_*^-\boldsymbol{SL}(2,\mathbb{C})$, $b : D \smallsetminus S \to \Lambda^+\boldsymbol{SL}(2,\mathbb{C})$ and $S \subset D$ is the discrete set where hg leaves the big cell. Calculating the meromorphic potential of the new complex extended frame $\tilde{g}$ we obtain:

$$\tilde{\xi} = \tilde{g}^{-1}\partial_z \tilde{g} = \mathrm{Ad}\, b_0^{-1}(\xi) = \lambda^{-1}\begin{pmatrix} 0 & 0 \\ \rho^2 p & 0 \end{pmatrix} dz\,, \tag{5.3}$$

where $b_0 = b_{|\lambda=0} = \mathrm{diag}(\rho,\, \rho^{-1})$, $\rho : D \smallsetminus S \to \mathbb{C}^*$. In particular, dressing preserves the class of minimal surfaces.

Next we calculate the stabilizer group Λ_0 of the "action" $g \mapsto \tilde{g} = (hg)_-$: let $g = \begin{pmatrix} 1 & 0 \\ \lambda^{-1}q & 1 \end{pmatrix}$ and $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then $hg = gb$ yields

$$\alpha + \lambda^{-1}\beta q = \tilde{\alpha}$$
$$\gamma + \lambda^{-1}\delta q = \tilde{\gamma} + \lambda^{-1}\tilde{\alpha} q$$

where $b = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$. Since h and b take values in $\Lambda^+\boldsymbol{SL}(2,\mathbb{C})$, the above relations imply

$$\tilde{\alpha}_0 = \alpha_0 = \delta_0 = \pm 1\,, \quad \beta_1 = 0$$

where $\alpha = \sum_{i \geqslant 0} \lambda^i \alpha_i$ and similar for the other coefficients. From this we conclude easily the following

Lemma 5.1.

$$\Lambda^+\boldsymbol{SL}(2,\mathbb{C})/\Lambda_0 \cong G$$

where

$$G = \{\begin{pmatrix} \alpha & \lambda\beta \\ 0 & \alpha^{-1} \end{pmatrix};\ \alpha \neq 0,\ \beta \in \mathbb{C}\}\,.$$

In particular, the dressing orbit of (the Gauss map of) a minimal surface is complex 2*–dimensional.*

This is a well-known fact and can be found in the existing literature in a much more general framework [14, 2, 4]. For an element $h = \begin{pmatrix} \alpha & \lambda\beta \\ 0 & \alpha^{-1} \end{pmatrix} \in G$ and a complex extended frame $g = \begin{pmatrix} 1 & 0 \\ \lambda^{-1}q & 1 \end{pmatrix}$, the dressing relation $hg = \tilde{g}b$, $b = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$ yields

$$\tilde{q} = \frac{q}{\alpha^2 + \alpha\beta q}\,. \tag{5.4}$$

From the discussion in Section 4 we know that $-q$ is the (stereographically) projected Gauss map, so that (5.4) is just the standard action of $\boldsymbol{SL}(2,\mathbb{C})$ on $S^2 = \mathbb{CP}^1$ restricted to G.

6. Symmetries of Minimal Surfaces

This chapter is largely in the spirit of [6]. We obtain a very specific description of symmetries for minimal surfaces. Let M be a complete, orientable minimal surface with meromorphic potential ξ and coordinate frame F and denote by ϕ the minimal immersion of M. We consider the group

$$\mathrm{Aut}\,\phi\,(M) = \{R \text{ proper rigid motion of } \mathbb{R}^3,\ R\,\phi\,(M) = \phi\,(M)\}\,.$$

From [6] we know

$$\text{For every } R \in \mathrm{Aut}\,\phi\,(M), \text{there exists a } g \in \mathrm{Aut}\,\mathbb{D} \text{ such that } \phi \circ g = R \circ \phi\,.$$

In particular, g is in the group

$$\mathrm{Aut}_\phi\,\mathbb{D} = \{g \in \mathrm{Aut}\,\mathbb{D}\,;\ \text{there exists } R \in \mathrm{Aut}\,\phi\,(M) : \phi \circ g = R \circ \phi\}.$$

Also we obtain [6]

$$(F \circ g)\,(z,\,\bar{z},\,\lambda) = \chi(g,\,\lambda)\;F(z,\,\bar{z},\,\lambda)\;k(g,\,z,\,\bar{z}), \tag{6.1}$$

where $\chi(g,\,\lambda) \in \Lambda\mathfrak{su}(2)$. Moreover, χ and k satisfy

$$\chi(g_2 \circ g_1, \lambda) = \epsilon(g_2, g_1)\chi(g_2, \lambda)\chi(g_1, \lambda) \tag{6.2a}$$

$$k(g_2 \circ g_1, z, \bar{z}) = \epsilon(g_2, g_1)\;k(g_2, g_1(z), \overline{g_1(z)})\;k(g_1, z, \bar{z}) \tag{6.2b}$$

where $\epsilon(g_2, g_1) = \pm 1$. Also, splitting $F = g_- F_+$, we obtain

$$(g_- \circ g)(z, \lambda) = \chi(g, \lambda) g_-(z, \lambda) p_+(g, z, \lambda) \tag{6.3}$$

where $p_+ \in \Lambda^+\mathfrak{sl}(2, \mathbb{C})$. For the meromorphic potential ξ this implies

$$g_*\xi(z,\,\lambda) = p_+^{-1}\xi p_+ + p_+^{-1}\,dp_+\,. \tag{6.4}$$

We have the following

Theorem 6.1. *Under the assumptions on M listed above and writing χ in the form $\chi = \left(\begin{smallmatrix} u & v \\ -\bar{v} & \bar{u} \end{smallmatrix}\right)$, we have*

$$\chi = \begin{pmatrix} u_0 & \lambda v_1 \\ -\lambda^{-1}\bar{v}_1 & \bar{u}_0 \end{pmatrix}, \tag{6.5}$$

$$p_+ = \begin{pmatrix} w_0(z) & \lambda p_1 \\ -\lambda^{-1} p_1 & t_0(z) \end{pmatrix}, \tag{6.6}$$

where $p_1 \in \mathbb{C}$ is independent of z.

Proof. Consider

$$g_-^{-1}(z,\lambda) = \chi^{-1}(g,\lambda)(g,\lambda)(g_- \circ g)(z,\lambda) =$$
$$= \begin{pmatrix} 1 & 0 \\ -\lambda^{-1}\check{b} & 1 \end{pmatrix} \begin{pmatrix} \bar{u} & -v \\ \bar{v} & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda^{-1}\check{b}\circ g & 1 \end{pmatrix} = \begin{pmatrix} \bar{u} & -v \\ -\lambda^{-1}\check{b}\bar{u}+\bar{v} & \lambda^{-1}\check{b}v+u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda^{-1}\check{b}\circ g & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \bar{u}-\lambda^{-1}\check{b}\circ g\, v & -v \\ -\lambda^{-1}\check{b}\bar{u}+\bar{v}+\lambda^{-2}\check{b}\check{b}\circ g v+ \lambda^{-1}u\check{b}\circ g & \lambda^{-1}\check{b}v+u \end{pmatrix}$$

This is an element of $\Lambda^+\mathfrak{sl}(2,\mathbb{C})$ if and only if

i. $v = v_1\lambda + v_3\lambda^3 + \ldots$
ii. $u = u_0 + u_2\lambda^2 + \ldots$

In the left lower corner of the matrix above, the coefficients of λ^{-1} and λ^{-3} must vanish. This implies respectively:

i. $-\check{b}\bar{u}_0 + \bar{v}_1 + \check{b}\check{b}\circ g v_1 + u_0\, \check{b}\circ g = 0$
ii. $-\check{b}\bar{u}_2 + \bar{v}_3 = 0$

Since v_3 and u_2 are independent of z, $v_3 = u_2 = 0$. Similarly, all higher–order terms vanish. This proves (6.5) and (6.6). □

Actually, (i) gives an important condition on $\check{b}$, namely

$$\check{b}\circ g = \frac{\bar{u}_0\check{b} - \bar{v}_1}{v_1\check{b} + u_0}\,. \tag{6.7}$$

Since v_1 and u_0 are constant and $(|u_0|^2 + |v_1|^2 = 1$ we obtain

Corollary 6.1.

$$\check{b}\circ g = T_g\check{b} \tag{6.8}$$

where

$$T_g = \begin{pmatrix} \bar{u}_0 & -\bar{v}_1 \\ v_1 & u_0 \end{pmatrix} \in \mathbf{SU}(2)\,. \tag{6.9}$$

Conversely, we have

Theorem 6.2. *Let $T_g = \begin{pmatrix} \bar{u}_0 & -\bar{v}_1 \\ v_1 & u_0 \end{pmatrix} \in \boldsymbol{SU}(2)$. Assume a meromorphic function $\check{b}$ satisfies* (6.8). *Then $g_- \circ g = \chi g_- p_+$, with χ given by the formula* (6.5).

Proof. A careful look at the proof of Theorem 6.1 shows that one can reverse the order of all arguments. □

Remark 1. Let us consider the stereographic projection from the point (0, 0, 1), $\sigma : S^2 \to \bar{\mathbb{C}}$ given by the formula

$$\sigma(x_1,\, x_2,\, x_3) = \frac{x_1 + ix_2}{1 - x_3}\,.$$

Note that since $N : D \to S^2$, the composite $\sigma \circ N : D \to \bar{\mathbb{C}}$ makes sense. Moreover, $\sigma \circ N = \nu$. (Recall that $(\mu,\, \nu)$ denotes the Weierstrass pair for the surface for which N is the Gauss map). Therefore, the relations (6.8) and $\nu = \frac{1}{\lambda^2 b}$ induce

$$\nu \circ g = \frac{u_0 \nu - \lambda^{-2}\nu_1}{\lambda^2 \overline{v_1}\nu + \bar{u}_0}\,, \tag{6.10}$$

which is to say

$$\nu \circ g = S_g^\lambda \circ \nu \tag{6.11}$$

where $S_g^\lambda = \begin{pmatrix} u_0 & -\lambda^2 v_1 \\ \lambda^2 \bar{v}_1 & \bar{u}_0 \end{pmatrix}$. Since $\sigma \circ N = \nu$ we also have

$$N \circ g = S_g^\lambda N\,. \tag{6.12}$$

7. Examples

To illustrate some of the results of this note, we list a number of well-known surfaces, all of which have the same Gauss map, represented by the map $\nu(z) = z$. From (3.5) we know that all these surfaces yield the same meromorphic potential, given by

$$a = b' = (-\frac{1}{\nu})' = \frac{1}{z^2}\,. \tag{7.1}$$

The catenoid can be obtained on $D = \mathbb{C} - \{0\}$ considering

$$\mu = \frac{1}{z^2}\, dz\,. \tag{7.2}$$

Since $\nu(z)^2\mu(z) = dz$ in this case, we can apply (3.5) and get $u_0 = u(0,\, 0) = 0$. Theorem 3.2 then gives the metric [1] and the Hopf differential:

$$e^u |dz|^2 = (1 + \frac{1}{z^2})^2 |dz|^2 \tag{7.3}$$

$$Q(z) = a(z)(dz)^2 = \frac{1}{z^2}(dz)^2\,. \tag{7.4}$$

Any minimal surface of revolution in $\mathbb{R}^3$ is (up to a rigid motion) part of a catenoid or part of a plane [1].

Another example is the helicoid which is given by

$$\mu(z) = -\frac{i}{z^2} dz\,. \tag{7.5}$$

The corresponding metric is the same as in the case of the catenoid (7.3). The importance of the helicoid among minimal surfaces is emphasized by the fact that any ruled minimal surface of $\mathbb{R}^3$ is, up to a rigid motion, part of a helicoid or part of a plane [1]. Theorem 1 in Section 2.4 showed that two orientable minimal surfaces with coordinate frames F and $\tilde{F}$ have the same Gauss map N if and only if the corresponding metrics $ds^2 = e^u\,|dz|^2$ and $ds^2 = e^{\tilde{u}}\,|dz|^2$ satisfy the property

[1] It is worthwhile to recall [1] that, given the Weierstrass pair $\mu = h(z)\,dz, \nu = \nu(z)$, the corresponding metric can be obtained *a posteriori* from it as $ds^2 = |h|^2(1 + |\nu|^2)^2|dz|^2$.

$\tilde{u} - u$ is harmonic. As already mentioned this is the case for the catenoid and helicoid. On the other hand there are examples of minimal surfaces having the same Gauss map but distinct metrics. Enneper's surface defined on $\mathbb{C}$ and given by Weierstrass pair $\mu(z) = dz$ (so $f(z) = 1$) and $\nu(z) = z$ has the same Gauss map as the helicoid and the catenoid, but the metric is $ds^2 = (1 + |z|^2)^2|dz|^2$. Obviously $u_E - u = ln(1 + |z|^2)^2 - ln(1 + \frac{1}{|z|^2})^2 = ln|z|^4 = 4\,ln|z|$ is a harmonic function (where u_E corresponds to the Enneper surface and u to the catenoid or helicoid).

Finally we mention Scherk's surface defined on the unit disc $D = \{z \in \mathbb{C},\ |z| < 1\}$ and given by

$$\mu(z) = \frac{4}{1 - z^4}\,dz \quad \nu(z) = z\,. \tag{7.6}$$

In this case the induced metric is

$$ds^2 = |\frac{4}{1 - z^4}|^2(1 + |z|^2)^2|dz|^2 = 16\frac{(1 + |z|^2)^2}{|1 - z^4|^2}|dz|^2\,. \tag{7.7}$$

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