Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras

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Introduction

Background

The study of harmonic maps of compact Riemann surfaces into symmetric spaces can be motivated by considerations from several areas of Geometry and Physics:

For example, minimal surfaces are (conformal) harmonic maps while constant mean curvature surfaces, constant Gauss curvature surfaces and Willmore surfaces all have Gauss maps which are harmonic maps into various symmetric spaces.

Again, the harmonic map equations on 2-dimensional domains enjoy a number of properties (conformal invariance, instanton solutions) analogous to those of the Yang–Mills equations in 4 dimensions. This observation has encouraged Physicists to study the (simpler) harmonic map equations in the hope of gaining insight into the Yang–Mills equations.

Much of the qualitative behaviour of the problem can be seen in embryo form in the simplest possible case: that of harmonic maps of a compact Riemann surface M into S^2.

If M is the Riemann sphere, all harmonic maps into S^2 are $\pm$-holomorphic and so are given by rational functions. A similar picture obtains when the target is a higher dimensional symmetric space [**6**,**7**,**11**,**34**,**35**]: in this setting, harmonic maps are obtained from holomorphic curves in some auxiliary complex manifold (a *twistor space*) such as a flag manifold [**6**,**11**] or a loop group [**34**].

A model for such results are the theorems of Calabi [**7**] and Eells–Wood [**11**] concerning S^{2n} and $\mathbb{C}P^n$ respectively which show that all harmonic 2-spheres in such spaces may be obtained by explicit algebraic procedures from holomorphic maps (in this regard, see also the results of Wood [**36**] concerning explicit construction of harmonic 2-spheres in complex Grassmannians). In these results, a key ingredient is that a harmonic map gives rise to a number of holomorphic differentials (the first being the $(2,0)$-part of the complexified metric) which, of course, vanish since the domain is a Riemann sphere. In case M has higher genus, it may also happen that for a particular harmonic map, these differentials vanish and then the harmonic map may again be constructed algebraically from holomorphic maps. Such harmonic maps are variously called *superminimal* [**4**], *pseudo-holomorphic* [**7**] or *isotropic* [**11**] and represent the first class of harmonic maps of higher

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genus surfaces to be systematically treated.

If M is a 2-torus then a harmonic map into S^2 is either holomorphic (which we view as the superminimal case) or nowhere conformal. The nowhere conformal harmonic maps are the Gauss maps of constant mean curvature tori in $\mathbb{R}^3$ and amount to doubly periodic solutions of the sinh–Gordon equation. These were studied by Pinkall–Sterling [**28**] who showed that all such could be obtained from solutions of a family of finite-dimensional completely integrable systems which linearise on the Jacobian of an algebraic curve, the *spectral curve*. These results enabled Bobenko [**2**], using methods from soliton theory, to provide an explicit description of these harmonic maps in terms of theta functions.

That such a picture persists for other symmetric targets was indicated by Hitchin's study [**18**] of harmonic 2-tori in S^3—here also such maps are obtained from linear flows on the Jacobian of a spectral curve. Finally, in the work of Ferus–Pedit–Pinkall–Sterling [**13**] on minimal non-superminimal 2-tori in S^4, the link between harmonic maps and Hamiltonian systems on loop algebras was made explicit. It is this theme that we take up in the present paper: indeed, our main result is that the "generic" harmonic 2-torus in a compact Riemannian symmetric space may be obtained from such a Hamiltonian system.

When M has genus greater than one, harmonic maps into S^2 (or any other symmetric space) are not understood in any systematic way although there are several classes of examples: the harmonic maps of rather symmetrical higher genus surfaces into S^2 due to Lemaire [**26**], the minimal surfaces in S^3 of Lawson [**25**] and Karcher–Pinkall–Sterling [**21**] and the Gauss maps of the constant mean curvature surfaces that Kapouleas [**20**] has obtained by analytic means.

Overview

Our main objects of study are pluriharmonic maps $\phi : \mathbb{C}^n/\Gamma \to G/K$ of a complex torus $\mathbb{C}^n/\Gamma$ into a symmetric space G/K, where G is a compact, usually semisimple, Lie group. In particular, when $n = 1$, these are precisely the harmonic 2-tori in G/K. We introduce a large class of such maps, *the pluriharmonic maps of finite type*, the description of which reduces to the integration of a family of commuting Hamiltonian flows on finite-dimensional subspaces of the algebra $\Omega\mathfrak{g}$ of based loops in the Lie algebra $\mathfrak{g}$ of G.

To obtain this description we reduce our problem in two ways: firstly, we lift our maps to the universal cover $\mathbb{C}^n$ of $\mathbb{C}^n/\Gamma$. Secondly, we avail ourselves of the Cartan embedding of our symmetric space G/K into its group of isometries G. This is a totally geodesic immersion $G/K \hookrightarrow G$ and therefore preserves pluriharmonicity. A large part of our study will therefore focus on pluriharmonic maps $\phi : \mathbb{C}^n \to G$.

Our analysis is based on the zero-curvature reformulation of the harmonic map equations due to Uhlenbeck [**34**] and extended to pluriharmonic maps by Ohnita–Valli [**27**]. To describe this, let θ be the Maurer–Cartan form of G and $\phi : \mathbb{C}^n \to G$ a smooth map. Denote by α' the $(1,0)$-part of $\phi^*\theta$ and by α'' the complex conjugate of α'. Now introduce a loop of gauge potentials (or, equivalently, an $\Omega\mathfrak{g}$-valued 1-form) by setting

$$A_\lambda = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha'', \tag{1}$$

for $\lambda \in S^1$. The point of this construction is that ϕ is pluriharmonic if and only if $d + A_\lambda$ is a flat connection for all $\lambda \in S^1$. Conversely, if $d + A_\lambda$ is a loop of flat connections of the form (1) then, $\mathbb{C}^n$ being simply connected, we may integrate the Maurer–Cartan equations to get an essentially unique pluriharmonic map $\phi : \mathbb{C}^n \to G$ with $\phi^*\theta = \alpha' + \alpha''$.

With this in hand, we are in a position to describe the class of pluriharmonic maps that will principally concern us. First, for $d \in \mathbb{N}$, let Ω_d be the finite dimensional subspace of $\Omega\mathfrak{g}$ consisting of loops of the form

$$\xi = \sum_{0<|n|\leq d} (1 - \lambda^n)\xi_n,$$

with each $\xi_n \in \mathfrak{g}^{\mathbb{C}}$.

We say that a pluriharmonic map $\phi : \mathbb{C}^n \to G$ is *of finite type* if, for some $d \in \mathbb{N}$, there is a map $\xi : \mathbb{C}^n \to \Omega_d$ such that

$$d\xi = [\xi, A_\lambda] \tag{2}$$

$$\alpha' = 4iV_k(\xi_d)dz^k, \tag{3}$$

where the $V_k : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ are $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial maps. These two conditions, taken together, imply two things: firstly, ξ is a solution to a Lax equation and, secondly, α' (and hence ϕ) can be recovered from ξ.

The Lax equations for ξ have a Hamiltonian formulation. From (3) we see that A_λ is constructed from suitable projections of gradients of adjoint invariants on $\Omega\mathfrak{g}$. Lax equations of this kind arise in the well-known Kostant–Adler–Symes theory of integrable systems (c.f., [**1**]) and the generalisation of that theory represented by the r-matrix formalism of Reyman–Semenov-Tian-Shansky [**32**]. In fact, the Lax equations (2) are obtained in a canonical way by applying the r-matrix formalism, which shows that ξ integrates a system of $2n$ commuting Hamiltonian flows on $\Omega\mathfrak{g}$.

We have therefore isolated a class of pluriharmonic maps into G that may be constructed by solving a finite dimensional system of commuting ordinary differential equations.

To find pluriharmonic maps of finite type into the symmetric space G/K, we must obtain conditions that ensure that pluriharmonic maps into G factor through the Cartan embedding and so give rise to pluriharmonic maps into G/K. This question has a pleasantly simple answer: it is necessary and sufficient that ξ be compatible with the involution defining the symmetric space.

Of course, the central issue is to discover sufficient conditions for a pluriharmonic map to be of finite type. Our main result in this direction is that a harmonic map of a (real) 2-torus into a symmetric space whose derivative along some holomorphic direction is regular semisimple at one (and hence every) point is of finite type. As a consequence of this, we prove that every non-conformal harmonic 2-torus in a rank one symmetric space is of finite type. In particular, taken together with [**13**], this accounts for all non-superminimal harmonic 2-tori in S^4.

For pluriharmonic maps of a complex n-torus, $n > 1$, our main result is also valid subject to a mild technical assumption—the *surjection property*—on the symmetric space which is satisfied by all group manifolds and all classical symmetric spaces.

To make contact with Algebraic Geometry and, in particular, recover Hitchin's results [**18**], it remains to show that our Hamiltonian flows are algebraically completely integrable i.e., that our flows are equivalent to linear flows on the Jacobian of some spectral curve. Part of this programme has already been carried out: there is a standard construction of a spectral curve and a dynamical system on its Jacobian from a Lax equation on a matrix loop algebra and it is shown in [**5**] that, in our case, this dynamical system is linear. To complete the programme, one must invert this construction and we shall return to this elsewhere.

Outline

In preparing this paper, our exposition was influenced by several considerations. Firstly, we have, to a certain extent, treated the special case of harmonic maps $\mathbb{C}/\Gamma \to G/K$ separately since, in some ways, this is the case of primary interest. Moreover, in this setting, the arguments are simpler, the results more complete and require less technicalities then in the general case.

Secondly, our ideas have been strongly influenced by the previous works [**13**,**28**] and we have attempted to present the sometimes *ad hoc* methods of those papers in their essential form.

As a consequence, there is a risk that the Reader may find our exposition insufficiently concise and we therefore present here a detailed outline of the paper.

As we have already seen, pluriharmonic maps $\mathbb{C}^n \to G$ are essentially the same as loops of flat connections $d + A_\lambda$ with A_λ of the form (1). The paper begins by describing a method of producing such connections from commuting flows on loop algebras. Choose $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial maps $V_1, \ldots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $d \in \mathbb{N}$ and define vectors fields X_1^k, X_2^k, $1 \leq k \leq n$, on Ω_d, the linear space of loops in $\mathfrak{g}$ of the form $\sum_{0<|n|\leq d}(1 - \lambda^n)\xi_n$, by

$$\tfrac{1}{2}(X_1^k - iX_2^k)(\xi) = [\xi, 2i(1 - \lambda)V_k(\xi_d)].$$

It is then an immediate consequence of the r-matrix theory enunciated in theorem 1.3 that the X_j^k are a commuting set of complete vector fields on Ω_d. If we then fix a base-point $\xi_0 \in \Omega_d$, we may simultaneously integrate along these vector fields to obtain a map $\xi : \mathbb{C}^n \to \Omega_d$ satisfying

$$\frac{\partial \xi}{\partial z^k} = [\xi, 2i(1 - \lambda)V_k(\xi_d)]$$
$$\xi(0) = \xi_0.$$

Moreover, as a further consequence of (1.3), we learn that such a ξ defines a loop of flat connections $d + A_\lambda$ of the form (1) by

$$A_\lambda = 2i(1 - \lambda)V_k(\xi_d)dz^k - 2i(1 - \lambda^{-1})\overline{V_k(\xi_d)}d\overline{z}^k$$

and thus a pluriharmonic map $\phi : \mathbb{C}^n \to G$ with

$$\alpha' = 4iV_k(\xi_d)dz^k.$$

The pluriharmonic maps so obtained are the pluriharmonic maps of *finite type*.

Matters simplify somewhat when $n = 1$ since then we can take the polynomial map V_1 to be the identity $\mathrm{id}_{\mathfrak{g}^{\mathbb{C}}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.

To treat pluriharmonic maps into a symmetric space G/K, we take the view that such a map is a pluriharmonic map into G which factors through the totally geodesic Cartan embedding $G/K \hookrightarrow G$. This embedding is given by

$$gK \mapsto g\tau(g)^{-1},$$

where τ is the involution of G fixing K. In section 1.4 we consider this situation and determine the image of the Cartan embedding. As a consequence, we give in theorem 1.12 conditions which ensure that pluriharmonic maps of finite type factor through the embedding. These conditions amount to demanding that

(*i*) the polynomial maps V_k defining the Lax equation are *τ-compatible* in the sense that $V_k \circ (-\tau) = -\tau \circ V_k$, for $1 \leq k \leq n$;

(*ii*) the initial condition $\xi_0 \in \Omega_d$ satisfies $\tau\xi_0(\lambda) = \xi_0(-\lambda)$, for $\lambda \in S^1$.

Again matters simplify when $n = 1$ since the first condition is always satisfied by $V_1 = \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}}$.

In the second and third chapters, which contain the heart of the paper, we discuss which pluriharmonic maps are of finite type. Clearly $\phi : \mathbb{C}^n \to G$ is of finite type precisely when, for some $d \in \mathbb{N}$, there is a map $\xi : \mathbb{C}^n \to \Omega_d$ satisfying

$$d\xi = [\xi, A_\lambda] \tag{2}$$

$$\alpha' = 4iV_k(\xi_d)dz^k \tag{3}$$

for some $\mathrm{Ad}\, G^{\mathbb{C}}$-equivariant polynomial maps $V_k : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$. We call a map $\xi : \mathbb{C}^n \to \Omega_d$ a *polynomial Killing field* for ϕ if ξ satisfies (2). Moreover, a polynomial Killing field is said to be *adapted* if (3) is replaced by the weaker condition that there exists a nowhere vanishing holomorphic vector field Z such that

$$\alpha'(Z) = \xi_d.$$

It is clear that when $n = 1$ the existence of an adapted polynomial Killing field is sufficient for $\phi : \mathbb{C}^n \to G$ to be of finite type. When $n > 1$, further conditions must be imposed in order that all of α' can be reconstructed from knowledge of α' along some holomorphic direction and thus from ξ_d.

We therefore ask when a pluriharmonic map admits an adapted polynomial Killing field. Certainly there are necessary conditions: a polynomial Killing field solves the Lax equation (2) from which it follows that ξ_d has image in a single $\mathrm{Ad}\, G^{\mathbb{C}}$-orbit in $\mathfrak{g}^{\mathbb{C}}$. Thus ϕ can only admit an adapted polynomial Killing field if $\alpha'(Z)$ has image in a single adjoint orbit.

The main point now is that for lifts of pluriharmonic maps of a complex torus $\mathbb{C}^n/\Gamma$ such a condition is almost sufficient: say that ϕ is *semisimple adapted* if $\alpha'(Z)$ has image in a single $\mathrm{Ad}\, G^{\mathbb{C}}$-orbit of *semisimple* elements of $\mathfrak{g}^{\mathbb{C}}$. We prove that a semisimple adapted pluriharmonic map $\mathbb{C}^n/\Gamma \to G$ admits an adapted polynomial Killing field. In particular, a semisimple adapted harmonic map $\mathbb{C}/\Gamma \to G$ is of finite type.

The proof of this result requires the introduction of an auxiliary object: an *adapted formal Killing field* for $\phi : \mathbb{C}^n/\Gamma \to G$ is a formal Laurent series $Y = \sum_{k \geq i} \lambda^{-k} Y_k$, with each $Y_k : \mathbb{C}^n/\Gamma \to \mathfrak{g}^{\mathbb{C}}$, satisfying

$$\begin{aligned} dY &= [Y, A_\lambda] \\ \alpha'(Z) &= Y_i, \end{aligned}$$

for some nowhere vanishing holomorphic vector field Z. If $\alpha'(Z)$ has image in a single adjoint orbit of semisimple elements then an adapted formal Killing field for ϕ can be constructed by a recursion argument (theorem 2.4). Moreover, each coefficient Y_k is a Jacobi field for ϕ (lemma 2.2) and so satisfies a linear elliptic partial differential equation. Since $\mathbb{C}^n/\Gamma$ is compact we deduce that the Y_k span a finite-dimensional space and this enables us to construct an adapted polynomial Killing field for ϕ from some of the Y_k.

This analysis provides a sufficient condition that a harmonic map $\phi : \mathbb{C}/\Gamma \to G/K$ be of finite type, namely: ϕ must be semisimple adapted. In general, when a pluriharmonic map $\phi : \mathbb{C}^n/\Gamma \to G/K$ is semisimple adapted, we have now produced an adapted polynomial Killing field ξ for ϕ but to deduce from this that ϕ is of finite type requires additional assumptions. Firstly, we must further restrict the orbit type of $\alpha'(Z) = \xi_d$ by demanding that it have image in an orbit of *regular* (with respect to G/K) semisimple elements. We call such a map *regular semisimple adapted*. This condition is to ensure that α' may be recovered as an $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial in $\alpha'(Z)$. Secondly, we require that the symmetric space G/K have the *surjection property* which amounts to demanding that $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ generate the G-equivariant polynomial maps $T(G/K) \to T(G/K)$. When these extra conditions are imposed, one finds (theorem 2.8) $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial maps $V_k : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that

$$\alpha' = 4iV_k(\xi_d)dz^k$$

so that ϕ is of finite type.

Finally, in chapter three, we give the "regular semisimple adapted" condition a more geometrical flavour. The results of Kostant and Kostant–Rallis [**23**,**24**] identify the orbits of regular semisimple elements with certain level sets of G-invariant polynomial maps $P : T(G/K)^{\mathbb{C}} \to \mathbb{C}^r$. Such maps, being parallel since invariant, pull back by a pluriharmonic ϕ to give holomorphic differentials $\phi^* P^{(k,0)}$ on $\mathbb{C}^n/\Gamma$ which are constant by Liouville's theorem. This means that the condition that ϕ be regular semisimple adapted can be verified by the examination of polynomials in $\partial\phi$ at a single point of $\mathbb{C}^n/\Gamma$.

In particular, if G/K has rank one, the invariant polynomials are generated by the Killing form (the metric of G/K) and, in this case, $\phi : \mathbb{C}/\Gamma \to G/K$ is regular semisimple adapted and so of finite type if and only if it is non-conformal.

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1 Harmonic maps and commuting Hamiltonians

1.1 The zero-curvature equations

We will study harmonic maps of a 2-torus into a compact Lie group G and, later on, pluriharmonic maps of higher dimensional tori into G. On two-dimensional domains, the harmonic map equations are conformally invariant and so it suffices to consider harmonic maps of $\mathbb{R}^2$ into G which are doubly periodic with respect to some lattice.

The starting point of our analysis is the well-known observation that the harmonic map equations for maps of a (simply-connected) surface into a Lie group are equivalent to the flatness of a certain loop of connections. We begin by recalling this development.

Let $\mathfrak{g}$ be the Lie algebra of G and θ the (left) Maurer–Cartan form of G. Let $\phi : M \to G$ be a map of a Riemannian manifold M into G and set $\alpha = \phi^*\theta$. Then α is a $\mathfrak{g}$-valued 1-form on M which satisfies the Maurer–Cartan equation:

$$d\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0. \tag{4}$$

Otherwise said, the connection $d + \alpha$ on the trivial principal bundle $M \times G$ is *flat*.

Conversely, when M is simply-connected, any $\mathfrak{g}$-valued 1-form α on M satisfying (4) can be integrated to give a map $\phi : M \to G$, unique up to left translation by a constant element of G, for which $\alpha = \phi^*\theta$. Otherwise said, ϕ gauges the flat connection $d + \alpha$ to the trivial connection.

Fix, once and for all, a bi-invariant metric on G. Then $\phi : M \to G$ is harmonic if and only if $\phi^*\theta$ is co-closed [**29**], i.e.

$$d^*\alpha = 0. \tag{5}$$

Thus the study of harmonic maps of (simply-connected) M into G reduces to the study of $\mathfrak{g}$-valued 1-forms on M satisfying (4) and (5).

Now we specialise to the case $M = \mathbb{R}^2$. Introduce a complex co-ordinate $z = x + iy$ on $\mathbb{R}^2$ and then we have a type decomposition:

$$\alpha = \alpha' + \alpha''$$

where α' is a $\mathfrak{g}^{\mathbb{C}}$-valued $(1,0)$-form, $\alpha'' = \overline{\alpha'}$ where the conjugation in $\mathfrak{g}^{\mathbb{C}}$ is with respect to the real form $\mathfrak{g}$. The exterior derivative also decomposes:

$$d = \partial + \overline{\partial}$$

and then (4) and (5) are equivalent to

$$\overline{\partial}\alpha' + \tfrac{1}{2}[\alpha' \wedge \alpha''] = \partial\alpha'' + \tfrac{1}{2}[\alpha' \wedge \alpha''] = 0 \tag{6}$$

The fundamental observation of Uhlenbeck [**34**] (c.f. also [**37**,**38**]) is that by introducing an auxiliary parameter λ (a *spectral* parameter), the equations (6) are equivalent to a single family of Maurer–Cartan equations.

Indeed, for $\lambda \in \mathbb{C}^*$, define a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form A_λ by

$$A_\lambda = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha''.$$

Note that, for $\lambda \in S^1$, A_λ is in fact $\mathfrak{g}$-valued. Now compare coefficients to see that

$$dA_\lambda + \tfrac{1}{2}[A_\lambda \wedge A_\lambda] = 0$$

for all λ if and only if $\alpha = A_{-1}$ satisfies equations (6) or, equivalently, if $\alpha = \phi^*\theta$ for a harmonic map $\phi : \mathbb{R}^2 \to G$.

Warning Our λ is the reciprocal of that in [**34**].

We provide a setting for this result by introducing the Lie algebra $\Omega\mathfrak{g}$ of based loops in $\mathfrak{g}$. Thus

$$\Omega\mathfrak{g} = \{\xi : S^1 \to \mathfrak{g} \colon \xi(1) = 0\}.$$

$\Omega\mathfrak{g}$ is a Lie algebra under pointwise bracket. Use the invariant inner product on $\mathfrak{g}$ to define the H^1 inner product on $\Omega\mathfrak{g}$,

$$(\xi_1, \xi_2)_{H^1} = \int_{S^1} (\xi_1', \xi_2'),$$

and then we have a Fourier decomposition on $\Omega\mathfrak{g}$. Indeed, any $\xi \in \Omega\mathfrak{g}$ may be written

$$\xi(\lambda) = \sum_{n\neq 0} (1-\lambda^n)\xi_n$$

with $\xi_n \in \mathfrak{g}^{\mathbb{C}}$ and $\xi_{-n} = \overline{\xi_n}$. With this in mind, we introduce a filtration of $\Omega\mathfrak{g}$ by finite dimensional subspaces

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega\mathfrak{g}$$

setting

$$\Omega_d = \{\xi \in \Omega\mathfrak{g} \colon \xi_n = 0 \text{ for } |n| > d\}$$

for $d \in \mathbb{N}$.

We also define

$$\Omega^+ = \{\xi \in \Omega\mathfrak{g}^{\mathbb{C}} : \xi_n = 0 \text{ for } n < 0\} \qquad \Omega^- = \{\xi \in \Omega\mathfrak{g}^{\mathbb{C}} : \xi_n = 0 \text{ for } n > 0\}.$$

Note that $\Omega^\pm$ are mutually conjugate subalgebras (with respect to the real form $\Omega\mathfrak{g}$) and

$$\Omega\mathfrak{g}^{\mathbb{C}} = \Omega^+ \oplus \Omega^-.$$

We may now summarize the above discussion in

Proposition 1.1 *Let $A = A_\lambda$ be an $\Omega\mathfrak{g}$-valued 1-form on $\mathbb{R}^2$ such that* (*i*) $A^{(1,0)}$ *has values in* $\Omega^+ \cap \Omega_1^{\mathbb{C}}$;

(*ii*) *the connection* $d + A$ *is flat.*

Then there is a harmonic map $\phi : \mathbb{R}^2 \to G$, *unique up to left translation by a constant, such that*

$$\phi^*\theta = A_{-1}$$

and all harmonic maps of $\mathbb{R}^2$ *into* G *arise this way.*

Our first result is that such A may be constructed by integrating a pair of ordinary differential equations.

1.2 Commuting flows

Fix $d \in \mathbb{N}$ and define vector fields X_1, X_2 on Ω_d by

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1-\lambda)\xi_d].$$

Observe that the X_i are indeed tangent to Ω_d since the coefficient of $1 - \lambda^{d+1}$ on the right hand side is $-2i[\xi_d, \xi_d] = 0$. Concerning these vector fields we have

Theorem 1.2 *The* X_i *are complete and commute. Thus, if* X_i^t *denotes the flow on* Ω_d *generated by* X_i, *then*

$$(t^1, t^2) \cdot \xi = X_1^{t^1} \circ X_2^{t^2}(\xi)$$

defines an action of $\mathbb{R}^2$ *on* Ω_d.

Fix an initial condition $\xi_0 \in \Omega_d$ *and let* $\xi : \mathbb{R}^2 \to \Omega_d$ *be given by*

$$\xi(t^1, t^2) = (t^1, t^2) \cdot \xi_0.$$

Then $A_\lambda = 2i(1-\lambda)\xi_d dz - 2i(1-\lambda^{-1})\xi_{-d} d\overline{z}$ *satisfies the Maurer–Cartan equation and so gives rise to a harmonic map* $\phi : \mathbb{R}^2 \to G$.

Otherwise said: if $\xi : \mathbb{R}^2 \to \Omega_d$ solves

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1-\lambda)\xi_d] \tag{7}$$

then $4i\xi_d dz = \phi^*\theta^{(1,0)}$ for a harmonic map $\phi : \mathbb{R}^2 \to G$ and solutions of (7) exist on all $\mathbb{R}^2$ for any choice of initial condition.

Proof We begin with an observation which is in any case fundamental to much of what follows; that is, the ordinary differential equations defining the flows of the X_i are of Lax type:

$$\dot{\xi} = [\xi, V_i(\xi)]$$

with $V_i : \Omega_d \to \Omega_d$.

The L^2 inner product on $\Omega\mathfrak{g}$ induces an invariant inner product on Ω_d so that

$$\begin{aligned}\frac{d}{dt^i}(\xi,\xi)_{L^2} &= 2(\dot{\xi},\xi)_{L^2}\\ &= 2([\xi,V_i(\xi)],\xi)_{L^2} = -2([\xi,\xi],V_i(\xi))_{L^2} = 0.\end{aligned}$$

Thus the flows evolve on spheres in Ω_d and so are complete.

Set $Z = \frac{1}{2}(X_1 - iX_2)$. The X_i commute if and only if $[Z,\overline{Z}]$ vanishes. For this we compute:

$$d_Z\overline{Z} = d_Z[\xi,-2i(1-\lambda^{-1})\xi_{-d}] = [Z,-2i(1-\lambda^{-1})\xi_{-d}] + [\xi,-2i(1-\lambda^{-1})Z_{-d}].$$

Using $(1-\lambda^n)(1-\lambda^m) = (1-\lambda^n)+(1-\lambda^m)-(1-\lambda^{n+m})$, we see that

$$Z_{-d}(\xi) = 2i[\xi_{-d},\xi_d]$$

so that

$$d_Z\overline{Z} = 4\{(1-\lambda)(1-\lambda^{-1})[[\xi,\xi_d],\xi_{-d}] + (1-\lambda^{-1})[\xi,[\xi_{-d},\xi_d]]\}.$$

Conjugating gives

$$d_{\overline{Z}}Z = 4\{(1-\lambda)(1-\lambda^{-1})[[\xi,\xi_{-d}],\xi_d] + (1-\lambda)[\xi,[\xi_d,\xi_{-d}]]\}$$

and then an application of the Jacobi identity shows that $[Z,\overline{Z}] = d_Z\overline{Z} - d_{\overline{Z}}Z$ vanishes.

Finally let $\xi : \mathbb{R}^2 \to \Omega_d$ be the orbit map through an initial condition ξ_0. Thus

$$\frac{\partial\xi}{\partial z} = [\xi,2i(1-\lambda)\xi_d] \tag{8'}$$

and, since ξ is real, we have the conjugate equation

$$\frac{\partial\xi}{\partial\overline{z}} = -[\xi,2i(1-\lambda^{-1})\xi_{-d}]. \tag{8''}$$

Otherwise put,

$$d\xi = [\xi,A].$$

Now take the $(1-\lambda^{-d})$ coefficient in $(8')$ and the $(1-\lambda^d)$ coefficient in $(8'')$ to get

$$\frac{\partial\xi_{-d}}{\partial z} = 2i[\xi_{-d},\xi_d] \qquad \frac{\partial\xi_d}{\partial\overline{z}} = -2i[\xi_d,\xi_{-d}]$$

which are easily checked to be equivalent to (6) for $\alpha = 4i\xi_d dz - 4i\xi_{-d}d\overline{z}$ and thus the Maurer–Cartan equation for A. □

1.3 r-matrices, higher flows and pluriharmonic maps

Theorem (1.2), although easily proved, is somewhat lacking in motivation. To remedy this, and to extend our theory to higher dimensional domains, let us briefly recall a development which is well known in the integrable systems literature (c.f. [**32**,**33**]).

Let $\mathcal{G}$ be a (possibly infinite-dimensional) Lie algebra. A linear map $R : \mathcal{G} \to \mathcal{G}$ is called an *r-matrix* if the bracket defined by

$$[\xi, \eta]_R \stackrel{\text{def}}{=} [R\xi, \eta] + [\xi, R\eta]$$

satisfies the Jacobi identity. This is certainly the case if R satisfies the (modified) classical Yang–Baxter equation:

$$R[\xi, \eta]_R - [R\xi, R\eta] = \alpha[\xi, \eta] \tag{9}$$

for some fixed $\alpha \in \mathbb{C}$.

Let $\mathcal{V} : \mathcal{G} \to \mathcal{G}$ be a vector field on $\mathcal{G}$. We say that $\mathcal{V}$ is ad-*equivariant* if

$$d\mathcal{V}_\xi([\xi, \eta]) = [\mathcal{V}(\xi), \eta]$$

for all ξ, $\eta \in \mathcal{G}$. We note that this forces $\mathcal{V}(\xi)$ to lie in the centre of the centraliser of ξ and so, in particular, any two such vector fields commute pointwise.

We remark that if $\mathcal{G}$ is the Lie algebra of a Lie group, then ad-equivariance is just the infinitesimal version of the condition that $\mathcal{V}$ should commute with the adjoint action of the group i.e. $\operatorname{Ad} g\mathcal{V}(\xi) = \mathcal{V}(\operatorname{Ad} g\xi)$. If, in addition, $\mathcal{G}$ is equipped with an invariant inner product, then the gradients of functions on $\mathcal{G}$ invariant under the adjoint action furnish examples of ad-equivariant vector fields.

An abstract setting for the results of the previous section is now provided by

Theorem 1.3 [**13**] *Let R be a solution to the modified classical Yang–Baxter equation* (9) *and $\mathcal{V}_1, \dots, \mathcal{V}_n$ be* ad-*equivariant vector fields on $\mathcal{G}$. Let μ_{ij} be real constants, $1 \le i, j \le n$ and define vector fields X_i on $\mathcal{G}$ by*

$$X_i(\xi) = [\xi, R\mathcal{V}_i(\xi)] = [\xi, R\mathcal{V}_i(\xi) + \mu_{ij}\mathcal{V}_j(\xi)].$$

Then the vector fields X_i mutually commute.

If, moreover, the X_i are complete, let X_i^t denote the flow on $\mathcal{G}$ generated by X_i. Then

$$(t^1, \dots, t^n) \cdot \xi = X_1^{t^1} \circ \cdots \circ X_n^{t^n}(\xi)$$

defines an action of $\mathbb{R}^n$ on $\mathcal{G}$. Fix an initial condition $\xi_0 \in \mathcal{G}$ and let $\xi : \mathbb{R}^n \to \mathcal{G}$ be the orbit map through ξ_0. Then the $\mathcal{G}$-valued 1*-form*

$$A = (R\mathcal{V}_i + \mu_{ij}\mathcal{V}_j)dt^i$$

satisfies the Maurer–Cartan equation.

To apply this to the case at hand, take $\mathcal{G} = \Omega\mathfrak{g}$ and define an r-matrix on $\Omega\mathfrak{g}$ by

$$R = \begin{cases} i & \text{on } \Omega^+, \\ -i & \text{on } \Omega^-. \end{cases}$$

That R satisfies the modified classical Yang–Baxter equation (9) is a simple consequence of the fact that $\Omega^\pm$ are subalgebras.

Define ad-equivariant vector fields $\mathcal{V}_1$, $\mathcal{V}_2$ by

$$\tfrac{1}{2}(\mathcal{V}_1 - i\mathcal{V}_2)(\xi) = \lambda^{1-d}\xi$$

and let the μ_{ij} be given by

$$(\mu_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

to get vector fields X_1, X_2 given by

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, (R+i)\lambda^{1-d}\xi]$$

which on Ω_d become

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1-\lambda)\xi_d]$$

since $R+i$ annihilates Ω^-.

Thus we may view (1.2) as a special case of (1.3) but in fact we can do more. Let $V : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be an $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial map and let

$$V = V^{(1)} + \cdots + V^{(k)}$$

be its decomposition into homogeneous maps with $V^{(i)}$ homogeneous of degree d_i. Each $V^{(i)}$ is also $\operatorname{Ad} G^{\mathbb{C}}$-equivariant. We define ad-equivariant vector fields $\mathcal{V}_1$, $\mathcal{V}_2$ on $\Omega\mathfrak{g}$ by

$$\tfrac{1}{2}(\mathcal{V}_1 - i\mathcal{V}_2)(\xi) = \sum_{i=1}^{k} (-1)^{d_i+1}\lambda^{1-dd_i}V^{(i)}(\xi)$$

and then, with the same choice of μ_{ij} as before, we obtain vector fields X_1, X_2 which on Ω_d are given by

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1-\lambda)V(\xi_d)].$$

(We note that the X_i are tangent to Ω_d since $V(\xi_d)$ lies in the centre of the centraliser of ξ_d.)

Just as in (1.2), the X_i are complete on Ω_d since their flows preserve the L^2 norm and so (1.3) gives

Theorem 1.4 *Let $V_1, \ldots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial maps and $d \in \mathbb{N}$. Then for any $\xi_0 \in \Omega_d$ there is a map $\xi : \mathbb{C}^n \to \Omega_d$ solving $\xi(0) = \xi_0$ and*

$$\frac{\partial \xi}{\partial z^k} = [\xi, 2i(1-\lambda)V_k(\xi_d)] \tag{10}$$

and then the $\Omega\mathfrak{g}$-valued 1-form on $\mathbb{C}^n$ defined by

$$A_\lambda = 2i(1-\lambda)V_k(\xi_d)dz^k - 2i(1-\lambda^{-1})\overline{V_k(\xi_d)}d\bar{z}^k$$

satisfies the Maurer–Cartan equation.

We have seen that for $n = 1$, such A give rise to harmonic maps $\phi : \mathbb{R}^2 \to G$. For higher dimensions, there is a similar interpretation due to Ohnita–Valli [**27**].

Let M be a complex manifold and N a Riemannian manifold. A map $\phi : M \to N$ is *pluriharmonic* if

$$\nabla^{\phi}_{\overline{X}} \partial\phi(Y) - \partial\phi(\overline{\partial}_{\overline{X}} Y) = 0 \tag{11}$$

for all X, $Y \in T^{1,0}M$. Here ∇^{ϕ} is the pull-back of the Levi-Civita connection on N, $\overline{\partial}$ is the Cauchy–Riemann operator on $T^{1,0}M$ and $\partial\phi : T^{1,0}M \to TN^{\mathbb{C}}$ is the $(1,0)$-part of the derivative of ϕ. If M is Kähler, this reduces to the condition

$$\nabla d\phi^{(1,1)} = 0.$$

It is well known that ϕ is pluriharmonic if and only if the restriction of ϕ to any holomorphic curve in M is harmonic.

Now take N to be a compact Lie group G, set $\alpha = \phi^*\theta$ with type decomposition $\alpha = \alpha' + \alpha''$. Then Ohnita–Valli prove:

Theorem 1.5 [**27**] *$\phi : M \to G$ is pluriharmonic if and only if*

$$d\alpha' + \tfrac{1}{2}[\alpha' \wedge \alpha''] = d\alpha'' + \tfrac{1}{2}[\alpha' \wedge \alpha''] = 0$$
$$[\alpha' \wedge \alpha'] = [\alpha'' \wedge \alpha''] = 0.$$

Further, in this case, the $\Omega\mathfrak{g}$-valued 1-form A on M given by

$$A_\lambda = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha''$$

satisfies the Maurer–Cartan equation.

Conversely, if M is simply connected and A is an $\Omega\mathfrak{g}$-valued 1-form on M such that

- *(i) $A^{(1,0)}$ has values in $\Omega^+ \cap \Omega_1^{\mathbb{C}}$;*
- *(ii) the connection $d + A$ is flat;*

then there is a pluriharmonic map $\phi : M \to G$, unique up to left translation by a constant, such that $\phi^*\theta = A_{-1}$.

Corollary 1.6 *In the situation of* (1.4), *we have*

$$4iV_k(\xi_d)dz^k - 4i\overline{V_k(\xi_d)}d\overline{z}^k = \phi^*\theta$$

for some pluriharmonic map $\phi : \mathbb{C}^n \to G$ unique up to left translation by a constant.

Thus the flows on each Ω_d that produce harmonic maps of $\mathbb{R}^2$ are part of a larger commuting family of flows that produce pluriharmonic maps of $\mathbb{C}^n$.

We now give a Hamiltonian interpretation of the ordinary differential equations that we have been discussing. For this, we return to the abstract setting of a Lie algebra $\mathcal{G}$ and an r-matrix $R : \mathcal{G} \to \mathcal{G}$.

An r-matrix equips $\mathcal{G}$ with a second Lie algebra structure and thus $\mathcal{G}^*$ acquires a second Poisson structure. For $f,\, g \in C^\infty(\mathcal{G}^*)$, set

$$\{f,g\}_R(x) = \langle x, [df_x, dg_x]_R\rangle$$

to get a Poisson bracket and thus a Lie algebra homomorphism $f \mapsto X_f$, $C^\infty(\mathcal{G}^*) \to C^\infty(T\mathcal{G}^*)$ given by $X_f g = \{f,g\}_R$.

Say that f is invariant if it is invariant under the co-adjoint action on $\mathcal{G}^*$. We have [**32**]

(*i*) if f, g are invariant they Poisson commute so that $[X_f, X_g]$ vanishes;

(*ii*) if f is invariant then, for $x \in \mathcal{G}^*$,

$$X_f(x) = -\operatorname{ad}^*(Rdf_x)(x).$$

Suppose now that $\mathcal{G}$ has an invariant inner product and use it to identify $\mathcal{G}^*$ with $\mathcal{G}$. Then f is invariant if and only if its gradient is ad-equivariant and then the corresponding Hamiltonian vector field is given by

$$X_f(\xi) = [\xi, R\nabla f(\xi)].$$

Thus, when our ad-equivariant vector fields are gradients, the vector fields X_i discussed above are Hamiltonian.

In the case at hand, take

$$(f_1 - if_2)(\xi) = \int_{S^1} \lambda^{1-d}(\xi,\xi)$$

to get Hamiltonians defining the flows of theorem (1.2). Similarly, if $P : \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ is an $\operatorname{Ad}(G^{\mathbb{C}})$-invariant homogeneous polynomial of degree r with gradient V, we take

$$\tfrac{1}{2}(f_1 - if_2)(\xi) = \int_{S^1} (-1)^{r+1}\lambda^{1-dr}P(\xi)$$

to get Hamiltonian flows with vector fields

$$\tfrac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1-\lambda)V(\xi_d)].$$

Thus pluriharmonic maps can be constructed from Hamiltonian flows given by invariant functions on an infinite-dimensional Lie algebra: such flows are a ubiquitous feature of completely integrable systems theory [**1**,**30**,**31**].

Remark This method of producing Poisson commuting functions on the dual of a Lie algebra is very similar to that of Kostant–Adler–Symes (see, in particular, [**1**, theorem 4.2]) but there are differences. In particular, in our case, the r-matrix is not skew with respect to an invariant inner product. As a consequence, the finite-dimensional subspaces Ω_d are not Poisson submanifolds for the R-Poisson structure although they are invariant under the Hamiltonian flows of invariant Hamiltonians.

In conclusion then, we have seen how pluriharmonic maps of $\mathbb{C}^n$ into compact Lie groups may be obtained from commuting Hamiltonian flows. Following [**28**] we christen such maps in the following definition.

Definition A pluriharmonic map $\phi : \mathbb{C}^n \to G$ is of *finite type* if there are $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial maps $V_1, \ldots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $d \in \mathbb{N}$ and $\xi : \mathbb{C}^n \to \Omega_d$ such that ξ satisfies (10) and

$$\phi^*\theta = 4iV_k(\xi_d)dz^k - 4i\overline{V_k(\xi_d)}d\overline{z}^k.$$

We shall address below the question of which pluriharmonic maps are of finite type but make some preliminary observations here.

First observe that the ordinary differential equations defining an orbit map $\xi : \mathbb{C}^n \to \Omega_d$ are of Lax type and so have many conserved quantities. Indeed, if $P : \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ is an $\operatorname{Ad} G^{\mathbb{C}}$-invariant polynomial, then $P(\xi)$ is a constant element of $\mathbb{C}[\lambda, \lambda^{-1}]$. Comparing coefficients then gives that $P(\xi_d)$ is constant. In fact, a slightly stronger result is true.

Proposition 1.7 *If $\xi : \mathbb{C}^n \to \Omega_d$ satisfies*

$$\frac{\partial \xi}{\partial z^k} = [\xi, 2i(1-\lambda)V_k(\xi_d)]$$

for $\operatorname{Ad} G^{\mathbb{C}}$-equivariant polynomial maps $V_1, \ldots, V_n$ then ξ_d takes values in a single $\operatorname{Ad} G^{\mathbb{C}}$ orbit in $\mathfrak{g}^{\mathbb{C}}$.

Proof Comparing coefficients, we see that

$$d\xi_d = -2i[\xi_{d-1}, V_k(\xi_d)]dz^k + 2i[\xi_d, \overline{V_k(\xi_d)}]d\overline{z}^k.$$

Since $V_k(\xi_d)$ lies in the centre of the centraliser of ξ_d, it is easy to see that $[V_k(\xi_d), \mathfrak{g}^{\mathbb{C}}] \subset [\xi_d, \mathfrak{g}^{\mathbb{C}}]$ so that $d\xi_d$ takes values in $[\xi_d, \mathfrak{g}^{\mathbb{C}}]$. Otherwise said, $d\xi_d$ takes values in the tangent space to the $\operatorname{Ad} G^{\mathbb{C}}$ orbit through ξ_d so that the assertion now follows from the uniqueness of solutions of ordinary differential equations. $\square$

Corollary 1.8 *If $\phi : \mathbb{C}^n \to G$ is a pluriharmonic map of finite type then each $\phi^*\theta(\frac{\partial}{\partial z^k})$ takes values in a single $\operatorname{Ad} G^{\mathbb{C}}$ orbit in $\mathfrak{g}^{\mathbb{C}}$.*

Proof $\phi^*\theta(\frac{\partial}{\partial z^k}) = 4iV_k(\xi_d)$ and V_k sends orbits to orbits. $\square$

In particular, a non-constant harmonic map of $\mathbb{R}^2$ into G has nowhere vanishing derivative if it is of finite type.

1.4 Maps into symmetric spaces

In Differential Geometry, interest is more often focussed on harmonic maps into symmetric spaces rather than just Lie groups. However, the (pluri)harmonic map equations are preserved by post-composition by a totally geodesic map and, as is well known [**9**], any symmetric space may be totally geodesically immersed in its group of isometries. Thus the symmetric space problem reduces to the Lie group problem. On the other hand, the above development produces pluriharmonic maps of $\mathbb{C}^n$ into a Lie group and it is natural to ask when these maps factor through a symmetric space.

In this section, we shall provide a simple necessary and sufficient condition for this to be the case. We begin by recalling the details of the Cartan embedding. Let N be a compact Riemannian symmetric space and set $G = I_0(N)$ with Lie algebra $\mathfrak{g}$. Fix $x \in N$ and let $\sigma_x \in I(N)$ be the involutive isometry with isolated fixed point x. Then $g \mapsto \sigma_x g \sigma_x$ is an involution of G which we call the *involution at* x and denote by τ_x. We now have a symmetric decomposition

$$\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x$$

of $\mathfrak{g}$ into ± 1-eigenspaces of (the derivative of) τ_x satisfying the familiar relations

$$[\mathfrak{k}_x, \mathfrak{k}_x] \subset \mathfrak{k}_x \qquad [\mathfrak{k}_x, \mathfrak{p}_x] \subset \mathfrak{p}_x \qquad [\mathfrak{p}_x, \mathfrak{p}_x] \subset \mathfrak{k}_x.$$

Moreover, $\mathfrak{k}_x$ is the Lie algebra of the isotropy group K_x of x and $(G^{\tau_x})_0 \subset K_x \subset G^{\tau_x}$.

The map $\mathfrak{g} \to T_x N$ given by $\eta \mapsto \frac{d}{dt}\big|_{t=0} \exp t\eta \cdot x$ is an isomorphism on $\mathfrak{p}_x$. Inverting this at each point gives a $\mathfrak{g}$-valued 1-form β on N which provides an isomorphism of TN with a subbundle $[\mathfrak{p}]$ of the trivial bundle $N \times \mathfrak{g}$ (c.f. [**6**]).

Now fix a basepoint $o \in N$, set $\tau = \tau_o$, $K = K_o$ etc. and define a map $i_o : N \to G$ by

$$i_o(g \cdot o) = g^{\tau} g^{-1}$$

which is certainly well defined. Concerning i_o we have

Proposition 1.9 *$i_o : N \to G$ is a totally geodesic immersion with*

$$i_o^* \theta = -2\beta.$$

Let $\widehat{N}$ be the image of i_o. Then $\widehat{N}$ is a globally Riemannian symmetric space totally geodesically embedded in G. Moreover $\widehat{N} = \exp \mathfrak{p}$ and further, $\widehat{N}$ is the component of $\{g \in G \colon g^{\tau} g = e\}$ containing the identity e of G.

Proof That i_o is a totally geodesic immersion is well known [**9**] and the formula for $i_o^* \theta$ is proved in [**6**, Lemma 8.1]. As for the remaining assertions, temporarily denote the component of $\{g \in G \colon g^{\tau} g = e\}$ containing e by F. Then F is the component containing e of the fixed set of the isometry $g \mapsto g^{-\tau}$ and so is a totally geodesic submanifold of G. Moreover, the derivative at e of this isometry is $\eta \mapsto -\tau\eta$ with fixed set $\mathfrak{p}$ so that $F = \exp \mathfrak{p}$. Thus F is a global Riemannian symmetric space by [**16**]. Finally $\widehat{N}$ is a compact subset of F and both F and $\widehat{N}$ have the same dimension $\dim \mathfrak{p}$ so that $\widehat{N}$ and F coincide. $\square$

Using this, we obtain a simple criterion for a map $\phi : M \to G$ to factor through $\widehat{N}$:

Lemma 1.10 *Let $\phi : M \to G$ be a map of a connected manifold M with $\phi(m) = e$ for some $m \in M$. Then ϕ has image in $\widehat{N}$ if and only if*

$$\tau \operatorname{Ad} \phi \; \phi^* \theta + \phi^* \theta = 0.$$

Proof By (1.9), it suffices to show that $\psi = \phi^{\tau} \phi$ is constant. However, a simple calculation gives

$$\psi^* \theta = \operatorname{Ad} \phi^{-1} \tau \phi^* \theta + \phi^* \theta$$

whence the result follows. $\square$

Let us now return to our orbit maps $\xi : \mathbb{C}^n \to \Omega_d$, that is to solutions of

$$d\xi = [\xi, 2i(1-\lambda)V_k(\xi_d)dz^k - 2i(1-\lambda^{-1})\overline{V_k(\xi_d)}d\overline{z}^k]. \tag{12}$$

To adapt this to the symmetric space setting, we make a

Definition Let $V : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be an $\operatorname{Ad} G^{\mathbb{C}}$-equivariant map and τ an involution of $\mathfrak{g}^{\mathbb{C}}$. Then V is *τ-compatible* if for all $\eta \in \mathfrak{g}^{\mathbb{C}}$

$$V(-\tau\eta) = -\tau V(\eta).$$

We note that if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the eigenspace decomposition of τ then, for τ-compatible V, we have $V(\mathfrak{p}) \subset \mathfrak{p}$. In general, any $\operatorname{Ad} G^{\mathbb{C}}$-equivariant V has a decomposition into $\operatorname{Ad} G^{\mathbb{C}}$-equivariant maps

$$V = V_+ \oplus V_-$$

with V_+ τ-compatible and V_- anti-commuting with $-\tau$. Thus $V_+ |_{\mathfrak{p}}$ is the $\mathfrak{p}$-part of $V |_{\mathfrak{p}}$. We also note that if τ is an inner automorphism then τ-compatibility reduces to the demand that the homogeneous components of V have *odd* degrees.

With this in mind, we have

Proposition 1.11 *Let $d \in \mathbb{N}$ be odd, τ an involution of $\mathfrak{g}$, $V_1, \dots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ τ-compatible $\operatorname{Ad} G^{\mathbb{C}}$-equivariant maps and $\xi : \mathbb{C}^n \to \Omega_d$ a solution of* (12). *Let $\phi : \mathbb{C}^n \to G$ satisfy $\phi^*\theta = 4iV_k(\xi_d)dz^k - 4i\overline{V_k(\xi_d)}d\overline{z}^k$. Define $\widetilde{\xi}$ by*

$$\widetilde{\xi}(\lambda) = \tau \operatorname{Ad}\phi\xi(-\lambda)\ ,\ \lambda \in S^1.$$

Then $\widetilde{\xi}$ also satisfies (12) *i.e.*

$$d\widetilde{\xi} = [\widetilde{\xi}, 2i(1-\lambda)V_k(\widetilde{\xi}_d)dz^k - 2i(1-\lambda^{-1})\overline{V_k(\widetilde{\xi}_d)}d\overline{z}^k].$$

Remark $\widetilde{\xi}$ need not take values in Ω_d since $\widetilde{\xi}(1)$ need not vanish but it does take values in $\Lambda_d = \{\sum_{|n|\leq d} \xi_n\lambda^n : \xi_n \in \mathfrak{g}^{\mathbb{C}}, \overline{\xi_n} = \xi_{-n}\}$ on which our vector fields are also defined.

Proof We compute:

$$\begin{aligned} d\widetilde{\xi}(\lambda) &= \tau d(\operatorname{Ad}\phi)\xi(-\lambda) + \tau \operatorname{Ad}\phi\, d\xi(-\lambda) \\ &= \tau \operatorname{Ad}\phi\,[\phi^*\theta, \xi(-\lambda)] + \tau \operatorname{Ad}\phi\, d\xi(-\lambda) \end{aligned}$$

using

$$d(\operatorname{Ad}\phi) = \operatorname{Ad}\phi \circ \operatorname{ad}\phi^*\theta.$$

Now substitute from (12) to get

$$\begin{aligned} d\widetilde{\xi}(\lambda) &= \tau \operatorname{Ad}\phi[4iV_k(\xi_d)dz^k - 4i\overline{V_k(\xi_d)}d\overline{z}^k, \xi(-\lambda)] \\ &\quad + \tau \operatorname{Ad}\phi[\xi(-\lambda), 2i(1+\lambda)V_k(\xi_d)dz^k - 2i(1+\lambda^{-1})\overline{V_k(\xi_d)}d\overline{z}^k] \\ &= \tau \operatorname{Ad}\phi[\xi(-\lambda), 2i(\lambda-1)V_k(\xi_d)dz^k - 2i(\lambda^{-1}-1)\overline{V_k(\xi_d)}d\overline{z}^k]. \end{aligned}$$

However, since d is odd we have

$$\widetilde{\xi}_d = -\tau \operatorname{Ad}\phi \xi_d$$

while $-\tau \operatorname{Ad}\phi$ commutes with both V_k and $\overline{V_k}$ (since both τ and $\operatorname{Ad}\phi$ are real). Thus

$$d\widetilde{\xi} = [\widetilde{\xi}, 2i(1-\lambda)V_k(\widetilde{\xi}_d)dz^k - 2i(1-\lambda^{-1})\overline{V_k(\widetilde{\xi}_d)}d\overline{z}^k].$$

as required. □

Before applying this, we briefly pause to introduce some notation. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an involution and set

$$\Lambda(\mathfrak{g}, \tau) = \{\xi : S^1 \to \mathfrak{g} \colon \xi(-\lambda) = \tau\xi(\lambda) \text{ for all } \lambda \in S^1\}.$$

Thus, if $\xi = \sum \lambda^n \xi_n$, then $\xi \in \Lambda(\mathfrak{g}, \tau)$ if and only if $\xi_n \in \mathfrak{p}^{\mathbb{C}}$ for n odd and $\xi_n \in \mathfrak{k}^{\mathbb{C}}$ for n even.

We now come to the main result of this section:

Theorem 1.12 *Let N be a compact Riemannian symmetric space with $I_0(N) = G$. Fix a basepoint $o \in N$, set $\tau = \tau_o$ and recall the immersion $i_o : N \to \widehat{N} \subset G$.*

Let $d \in \mathbb{N}$ be odd, $V_1, \dots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ τ-compatible $\operatorname{Ad} G^{\mathbb{C}}$-equivariant maps and $\xi_0 \in \Omega_d \cap \Lambda(\mathfrak{g}, \tau)$. Let $\xi : \mathbb{C}^n \to \Omega_d$ be a solution of (12) with $\xi(0) = \xi_0$ and let $\phi : \mathbb{C}^n \to G$ be the pluriharmonic map satisfying $\phi(0) = e$ and $\phi^*\theta = 4iV_k(\xi_d)dz^k - 4i\overline{V_k(\xi_d)}d\overline{z}^k$.

Then ϕ has image in $\widehat{N}$ and moreover, $\xi(x) \in \Lambda(\mathfrak{g}, \tau_{\phi(x)})$ for each $x \in \mathbb{C}^n$ where $\tau_{\phi(x)}$ is the involution at $\phi(x) \in \widehat{N}$.

Proof Set $\widetilde{\xi}(\lambda) = \tau \operatorname{Ad}\phi\, \xi(-\lambda)$ then by (1.11), $\widetilde{\xi}$ also solves (12). But since $\xi_0 \in \Lambda(\mathfrak{g}, \tau)$

$$\widetilde{\xi}(\lambda)(0) = \tau\xi(-\lambda)(0) = \xi(\lambda)(0)$$

so that, by uniqueness of solutions of ordinary differential equations, we get $\xi = \widetilde{\xi}$. Comparing coefficients at λ^d then gives

$$-\tau \operatorname{Ad}\phi\, \xi_d = \xi_d$$

whence, for each k

$$V_k(\xi_d) = -\tau \operatorname{Ad}\phi\, V_k(\xi_d)$$

and hence,

$$\tau \operatorname{Ad}\phi\ \phi^*\theta + \phi^*\theta = 0.$$

Thus ϕ has image in $\widehat{N}$ by (1.10).

Finally, if $\phi(x) = i_o(g \cdot o)$ then the involution at $\phi(x)$ is given by

$$\tau_{\phi(x)} = \operatorname{Ad} g\, \tau \operatorname{Ad} g^{-1} = \tau \operatorname{Ad}(g^\tau g^{-1}) = \tau \operatorname{Ad}\phi(x)$$

so that $\xi = \widetilde{\xi}$ means that $\xi \in \Lambda(\mathfrak{g}, \tau_\phi)$. □

Thus we conclude that the question of whether our pluriharmonic maps factor through a symmetric space is simply a question of choosing τ-compatible flows and the right initial condition.

For the case where the domain is $\mathbb{R}^2$, the situation is even simpler, there we take $V = \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}}$ as our $\operatorname{Ad} G^{\mathbb{C}}$-equivariant map which is always τ-compatible and then we produce harmonic maps into symmetric spaces whenever we have chosen an appropriate initial condition.

2 Polynomial Killing fields and harmonic maps of finite type

2.1 Polynomial Killing fields

Let $\phi : \mathbb{C}^n \to G$ be a pluriharmonic map and recall the $\Omega\mathfrak{g}$-valued 1-form A_λ given by

$$A_\lambda = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha''$$

where $\phi^*\theta = \alpha' + \alpha''$. We have seen that ϕ is of finite type precisely when, for some $d \in \mathbb{N}$, there is a map $\xi : \mathbb{C}^n \to \Omega_d$ such that

$$d\xi = [\xi, A_\lambda] \tag{13}$$

and

$$\alpha' = 4iV_k(\xi_d)dz^k \tag{14}$$

for some $\mathrm{Ad}\, G^{\mathbb{C}}$-equivariant polynomial maps $V_k : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.

This motivates the following definition in which our terminology is taken from [**13**].

Definition Let $\phi : M \to G$ be a pluriharmonic map of a complex manifold into G with A_λ as above. A *polynomial Killing field* for ϕ is a map $\xi : M \to \Omega_d$, some $d \in \mathbb{N}$, satisfying (13).

If, in addition, there is a nowhere vanishing holomorphic vector field Z on M such that

$$\xi_d = \alpha'(Z),$$

we say that ξ is an *adapted* polynomial Killing field.

Of course, the situation we have in mind is when Z is a holomorphic co-ordinate vector field on a complex torus or Euclidean space. In particular, if M is $\mathbb{R}^2$ or a 2-torus, we may assume that $Z = \frac{\partial}{\partial z}$ and then an adapted Killing field is an orbit map with $V = \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}}$ and we conclude:

Lemma 2.1 *If $\phi : \mathbb{R}^2 \to G$ admits an adapted polynomial Killing field then ϕ is of finite type.*

In fact, it will be shown in section (2.4) that a similar result is true for a large class of pluriharmonic maps. The question therefore arises as to which pluriharmonic maps admit such a polynomial Killing field. For maps that factor through a complex torus, it suffices to consider a more general kind of field which we now describe.

2.2 Formal Killing fields

The idea is to consider an analogue of polynomial Killing fields taking values in a space of $\mathfrak{g}^{\mathbb{C}}$-valued formal Laurent series rather than in some Ω_d.

Definition Let $\phi : M \to G$ be a pluriharmonic map of a complex manifold into G with A_λ as above. A *formal Killing field* for ϕ is a formal Laurent series $Y = \sum_{k\geq i} \lambda^{-k} Y_k$ with each $Y_k : M \to \mathfrak{g}^{\mathbb{C}}$ satisfying

$$dY = [Y, A_\lambda]. \tag{15}$$

If, in addition, there is a nowhere vanishing holomorphic vector field Z on M for which the top term of Y satisfies

$$Y_i = \alpha'(Z)$$

we say that Y is *adapted*.

It will be convenient to write our equations in a slightly different way: if we identify $\phi^{-1}TG$ with $M \times \mathfrak{g}$ via $\phi^{-1}\theta$, then the pull-back of the Levi-Civita connection on G is given by (c.f. [**6**])

$$\nabla = d + \tfrac{1}{2} \operatorname{ad} \alpha.$$

In terms of ∇, the formal Killing field equations (15) read

$$\nabla Y = \tfrac{1}{2}[\lambda\alpha' + \lambda^{-1}\alpha'', Y] \tag{16}$$

while the pluriharmonic map equations of theorem (1.5) read

$$d_\nabla \alpha' = d_\nabla \alpha'' = 0 \tag{17}$$

$$[\alpha' \wedge \alpha'] = [\alpha'' \wedge \alpha''] = 0 \tag{18}$$

where d_∇ is the induced exterior derivative on $\mathfrak{g}$-valued forms.

The technical fact that will allow us to pass from formal Killing fields to polynomial Killing fields is that the coefficients of a formal Killing field are solutions of a linear elliptic partial differential equation.

Definition Let $\phi : M \to G$ be a pluriharmonic map of a Hermitian manifold. A map $v : M \to \mathfrak{g}^{\mathbb{C}}$ is a *Jacobi field* for ϕ if it solves the linear elliptic equation

$$J_\phi v = (\partial_\nabla \overline{\partial}_\nabla - \overline{\partial}_\nabla \partial_\nabla) v(Z_k, \overline{Z}_k) - \tfrac{1}{4}[\alpha(\overline{Z}_k), [\alpha(Z_k), v]] - \tfrac{1}{4}[\alpha(Z_k), [\alpha(\overline{Z}_k), v]] = 0$$

where $\{Z_k\}$ is a local unitary frame for $T^{1,0}M$.

Remark If M is Kähler, then J_ϕ is the Jacobi operator of ϕ, i.e. the linearization of the harmonic map equation, whence our terminology.

Lemma 2.2 *Let $\phi : M \to G$ be a pluriharmonic map of a Hermitian manifold and Y a formal Killing field for ϕ. Then each coefficient Y_k is a Jacobi field.*

Proof We prove a stronger result: we have from (16)

$$\begin{aligned}
\partial_\nabla \overline{\partial}_\nabla Y &= \tfrac{1}{2}\partial_\nabla[\lambda^{-1}\alpha'', Y] = \tfrac{1}{2}\lambda^{-1}[\partial_\nabla \alpha'', Y] - \tfrac{1}{2}[\lambda^{-1}\alpha'' \wedge \partial_\nabla Y] \\
&= \tfrac{1}{2}\lambda^{-1}[\partial_\nabla \alpha'', Y] - \tfrac{1}{4}[\alpha'' \wedge [\alpha', Y]] \\
&= -\tfrac{1}{4}[\alpha'' \wedge [\alpha', Y]],
\end{aligned}$$

since $\partial_\nabla \alpha''$ vanishes by the pluriharmonic map equation (17). Similarly

$$\overline{\partial}_\nabla \partial_\nabla Y = -\tfrac{1}{4}[\alpha' \wedge [\alpha'', Y]]$$

so that

$$(\partial_\nabla \overline{\partial}_\nabla - \overline{\partial}_\nabla \partial_\nabla) Y + \tfrac{1}{4}[\alpha'' \wedge [\alpha', Y]] - \tfrac{1}{4}[\alpha' \wedge [\alpha'', Y]] = 0.$$

Taking the appropriate trace in this and comparing coefficients now establishes the lemma. $\square$

With this understood, we now have

Theorem 2.3 *Let $\phi : M \to G$ be a pluriharmonic map of a* compact *complex manifold. If ϕ admits a non-zero formal Killing field Y then ϕ admits a polynomial Killing field ξ. Moreover, if Y is adapted so is ξ.*

Finally, if ϕ factors through a symmetric space $\widehat{N} \subset G$ as in section (1.4) then

$$\xi(x) \in \Lambda(\mathfrak{g}, \tau_{\phi(x)})$$

for all $x \in M$ where $\tau_{\phi(x)}$ is the involution at $\phi(x) \in \widehat{N}$.

Proof We begin by introducing some notation: if $Y = \sum_{k \geq i} \lambda^{-k} Y_k$ is a formal Laurent series, let

$$Y^+ = \sum_{0 \geq k \geq i} \lambda^{-k} Y_k$$

denote its polynomial part. We define a vector space $\mathcal{K}^+$ by

$$\mathcal{K}^+ = \{Y^+ : Y \text{ a formal Killing field for } \phi\}.$$

Further, we define linear maps on $\mathcal{K}^+$ by

$$\begin{aligned} \mathcal{D} Y^+ &= \nabla Y^+ - \tfrac{1}{2}[\lambda\alpha' + \lambda^{-1}\alpha'', Y^+] \\ \mathcal{E} Y^+ &= \mathrm{ev}_1 \circ Y^+ \end{aligned}$$

where ev_1 is evaluation at $\lambda = 1$.

We claim that both $\mathcal{D}$ and $\mathcal{E}$ have finite rank. For this, introduce a Hermitian metric on M so that, by lemma (2.2), the coefficients of any formal Killing field are Jacobi fields and thus lie in a finite-dimensional space by the ellipticity of J_ϕ and the compactness of M. Now

$$\mathcal{E} Y^+ = \sum_{0 \geq k \geq i} Y_k$$

is a Jacobi field so that $\mathcal{E}$ has finite rank. Moreover,

$$\begin{aligned} \nabla Y^+ = (\nabla Y)^+ &= \tfrac{1}{2}[\lambda\alpha' + \lambda^{-1}\alpha'', Y]^+ \\ &= \tfrac{1}{2}[\lambda\alpha' + \lambda^{-1}\alpha'', Y^+] + \tfrac{1}{2}[\alpha', Y_{-1}] - \tfrac{1}{2}[\lambda^{-1}\alpha'', Y_0]. \end{aligned}$$

Thus $\mathcal{D} Y^+$ lies in the sum of the images of the Jacobi fields under the linear maps $\tfrac{1}{2}\,\mathrm{ad}\,\alpha'$ and $\tfrac{1}{2}\,\mathrm{ad}\,\lambda^{-1}\alpha''$ and so $\mathcal{D}$ also has finite rank and the claim is proved.

Consider now a non-zero formal Killing field Y. For each $N \in \mathbb{N}$, $\lambda^N Y$ is also a formal Killing field and, moreover, for N large enough, the truncations $(\lambda^N Y)^+$ are linearly independent. Thus some linear combination of these, ξ' say, lies simultaneously in the kernels of $\mathcal{D}$ and $\mathcal{E}$. Since these equations are both real, the Laurent polynomial $\xi = \xi' + \overline{\xi'}$ satisfies

$$\xi(1) = 0, \qquad \nabla\xi = \tfrac{1}{2}[\lambda\alpha' + \lambda^{-1}\alpha'', \xi].$$

The first equation means that ξ lies in some Ω_d while the second says that ξ is a polynomial Killing field.

Moreover, if Y is adapted, so is each $\lambda^N Y$ and, since the top term of ξ is a non-zero multiple of that of Y, after multiplication by a suitable constant, we conclude that ξ is also adapted.

Finally, suppose ϕ factors through a symmetric space $\widehat{N} \subset G$ and let τ_ϕ be the field of involutions where $\tau_{\phi(x)}$ is the involution at $\phi(x) \in \widehat{N}$. Then τ_ϕ is ∇-parallel and $\tau_\phi\alpha = -\alpha$. From this it is easy to check that if $Y = \sum_{k\geq -1} \lambda^{-k} Y_k$ is a formal Killing field, so is

$$\widehat{Y} = \sum_{k\geq -1} \lambda^{-k}(-1)^k \tau_\phi Y_k$$

and thus also $\widetilde{Y} = \frac{1}{2}(Y + \widehat{Y})$. Then each $(\lambda^{2N}\widetilde{Y})^+$ lies in $\Lambda(\mathfrak{g}^{\mathbb{C}}, \tau_\phi)$ and we apply the above argument to these to get a polynomial Killing field ξ with each $\xi(x) \in \Lambda(\mathfrak{g}, \tau_{\phi(x)})$ as required. $\square$

2.3 Existence of formal Killing fields

Now we come to the heart of the matter and introduce a hypothesis for pluriharmonic maps that ensures the existence of a formal Killing field.

Definition Let $\phi : M \to G$ be a pluriharmonic map of a complex manifold with $\phi^*\theta = \alpha$. We say that ϕ is *semisimple adapted* if there is a non-zero holomorphic vector field Z on M such that $\alpha(Z)$ takes values in a single $\operatorname{Ad} G^{\mathbb{C}}$-orbit of semisimple elements of $\mathfrak{g}^{\mathbb{C}}$.

Remark In general, the notion of semisimplicity of an element of a Lie algebra depends on a representation of that algebra. However, in what follows, we shall only consider semisimple $\mathfrak{g}^{\mathbb{C}}$ where the notion is unambiguous or take $\mathfrak{g}^{\mathbb{C}} = \mathbf{gl}(n, \mathbb{C})$ with its fundamental representation. For semisimple subalgebras of $\mathbf{gl}(n, \mathbb{C})$, the two notions coincide (c.f. [**19**]).

We shall see below that a large class of pluriharmonic maps of complex n-tori are semisimple adapted.

The theorem we shall prove is

Theorem 2.4 *Let $\phi : M \to G$ be a semisimple adapted pluriharmonic map of a complex manifold where G is semisimple or a unitary group. Then ϕ admits an adapted formal Killing field.*

The proof of this will occupy us for the rest of this section. Before embarking on this, let us pause to extract an important corollary: from (2.1), (2.3) and (2.4) we get

Corollary 2.5 *Let $\phi : T^2 \to G$ be a semisimple adapted harmonic map of a 2-torus into a compact semisimple Lie group. Then ϕ is of finite type.*

Now let us turn to the proof of (2.4): it will transpire that it is enough to prove the result for the case that G is a unitary group so that is where we shall begin.

Let $\phi : M \to \mathbf{U}(n)$ be a pluriharmonic map with Z a holomorphic vector field on M such that $\alpha(Z)$ takes values in a single $\operatorname{Ad}\mathbf{GL}(n,\mathbb{C})$-orbit of semisimple elements of $\mathbf{gl}(n,\mathbb{C})$. We seek a formal Laurent series $Y = \sum_{k\geq 0} \lambda^{-k} Y_k$ with $Y_k : M \to \mathbf{gl}(n,\mathbb{C})$; $Y_0 = \alpha(Z)$ and

$$dY + [A, Y] = 0,$$

where as usual $A = \frac{1-\lambda}{2}\alpha' + \frac{1-\lambda^{-1}}{2}\alpha''$.

Set $B = \alpha(Z) : M \to \mathbf{gl}(n,\mathbb{C})$. We define a bundle decomposition

$$M \times \mathbf{gl}(n,\mathbb{C}) = V \oplus V^{\perp}$$

by setting $V = \ker \operatorname{ad} B$, $V^{\perp} = \operatorname{im} \operatorname{ad} B$ and note the relations

$$VV \subset V, \qquad V^{\perp}V \subset V^{\perp}, \qquad VV^{\perp} \subset V^{\perp}. \tag{19}$$

Since B has image in a single conjugacy class, dB takes values in $V^{\perp}$ whence $\nabla B = dB + \frac{1}{2}[\alpha, B]$ also takes values in $V^{\perp}$. Now $\operatorname{ad} B$ is invertible on $V^{\perp}$ so we may define a $V^{\perp}$-valued 1-form Q by

$$\nabla B = \tfrac{1}{2}[B, Q]$$

and thus a new connection $D = \nabla + \frac{1}{2}\operatorname{ad} Q$ for which B and hence V and $V^{\perp}$ are parallel. In fact, it follows from the pluriharmonic map equations (17) that Q is a $(1,0)$-form. We use D to write the formal Killing field equations (16) as

$$DY = \tfrac{1}{2}[Q + \lambda\alpha' + \lambda^{-1}\alpha'', Y]. \tag{20}$$

Remark The $(1,0)$-part of this equation is essentially that considered in integrable systems theory for constructing hierarchies of nonlinear evolution equations (c.f. [**14**]).

We will construct Y via the following ansatz (c.f. [**12**,**13**]): we seek Y of the form

$$Y = (1 + W)^{-1} B (1 + W)$$

where the formal gauge transformation $W = \sum_{k\geq 1} \lambda^{-k} W_k$ has coefficients which are sections of $V^{\perp}$. Now

$$(1 + W) D((1 + W)^{-1}\sigma(1 + W))(1 + W)^{-1} = D\sigma + [\sigma, DW(1 + W)^{-1}]$$

so that such a Y will satisfy (20) if and only if

$$[DW(1 + W)^{-1} + \tfrac{1}{2}(1 + W)(Q + \lambda\alpha' + \lambda^{-1}\alpha'')(1 + W)^{-1}, B] = 0$$

since DB vanishes. Thus we seek W such that

$$DW(1 + W)^{-1} + \tfrac{1}{2}(1 + W)(Q + \lambda\alpha' + \lambda^{-1}\alpha'')(1 + W)^{-1} = \omega$$

for a 1-form ω taking values in (formal Laurent series in) V. Otherwise put, we want

$$DW + \tfrac{1}{2}(1+W)(Q + \lambda\alpha' + \lambda^{-1}\alpha'') = \omega(1+W).$$

We can find such a W for which this holds at least in the Z direction: evaluate the above on Z to get

$$D_Z W + \tfrac{1}{2}(1+W)(Q(Z) + \lambda B) = \omega(Z)(1+W)$$

and project onto V and $V^\perp$ to get

$$(WQ(Z))^V + \lambda B = 2\omega(Z) \tag{21}$$

$$2D_Z W + Q(Z) + (WQ(Z))^\perp + \lambda W B = 2\omega(Z) W. \tag{22}$$

Here ^{V}, ${}^\perp$ denote the projections onto V and $V^\perp$ respectively and we have used (19) and the fact that V and $V^\perp$ are D-parallel. Now substitute (21) into (22) to get an equation for W:

$$\lambda[B, W] = 2D_Z W + Q(Z) + (WQ(Z))^\perp - (WQ(Z))^V W$$

which can be recursively solved. Indeed, in terms of coefficients, it reads

$$\begin{aligned}[B, W_1] &= Q(Z)\\ [B, W_{k+1}] &= 2D_Z W_k + (W_k Q(Z))^\perp - \sum_{i+j=k} (W_i Q(Z))^V W_j,\end{aligned}$$

for $k \geq 1$, which uniquely define the W_k since $\operatorname{ad} B$ is invertible on $V^\perp$. With W so defined, we take

$$Y = (1+W)^{-1} B (1+W)$$

and conclude that

$$D_Z Y = \tfrac{1}{2}[Q(Z) + \lambda\alpha(Z), Y]$$

so that Y satisfies the formal Killing field equations (20) in the Z direction.

It remains to show that Y satisfies (20) in all other directions. This can be accomplished by a direct but extremely lengthy computation, however, we shall present a shorter indirect argument.

Let $\widetilde{Z}$ be some local vector field on M commuting with Z. In terms of the flat $\Omega\mathfrak{g}$-connection, $\mathcal{D} = d + \operatorname{ad} A$, the Killing field equations we wish to establish read

$$\mathcal{D}Y = 0.$$

We have $\mathcal{D}_Z Y = 0$ so that, by the flatness of $\mathcal{D}$, we get

$$\mathcal{D}_Z \mathcal{D}_{\widetilde{Z}} Y = \mathcal{D}_{\widetilde{Z}} \mathcal{D}_Z(Y) = 0.$$

Setting $\mathcal{D}_{\widetilde{Z}} Y = (1+W)^{-1}\sigma(1+W)$ this means

$$D_Z \sigma = [\omega(Z), \sigma]$$

with $\omega(Z)$ given by (21). Using $(1+W)Y(1+W)^{-1} = B$, we see that

$$\sigma = D_{\widetilde{Z}} B - [D_{\widetilde{Z}}(1+W)^{-1} + \tfrac{1}{2}(1+W)(Q(\widetilde{Z}) + \lambda\alpha'(\widetilde{Z}) + \lambda^{-1}\alpha''(\widetilde{Z}))(1+W)^{-1}, B]$$

so that, since B is D-parallel, we conclude that the coefficients of σ are sections of $V^\perp$. That $\mathcal{D}_{\widetilde{Z}} Y$ vanishes is now a consequence of the following lemma:

Lemma 2.6 *Let $\sigma = \sum_{k\geq i} \lambda^{-k}\sigma_k$ be a solution of*

$$D_Z\sigma = [\omega(Z), \sigma] \tag{23}$$

with each coefficient σ_k a section of $V^\perp$. Then σ vanishes identically.

Proof Recall that $\omega(Z) = \frac{1}{2}(\lambda B + (WQ(Z))^V)$ and observe that $WQ(Z)$ involves only negative powers of λ. If σ does not vanish then the top term σ_i is not identically zero but comparing coefficients of λ^{-i+1} in (23) gives

$$0 = \tfrac{1}{2}[B, \sigma_i]$$

so that σ_i is simultaneously a section of V and $V^\perp$ and so vanishes identically. □

Thus we conclude that $\mathcal{D}_{\widetilde{Z}}Y = 0$ whence Y is a formal Killing field. Moreover, $Y_0 = B = \alpha(Z)$ so that Y is adapted.

This establishes theorem (2.4) in the case that $G = \mathbf{U}(n)$. If G is semisimple, we choose a faithful unitary representation and thus an inclusion $G \subset \mathbf{U}(n)$. If we view $\phi : M \to G$ as a map into $\mathbf{U}(n)$ then ϕ remains pluriharmonic and semisimple adapted. Thus we have a $\mathbf{gl}(n, \mathbb{C})$-formal Killing field $\widehat{Y}$ adapted to ϕ. However, $\mathfrak{g}$ being semisimple, we can find an $\mathrm{ad}_{\mathbf{gl}(n,\mathbb{C})}\, \mathfrak{g}^{\mathbb{C}}$-invariant complement to $\mathfrak{g}^{\mathbb{C}} \subset \mathbf{gl}(n, \mathbb{C})$ projection of $\widehat{Y}$ along which produces a $\mathfrak{g}^{\mathbb{C}}$-formal Killing field adapted to ϕ. This completes the proof of (2.4).

2.4 Reconstruction and factorisation

In lemma (2.1), we saw that for a harmonic map $\phi : M \to G$ to be of finite type, it is necessary and sufficient that ϕ admit an adapted polynomial Killing field. In this section, we address the corresponding question for pluriharmonic maps into Lie groups and symmetric spaces where the issue is more subtle. Throughout this section, we take G to be semisimple.

Recall that a pluriharmonic map $\phi : \mathbb{C}^n \to G$ is of finite type if we can find a polynomial Killing field $\xi : \mathbb{C}^n \to \Omega_d$ from which we can reconstruct the derivative $\phi^*\theta = \alpha' + \alpha''$ by

$$\alpha' = 4iV_k(\xi_d)dz^k$$

for some $\mathrm{Ad}\, G^{\mathbb{C}}$-equivariant polynomial maps $V_1, \ldots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$. Moreover, when ϕ factors through a symmetric space, we further require that ξ take values in $\Lambda(\mathfrak{g}, \tau_\phi)$, d be odd and that the V_k be τ-compatible (c.f. section (1.4)). To achieve such a reconstruction, we will have to place conditions on the orbit in which $\xi_d : \mathbb{C}^n \to \mathfrak{g}^{\mathbb{C}}$ has its image (c.f. (1.7)).

In what follows, most of the technical difficulties arise in treating the symmetric space case. To avoid dealing with this case separately, we will unify the exposition by changing our viewpoint somewhat. That is, we shall regard Lie groups G as type II symmetric spaces $G \times G/\Delta G$ (c.f. [**16**]) and restrict attention to pluriharmonic maps into symmetric spaces.

So let G be a compact semisimple Lie group and $N = G/K$ be a compact symmetric space with involution τ and symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ at the basepoint $o = eK \in N$. We begin by recalling some generalities about the orbit structure of the $\mathrm{Ad}\, K^{\mathbb{C}}$-action on $\mathfrak{p}^{\mathbb{C}}$. Let $\mathbb{C}[\mathfrak{p}]$ denote the polynomial ring over $\mathfrak{p}^{\mathbb{C}}$ and let $\mathbb{C}[\mathfrak{p}]^K$ denote the ring of $\mathrm{Ad}\, K^{\mathbb{C}}$-invariant polynomials. Then

$\mathbb{C}[\mathfrak{p}]^K$ is a polynomial ring on $r = \operatorname{rank} N$ homogeneous generators, see [**17**]. In the case that N is a group manifold, this reduces to the more familiar assertion that the ring $\mathbb{C}[\mathfrak{g}]^G$ of $\operatorname{Ad} G^{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ is a polynomial ring on $\operatorname{rank} G$ generators.

Following Kostant–Rallis [**24**], we call an element $\xi \in \mathfrak{p}^{\mathbb{C}}$ *regular* if its $\operatorname{Ad} K^{\mathbb{C}}$-orbit has maximal possible dimension. Such orbits turn out to be easy to characterise, at least when ξ is semisimple (in the sense that $\operatorname{ad} \xi$ is semisimple on $\mathfrak{g}^{\mathbb{C}}$). Indeed, let $\mathcal{O} = \operatorname{Ad} K^{\mathbb{C}} \cdot \xi$ be the orbit of a regular semisimple element $\xi \in \mathfrak{p}^{\mathbb{C}}$ and let $\mathbb{C}[\mathfrak{p}]^K = \mathbb{C}[p_1, \dots, p_r]$. From [**24**] we have

(*i*) the centraliser $C_{\mathfrak{p}}(\xi)$ of ξ in $\mathfrak{p}^{\mathbb{C}}$ is a maximal abelian subspace of $\mathfrak{p}^{\mathbb{C}}$;

(*ii*) $\mathcal{O}$ is the level set of a regular value of the map $(p_1, \dots, p_r) : \mathfrak{p}^{\mathbb{C}} \to \mathbb{C}^r$.

From this, we see immediately that $C_{\mathfrak{p}}(\xi)$ (which is the Killing orthogonal complement to $T_\xi \mathcal{O} = [\xi, \mathfrak{k}^{\mathbb{C}}]$) has basis given by the gradients of the p_j at ξ.

Remark When N is a group manifold, these results describe the $\operatorname{Ad} G^{\mathbb{C}}$-orbits of regular semisimple elements of $\mathfrak{g}^{\mathbb{C}}$ and are due to Kostant [**23**].

Our analysis of pluriharmonic maps of finite type involves τ-compatible gradients of $\operatorname{Ad} G^{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ rather than $\operatorname{Ad} K^{\mathbb{C}}$-invariant polynomials on $\mathfrak{p}^{\mathbb{C}}$ and this prompts the following definition.

Definition A symmetric space has the *surjection property* if the restriction map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{p}]^K$ is surjective.

We see from the following examples that almost all symmetric spaces have this property:

(*i*) all rank one symmetric spaces have the surjection property: indeed, $\mathbb{C}[\mathfrak{p}]^K$ is generated by the restriction of the Killing form;

(*ii*) all symmetric spaces of type II (i.e. group manifolds) have the surjection property: indeed, if p is a homogeneous $\operatorname{Ad} G^{\mathbb{C}}$-invariant polynomial of degree d on $\mathfrak{g}^{\mathbb{C}}$, then p is the restriction of the $\operatorname{Ad}(G \times G)^{\mathbb{C}}$-invariant polynomial on $(\mathfrak{g} \times \mathfrak{g})^{\mathbb{C}}$ given by

$$\widehat{p}((\xi_1, \eta_1), \dots, (\xi_n, \eta_n)) = \tfrac{1}{2}(p(\xi_1, \dots, \xi_n) + (-1)^d p(\eta_1, \dots, \eta_n)).$$

(*iii*) Finally, Helgason [**15**] shows by a case by case analysis that if G and K are classical then G/K has the surjection property. He also shows that the surjection property fails for certain exceptional symmetric spaces.

We now globalise this situation by translating it around our symmetric space $N = G/K$. For this, recall from section (1.4) that we may identify TN with a subbundle $[\mathfrak{p}]$ of the trivial bundle $N \times \mathfrak{g}$ where

$$[\mathfrak{p}]_x = \mathfrak{p}_x.$$

Under this identification, the G-action on TN becomes the restriction of the adjoint action of G on $N \times \mathfrak{g}$. Let $\mathcal{O}$ be an $\operatorname{Ad} K^{\mathbb{C}}$-orbit in $\mathfrak{p}^{\mathbb{C}} = T_o N^{\mathbb{C}}$. We define the *$G$-translate* of $\mathcal{O}$, denoted $\mathcal{O}_G \subset TN^{\mathbb{C}}$, by

$$(\mathcal{O}_G)_{g \cdot o} = \operatorname{Ad} g(\mathcal{O}).$$

This is clearly well-defined since $\operatorname{Ad} K(\mathcal{O}) = \mathcal{O}$ and $\mathcal{O}_G$ is a sub-fibre-bundle of $TN^{\mathbb{C}}$ invariant under the G-action. We note that, viewing $\mathcal{O}_G$ as a subset of $N \times \mathfrak{g}^{\mathbb{C}}$, its image under projection onto $\mathfrak{g}^{\mathbb{C}}$ lies in a single $\operatorname{Ad} G^{\mathbb{C}}$-orbit.

Example If N is a Lie group G and $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ is an $\operatorname{Ad} G^{\mathbb{C}}$-orbit, then, identifying TN with $G \times \mathfrak{g}$ via the left Maurer–Cartan form, $\mathcal{O}_{G \times G}$ is just $G \times \mathcal{O}$.

Now let $f \in \mathbb{C}[\mathfrak{p}]^K$. We define a symmetric form $\widetilde{f} : TN^{\mathbb{C}} \to \mathbb{C}$ by

$$\widetilde{f}_{g \cdot o} = f \circ \operatorname{Ad} g^{-1}.$$

Again, the $\operatorname{Ad} K^{\mathbb{C}}$-invariance of f means that $\widetilde{f}$ is well-defined and G-invariant (and hence parallel for the Levi-Civita connection on N).

Example If N has rank one, then $\mathbb{C}[\mathfrak{p}]^K$ is generated by the Killing form whose extension to $TN^{\mathbb{C}}$ is just the (complexified) metric of N.

With this in hand, we reformulate the above discussion of orbit structure in

Proposition 2.7 *Let $N = G/K$ be a symmetric space of rank r, $\mathbb{C}[\mathfrak{p}]^K = \mathbb{C}[p_1, \dots, p_r]$ and $\mathcal{O} \subset \mathfrak{p}^{\mathbb{C}}$ a regular semisimple $\operatorname{Ad} K^{\mathbb{C}}$-orbit. Then*

- *(i) $\mathcal{O}_G$ is a level set of the map $(\widetilde{p}_1, \dots, \widetilde{p}_r) : TN^{\mathbb{C}} \to \mathbb{C}^r$;*
- *(ii) if, in addition, N has the surjection property, there exist $\operatorname{Ad} G^{\mathbb{C}}$-equivariant, τ-compatible polynomial maps $J_1, \dots, J_r : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that, for $\eta \in (\mathcal{O}_G)_x$, the centraliser of η in $\mathfrak{p}^{\mathbb{C}}_x$ has basis $J_1(\eta), \dots, J_r(\eta)$.*

We have now assembled what we need to discuss whether a pluriharmonic map is of finite type. So let $\phi : M \to N$ be a pluriharmonic map of a complex manifold. Suppose that ϕ admits a polynomial Killing field $\xi : M \to \Omega_d$ with d odd and $\xi(x) \in \Lambda(\mathfrak{g}, \tau_{\phi(x)})$ for all $x \in N$. (We recall from (2.3) and (2.4) that such a ξ exists if ϕ is semisimple adapted and M is compact). Since d is odd, $\xi_d : M \to \mathfrak{g}^{\mathbb{C}}$ is a section of $[\mathfrak{p}]^{\mathbb{C}}$ along ϕ and we say ξ is of *regular semisimple orbit type* if ξ_d has image in the G-translate of a single $\operatorname{Ad} K^{\mathbb{C}}$-orbit of regular semisimple elements of $\mathfrak{p}^{\mathbb{C}}$.

Theorem 2.8 *Let $M = \mathbb{C}^n/\Gamma$ be a complex torus, N a compact symmetric space with the surjection property and $\phi : M \to N$ a pluriharmonic map admitting a polynomial Killing field $\xi : M \to \Omega_d$ of regular semisimple orbit type. Then ϕ is of finite type, that is, there exist $\operatorname{Ad} G^{\mathbb{C}}$-equivariant, τ-compatible maps $V_1, \dots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that*

$$\alpha' = 4i V_k(\xi_d) dz^k$$

where $\phi^*\theta = \alpha' + \alpha''$.

Proof Comparing coefficients in the Killing field equations (13) gives

$$0 = [\xi_d, \alpha'] \tag{24}$$

$$d\xi_d = [\xi_d, \tfrac{1}{2}\alpha''] - [\xi_{d-1}, \tfrac{1}{2}\alpha']. \tag{25}$$

Since ξ is of regular semisimple orbit type and N has the surjection property, we know from (2.7) that there are $r = \operatorname{rank} N$ $\operatorname{Ad} G^{\mathbb{C}}$-equivariant, τ-compatible polynomial maps $J_1, \ldots, J_r : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that for each $x \in M$, the $\{J_k(\xi_d(x))\}$ span the centraliser of $\xi_d(x)$ in $\mathfrak{p}^{\mathbb{C}}_{\phi(x)}$. Thus, from (24) we see that

$$\alpha' = J_j(\xi_d)c^j \tag{26}$$

for $(1,0)$ forms $c^1, \ldots, c^r$ on M. We claim that the c^j are closed: indeed

$$d\alpha' = d(J_j(\xi_d)) \wedge c^j + J_j(\xi_d)dc^j.$$

Using the $\operatorname{Ad} G^{\mathbb{C}}$-equivariance of J_j we have

$$\begin{aligned} d(J_j(\xi_d)) &= (dJ_j)_{\xi_d}(d\xi_d) \\ &= (dJ_j)_{\xi_d}([\xi_d, \tfrac{1}{2}\alpha'']) - (dJ_j)_{\xi_d}([\xi_{d-1}, \tfrac{1}{2}\alpha']) \\ &= [J_j(\xi_d), \tfrac{1}{2}\alpha''] - (dJ_j)_{\xi_d}([\xi_{d-1}, \tfrac{1}{2}\alpha']). \end{aligned}$$

Now, d being odd, for $x \in M$ we have $\xi_{d-1}(x) \in \mathfrak{k}^{\mathbb{C}}_{\phi(x)}$ while α'_x takes values in the (abelian) centraliser of $\xi_d(x)$ so that

$$[\xi_{d-1}(x), \alpha'_x] \in [\mathfrak{k}^{\mathbb{C}}_{\phi(x)}, C_{\mathfrak{p}}\ (\xi_d(x))] = [\xi_d(x), \mathfrak{k}^{\mathbb{C}}_{\phi(x)}]$$

whence

$$(dJ_j)_{\xi_d(x)}([\xi_{d-1}(x), \tfrac{1}{2}\alpha'_x]) \in [J_j(\xi_d(x)), \mathfrak{k}^{\mathbb{C}}_{\phi(x)}] = [\xi_d(x), \mathfrak{k}^{\mathbb{C}}_{\phi(x)}]$$

where we have used the fact that, since $C_{\mathfrak{p}}\ (\xi_d(x))$ is abelian,

$$[\mathfrak{k}^{\mathbb{C}}_{\phi(x)}, C_{\mathfrak{p}}\ (\xi_d(x))] = [\mathfrak{k}^{\mathbb{C}}_{\phi(x)}, \xi_d(x)].$$

Thus,

$$d\alpha' = [J_j(\xi_d), \tfrac{1}{2}\alpha''] \wedge c^j - [\xi_d, \omega] + J_j(\xi_d)dc^j$$

for some $[\mathfrak{k}]^{\mathbb{C}}$-valued 1-form ω, or, using (26),

$$d\alpha' + \tfrac{1}{2}[\alpha' \wedge \alpha''] = -[\xi_d, \omega] + J_j(\xi_d)dc^j.$$

However, by theorem (1.5), the left hand side of this vanishes giving

$$[\xi_d, \omega] = J_j(\xi_d)dc^j$$

in which the right hand side takes values in $C_{\mathfrak{p}}(\xi_d)$ and the left in $\operatorname{im} \operatorname{ad} \xi_d$, which are complementary since ξ_d is semisimple. Thus $J_j(\xi_d)dc^j$ vanishes, whence, since $J_j(\xi_d)$ are linearly independent, the c^j are closed.

We remark that so far we have used no facts concerning the nature of M. Now using the fact that M is a complex torus, we conclude that the c^j are constant:

$$c^j = a^j_k dz^k,$$

for constants $a^j_k \in \mathbb{C}$, so that defining $\mathrm{Ad}\, G^{\mathbb{C}}$-equivariant τ-compatible polynomial maps $V_1, \ldots, V_n : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ by

$$V_k = \frac{1}{4i} J_j a^j_k$$

we get

$$\alpha' = 4i V_k(\xi_d) dz^k$$

as required. □

We are now in a position to state the analogue of (2.5) for pluriharmonic maps. First a definition.

Definition Let $\phi : M \to N$ be a pluriharmonic map of a complex manifold into a compact symmetric space. Say ϕ is *regular semisimple adapted* if there is a holomorphic vector field Z on M such that $\phi_* Z$ takes values in the G-translate of a single regular semisimple $\mathrm{Ad}\, K^{\mathbb{C}}$-orbit in $\mathfrak{p}^{\mathbb{C}}$.

Example In case $N = G$, this reduces to the demand that $\alpha(Z)$ take values in a single regular semisimple $\mathrm{Ad}\, G^{\mathbb{C}}$-orbit in $\mathfrak{g}^{\mathbb{C}}$.

We remark that if $\phi : M \to N$ is regular semisimple adapted then $\alpha(Z)$ takes values in a single semisimple orbit in $\mathfrak{g}^{\mathbb{C}}$ i.e. the map $M \to \widehat{N} \subset G$ is semisimple adapted so that (2.4) applies. Then an application of (2.3) provides a polynomial Killing field of regular semisimple orbit type and (2.8) gives:

Corollary 2.9 *Let $\phi : \mathbb{C}^n/\Gamma \to N$ be a regular semisimple adapted pluriharmonic map of a complex torus into a compact symmetric space with the surjection property. Then ϕ is of finite type.*

In the next section we shall see that the "generic" pluriharmonic map of a torus is regular semisimple adapted. In the meantime we conclude this section with a reformulation of (2.8) which gives a factorisation theorem for pluriharmonic maps of simply connected domains with a polynomial Killing field of regular semisimple orbit type. It is easy to see that such a map has rank not exceeding $2\,\mathrm{rank}\, N$ but in fact a stronger result is true: such a map factors holomorphically through $\mathbb{C}^r$. This result is reminiscent of those obtained by Carlson and Toledo for pluriharmonic maps into (quotients of) *negatively curved* symmetric spaces [**8**].

Theorem 2.10 *Let M be a simply connected complex manifold, N a compact symmetric space of rank r with the surjection property and $\phi : M \to N$ a pluriharmonic map admitting a polynomial Killing field of regular semisimple orbit type.*

Then there exists a holomorphic map $h : M \to \mathbb{C}^r$ and a pluriharmonic map of rank $2r$ *$\psi : \mathbb{C}^r \to N$ of finite type such that $\phi = \psi \circ h$.*

Proof The argument of (2.8) gives $\operatorname{Ad} G^{\mathbb{C}}$-equivariant, τ compatible polynomial maps $J_1, \ldots, J_r$ and closed $(1,0)$ forms $c^1, \ldots, c^r$ such that

$$\alpha' = J_j(\xi_d)c^j.$$

Fix a base point x_0 in M with $\phi(x_0) = o \in N$. Then we can integrate to get a holomorphic map $h = (h^1, \ldots, h^r) : M \to \mathbb{C}^r$ with

$$dh^j = c^j, \qquad h(x_0) = 0.$$

Now let $u^1, \ldots, u^r$ be the co-ordinates on $\mathbb{C}^r$ and $\eta : \mathbb{C}^r \to \Omega_d$ the solution to

$$\begin{gathered} d\eta = [\eta, \tfrac{1}{2}(1-\lambda)J_j(\eta_d)du^j + \tfrac{1}{2}(1-\lambda^{-1})\overline{J_j(\eta_d)}d\overline{u}^j] \\ \eta(0) = \xi(x_0). \end{gathered}$$

Then, by (1.12), there is a pluriharmonic map $\psi : \mathbb{C}^r \to \widehat{N} \in G$ satisfying

$$(\psi^*\theta)' = J_j(\eta_d)du^j, \qquad \psi(0) = e.$$

Note that ψ has rank $2r$ since the $J_j(\eta_d)$ are linearly independent at 0 and hence everywhere by (1.7).

We claim that $\phi = \psi \circ h$. Certainly these maps agree at x_0 so it suffices to show that $\phi^*\theta = (\psi \circ h)^*\theta$ (we are of course viewing ϕ as a map to $\widehat{N} \subset G$). Now

$$(\phi^*\theta)' = J_j(\xi_d)c^j$$

while

$$((\psi \circ h)^*\theta)' = h^*(J_j(\eta_d)du^j) = J_j(\eta_d \circ h)h^*du^j = J_j(\eta_d \circ h)c^j$$

so it suffices to prove that $\eta \circ h = \xi$. However,

$$d\xi = [\xi, \tfrac{1}{2}(1-\lambda)J_j(\xi_d)c^j + \tfrac{1}{2}(1-\lambda^{-1})\overline{J_j(\xi_d)}\overline{c}^j]$$

while

$$\begin{aligned} d(\eta \circ h) &= h^*d\eta = h^*[\eta, \tfrac{1}{2}(1-\lambda)J_j(\eta_d)du^j + \tfrac{1}{2}(1-\lambda^{-1})\overline{J_j(\eta_d)}d\overline{u}^j] \\ &= [\eta \circ h, \tfrac{1}{2}(1-\lambda)J_j(\eta_d \circ h)c^j + \tfrac{1}{2}(1-\lambda^{-1})\overline{J_j(\eta_d \circ h)}\overline{c}^j]. \end{aligned}$$

Moreover $\eta \circ h(x_0) = \eta(0) = \xi(x_0)$ so that $\xi = \eta \circ h$ by the uniqueness of solutions to ordinary differential equations. $\square$

3 Conclusions

In this section, we shall tie together the preceding results and, in particular, show how simple geometric conditions ensure that a (pluri)harmonic map from a complex torus is of finite type and so arises from commuting flows. Thus we obtain an explicit construction of a substantial class of pluriharmonic tori in Lie groups and symmetric spaces. As in section (2.4), we shall unify the exposition by viewing a Lie group G as a symmetric $G \times G$-space.

Recall that a harmonic map of a 2-torus into a symmetric space is of finite type if it is semisimple adapted (2.5) while a pluriharmonic map of a higher dimensional complex torus into a symmetric space with the surjection property is of finite type if it is regular semisimple adapted (2.9). Thus we need a criterion for detecting when a map is (regular) semisimple adapted. For this we use (2.7) and a familiar holomorphic differentials argument going back to [**10**]: recall that an $\operatorname{Ad} K^{\mathbb{C}}$-invariant polynomial $p : \mathfrak{p}^{\mathbb{C}} \to \mathbb{C}$ gives rise to a G-invariant (and hence covariant constant) map $\widetilde{p} : TN^{\mathbb{C}} \to \mathbb{C}$. Concerning such maps, we have

Lemma 3.1 *Let $\phi : M \to N$ be a pluriharmonic map of a complex manifold into a Riemannian manifold and let P be a covariant constant section of $S^d(T^*N^{\mathbb{C}})$. Then $(\phi^*P)^{(d,0)}$ is a holomorphic differential of degree d on M.*

Proof Let $Z_1, \dots, Z_d$ be (local) holomorphic vector fields on M. We must show that

$$\overline{\partial}(\phi^* P(Z_1, \dots, Z_d)) = 0.$$

But if ${}^{\phi}\nabla$ denotes the pull-back of the Levi-Civita connection on N then, since P is covariant constant, we have

$$\overline{\partial}(\phi^* P(Z_1, \dots, Z_d)) = \sum_{i=0}^{d} P(\phi_* Z_1, \dots, {}^{\phi}\nabla^{(0,1)} \phi_* Z_i, \dots, \phi_* Z_d)$$

which vanishes since each ${}^{\phi}\nabla^{(0,1)}\phi_* Z_i$ vanishes by the pluriharmonic map equations (11). □

We apply this to the case at hand and obtain the main result of the paper:

Theorem 3.2 *Let $\phi : \mathbb{C}^n/\Gamma \to N = G/K$ be a pluriharmonic map of a complex torus into a compact symmetric space which has the surjection property if $n > 1$.*

Let Z be a holomorphic vector field on $\mathbb{C}^n/\Gamma$ and suppose that, for some $x_0 \in \mathbb{C}^n/\Gamma$, $\phi_* Z_{x_0} \in \mathfrak{p}^{\mathbb{C}}_{\phi(x_0)}$ is regular semisimple. Then ϕ is of finite type.

Proof It suffices to show that ϕ is regular semisimple adapted. Now $\phi_* Z_{x_0}$ is an element of the G-translate $\mathcal{O}_G$ of some $\operatorname{Ad} K^{\mathbb{C}}$-orbit of regular semisimple elements. From (2.7), we see that there are $r = \operatorname{rank} N$ covariant constant homogeneous maps $(\widetilde{p}_1, \dots, \widetilde{p}_r) : TN^{\mathbb{C}} \to \mathbb{C}^r$ for which $\mathcal{O}_G$ is a common level set. From (3.1), we conclude that each $\widetilde{p}_j(\phi_* Z)$ is holomorphic on $\mathbb{C}^n/\Gamma$ and thus constant by Liouville's Theorem. Thus $\phi_* Z$ takes values in the level set of $(\widetilde{p}_1, \dots, \widetilde{p}_r)$ containing $\phi_* Z_{x_0}$ which is $\mathcal{O}_G$ whence ϕ is regular semisimple adapted as required. □

Thus, for pluriharmonic maps of complex tori, it suffices to verify the regular semisimple adapted condition at a single point. We remark that the regular semisimple elements form an open dense subset of $\mathfrak{p}^{\mathbb{C}}$ so that in some sense the regular semisimple adapted pluriharmonic tori are "generic".

Let us extract a corollary of (3.2) concerning harmonic 2-tori in rank one symmetric spaces: in this case, the $\operatorname{Ad} K^{\mathbb{C}}$-invariant polynomials are generated by the Killing form whose extension to $TN^{\mathbb{C}}$

is just the complexified metric. Moreover, $\xi \in \mathfrak{p}^{\mathbb{C}}$ is regular semisimple if and only if the Killing form is non-zero on ξ. Thus a harmonic map $\phi : T^2 \to N$ is regular semisimple adapted if and only if it is non-conformal at a single point (and hence everywhere). This gives

Corollary 3.3 *Let $\phi : T^2 \to N$ be a non-conformal harmonic map of a 2-torus into a compact rank one symmetric space. Then ϕ is of finite type.*

Taken together with the results of Calabi [**7**] and Bryant [**4**] for superminimal 2-tori in S^4 and those of Ferus–Pedit–Pinkall–Sterling [**13**] for non-superminimal minimal 2-tori in S^4, the above result accounts for all harmonic 2-tori in S^4 and thus in $S^3 = \mathbf{SU}(2)$. The Reader is invited to compare our results with those of Hitchin [**18**] concerning harmonic 2-tori in S^3.

For other rank one symmetric spaces, it remains to deal with the conformal harmonic (i.e. branched minimal) 2-tori. So far, a complete theory exists only for certain minimal tori in S^n and $\mathbb{C}P^n$. For these, the isotropic ones are dealt with by Calabi [**7**] in the S^n case and by Eells–Wood [**11**] in the $\mathbb{C}P^n$ case. Another class, characterised by the vanishing of all but one of a series of holomorphic differentials, have been dealt with in [**3**] using methods similar to our own. These latter maps correspond to doubly periodic solutions of the Toda field equations for $\mathbf{SO}(n+1)$ and $\mathbf{SU}(n+1)$. As special cases of these results, all almost complex tori in S^6 and all minimal non-isotropic 2-tori in $\mathbb{C}P^2$ are shown to be of finite type.

When the rank of N is greater then one, it is not so easy to characterise the condition on the derivative that leads to ϕ being regular semisimple adapted. However, a result of Kostant implies that, for higher rank Lie groups, there are minimal tori that are regular semisimple adapted. In fact, let $I_1, \ldots, I_r$ be homogeneous algebraically independent generators of $\mathbb{C}[\mathfrak{g}]^G$ arranged in increasing order of degree. If $\xi \in \mathfrak{g}^{\mathbb{C}}$ satisfies

$$I_1(\xi) = \cdots = I_{r-1}(\xi) = 0$$
$$I_r(\xi) \neq 0$$

then we have from [**22**, corollary 9.2] that ξ is regular semisimple. Following Kostant we call such ξ *cyclic*. If we now take I_1 to be the Killing form, we get

Corollary 3.4 *Let $\phi : T^2 \to G$ be a minimal torus in a Lie group of rank ≥ 2. If $\phi^*\theta(\partial/\partial z)$ is cyclic at a single point, it is of finite type.*

Finally, using (2.10) rather than (2.9), we get a mild generalisation of (3.2):

Theorem 3.5 *Let $\phi : M \to N$ be a pluriharmonic map of a compact complex manifold into a compact symmetric space of rank r which has the surjection property if* $\dim_{\mathbb{C}} M > 1$.

Let Z be a holomorphic vector field on M and suppose that, for some $x_0 \in M$, $\phi_* Z_{x_0} \in \mathfrak{p}^{\mathbb{C}}_{\phi(x_0)}$ is regular semisimple. Then the lift $\widetilde{\phi} : \widetilde{M} \to N$ to the universal cover of M factors as $\widetilde{\phi} = \psi \circ h$ where $h : \widetilde{M} \to \mathbb{C}^r$ is holomorphic and $\psi : \mathbb{C}^r \to N$ is a rank $2r$ pluriharmonic map of finite type.

Proof As in (3.2), we see that ϕ is regular semisimple adapted and so admits a polynomial Killing field of regular semisimple orbit type. Thus $\widetilde{\phi}$ also admits such a polynomial Killing field so that we may apply (2.10). □

We remark that under the hypotheses of (3.5), the holomorphic vector field Z is nowhere vanishing which imposes severe restrictions on the nature of M.

We conclude with another, simpler, version of (3.5) for harmonic 2-tori which shows that any regular semisimple adapted harmonic 2-torus may be extended on the universal cover to a pluriharmonic map. One may view this map as a "coherent" collection of harmonic maps of $\mathbb{R}^2$ commuting with the original one.

Theorem 3.6 *Let $\phi : \mathbb{C}/\Gamma \to N$ be a regular semisimple adapted harmonic map of a 2-torus into a compact symmetric space of rank r with the surjection property. Then there is a pluriharmonic map of finite type and rank $2r$ $\tilde{\phi} : \mathbb{C}^r = \mathbb{C} \times \mathbb{C}^{r-1} \to N$ such that*

$$\tilde{\phi}|_{\mathbb{C}} = \phi.$$

Proof From (2.5), we know that there is a map $\xi : \mathbb{C} \to \Omega_d$, for some odd d, satisfying

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1-\lambda)\xi_d]$$
$$4i\xi_d dz = \alpha'.$$

Moreover, since N has the surjection property, there are r $\operatorname{Ad} G^{\mathbb{C}}$-equivariant, τ-compatible polynomial maps $V_1, \dots, V_r : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ with $\{V_j(\xi_d)\}$ linearly independent. Without loss of generality, we may take $V_1 = \mathrm{id}_{\mathfrak{g}^{\mathbb{C}}}$. Now let $\tilde{\xi} : \mathbb{C}^r \to \Omega_d$ solve

$$\frac{\partial \tilde{\xi}}{\partial z^k} = [\tilde{\xi}, 2i(1-\lambda)V_k(\tilde{\xi}_d)]$$
$$\tilde{\xi}(0) = \xi(0).$$

Then we conclude that $\tilde{\xi}|_{\mathbb{C}} = \xi$ and take $\tilde{\phi}$ to be the pluriharmonic map arising from $\tilde{\xi}$ in (1.6). □

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