CURVES AND IMMERSIONS

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1. Introduction

Given C, an immersed circle with transverse intersections in $\mathbb{R}^2$, we propose a method of determining whether C bounds an immersed manifold with boundary, M. This procedure also identifies the genus of the manifold and the number of distinct immersions of this manifold to the given curve.

Let $\widetilde{C} = \mathbb{R}^2 \setminus C$ and let $R_1, \ldots, R_n$ be the open connected subsets of $\widetilde{C}$. We assign to each region R_i a *degree number*. The degree number can be found as follows: first pick a point on the outermost edge of C that is not a point where C intersects itself, then choose an orientation for the curve. As a convention we will always go counter-clockwise from our starting point. Assign to the region on the right of the curve degree number zero and assign to the region on the left degree number one. This means that the surface always lies on the left of the curve. Follow C all the way around in the direction of its orientation assigning a number to each region as you go along side it. Add one to a region each time C is crossed left to right and subtract one each time C is crossed right to left (See Figure 1). Since the numbering depends only on the orientation of the curve, it is obviously unique given an orientation. We say a curve cannot be numbered if, after following the algorithm above, the curve has any regions with negative degree numbers or any region has been assigned two different numbers.

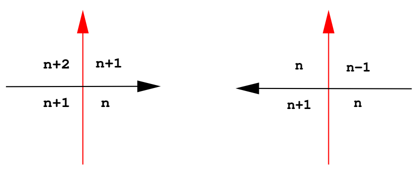

Figure 1

If C bound an immersed manifold, the degree number of any region in $\widetilde{C}$ represents the number of preimages each point in this region has on the original manifold. The reason is as follows: Since the surface which C bounds always lies on the left of C, any time the curve is crossed from left to right, you are adding a layer of surface. Similarly when the curve is crossed right to left you are losing a layer of the surface.

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Given a curve C with non-negative degree numbers, we will determine whether one can construct a graph following a detailed set of rules. If such a graph can be constructed, then C bounds an immersed manifold. If multiple graphs exist, then there are multiple immersions from the manifold to the surface cut out by C (one for each graph). By examining the Euler characteristic of the manifold we can classify the manifold by its genus.

2. Curves and Immersions

Theorem. *Let $F : M \longrightarrow \mathbb{R}^2$ be an immersion of a two-dimensional simply connected manifold M with boundary into $\mathbb{R}^2$ and suppose that $C = F(\partial M)$ is a closed curve. Then there exists a unique graph $C(G)$ which follows rules 1–3 below.*

Rule 1. The set of vertices of $G(C)$ is the disjoint union of subsets assigned to the regions $R_1, \ldots, R_n$ of $\widetilde{C}$. The number of vertices assigned to a region is equal to the degree number of that region. If a vertex is assigned to a region we say it is in that region.

Rule 2. Let R and S be two regions in $\widetilde{C}$ which share a common boundary. If R has $n+1$ vertices and S has n vertices, there is a single edge from each vertex is S to an individual vertex in R (Figure 2).

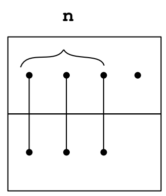

Figure 2

Rule 3a. At any intersection on the curve, each vertex must b e connected to another vertex in at least one of the adjacent regions (Figure 3).

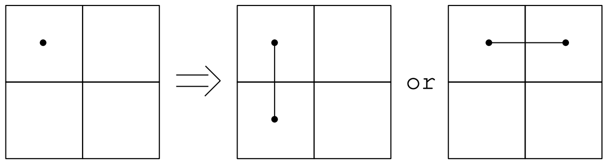

Figure 3

Rule 3b. At any intersection on the curve, if a vertex in one region is connected to vertices in the two adjacent regions, then these two vertices must be connected to a common vertex in the fourth region (Figure 4).

Proof. Given a curve C arising as $F(\partial M)$, we can construct a graph $G(C)$ as follows. The set inverse of the image of the boundary of M, $F^{-1}(F(\partial M))$, shows all points in M which are mapped by F to C. Take the dual graph

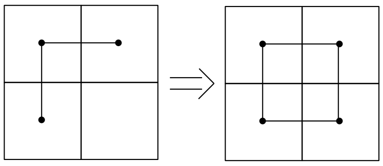

FIGURE 4

of $F^{-1}(F(\partial M))$, called Γ, which gives each distinct region one vertex and connects all adjacent regions through all possible boundaries. $F(\Gamma)$ is our graph $G(C)$ (Figure 5).

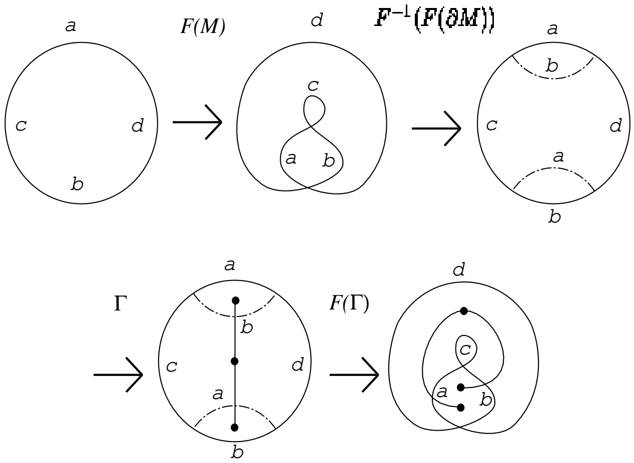

FIGURE 5

We must now verify that the graph $G(C)$ obeys the rules 1–3.

Proof of rule 1: For each region of $F^{-1}(F(\partial M))$ that maps to a given region of C , one vertex of Γ maps to one vertex of $G(C)$. If a region has n preimages on M there will be n vertices of Γ that map to that region in $\widetilde{C}$. Since the degree number of a region is the number of preimages of that region on M, this will be equal to the degree number.

Proof of rule 2: The boundary between a region of degree n and a region of degree $n+1$ has $n+1$ preimages in $F^{-1}(F(\partial M))$. Only one of these can be part of ∂M. If two segments of ∂M were mapped to the same segment of C, C would have a tangent which is not allowed. Every edge on $F^{-1}(F(\partial M))$ that is not a segment of ∂M will have an edge of Γ across it (Figure 5)

Under F each of the $n+1$ individual regions on the left side of Figure 6 are mapped to the same region of $\widetilde{C}$ and show in the right side. Here, each one of the vertices in the $n+1$ region is connected to a distinct vertex in the adjacent n region, except for the vertex originally along ∂M.

Proof of rule 3a: Each intersection on the curve corresponds to an intersection on $F^{-1}(F(\partial M))$. Consider any region cutout by $F^{-1}(F(\partial M))$.

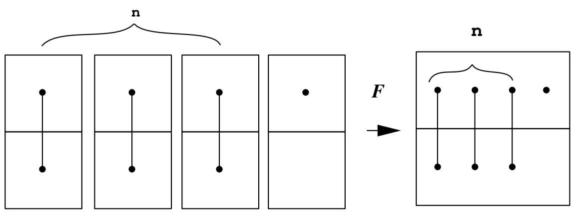

FIGURE 6

By definition the vertex of the dual graph of this region will connect across every possible boundary to the vertices of the adjacent regions. The only time the dual graph will not cross the boundary of a region is when that boundary is a segment of ∂M.

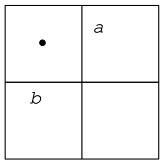

FIGURE 7

In Figure 7, since the vertex does not connect through either a or b then both a and b must be segments of ∂M. However, M is a smooth manifold and ∂M does not intersect itself, so having two segments of ∂M meeting at a point is impossible. The vertex must therefore connect through one of the two edges.

Proof of rule 3a: Since vertices B and D in Figure 8 do not connect to C both a and b are segments of ∂M. Since a and b cannot meet at a point either B or D must be connected to C.

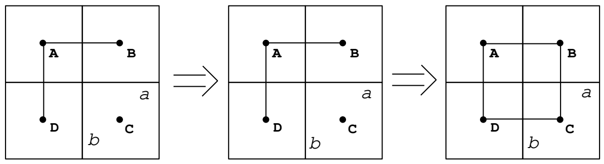

FIGURE 8

Assume without loss of generality that B is connected to C but D is not. This implies that C is tangent with itself along a. This cannot happen so D must also be connected to C. $\square$

Theorem. *Given a curve C with a connection graph $G(C)$ constructed under rules 1–3, there exists a manifold M with boundary and an immersion F of M into the plane such that $F(\partial M) = C$.*

Let C be a closed curve in $\mathbb{R}^2$, and let $G(C)$ be a graph connecting the regions of $\widetilde{C}$ that follows rules 1–3. From C and $G(C)$, we can construct a manifold M and an immersion $F : M \longrightarrow \mathbb{R}^2$, such that $F(\partial M) = C$.

Let $D_1 \ldots D_n$ be a set of N disks, one for each vertex of the graph. Each disk is a copy of the region of $\widetilde{C}$ in which its vertex is found.

Disks D_i and D_j are connected if vertices i and j are joined on the graph. If D_i to D_j are connected we consider them "glued" along the segment of their boundaries crossed by the edge of the graph connecting i and j. The manifold M is exactly this collection of disks glued as per the instructions of the graph.

We claim that M is smooth and has a smooth boundary. The boundary of M is just pieces of C glued together. In fact, since the disks are glued along C (the same way the regions of $\widetilde{C}$ are connected), locally the boundary of M is identical to C. Since C is smooth, ∂M must be smooth.

To show that M is smooth, we need only look at the intersections between these disks. Furthermore, because these disks are copies of regions of $\mathbb{R}^2$ which fit together perfectly, the manifold must be smooth except at corners where two or more disks meet. Thus we need only concern ourselves with the corners of the disks.

We eliminate the case where five of more disks meet at a common vertex by asserting that no such cases could follow all the above rules. The two ways that five disks could meet at a vertex are shown in Figures 9A and B.

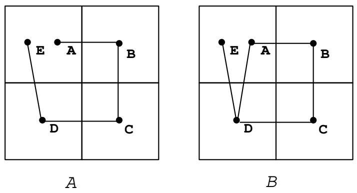

FIGURE 9

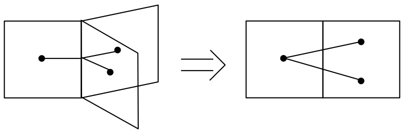

FIGURE 10

The configuration shown in Figure 9A breaks rule 3b, which stipulates that points A and D must be connected. The configuration in Figure 9B breaks rule 2, which does not allow two points in one region to connect to the same point in a third region. Rule 2 also eliminates all other cases where two disks are glued to the same third disk along the same edge (Figure 10).

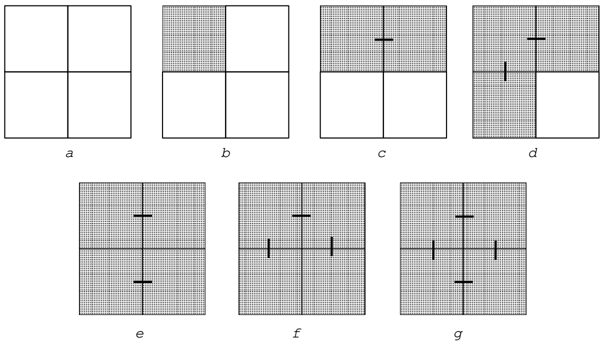

FIGURE 11

We therefore need only consider the cases where four or fewer disks meet in the plane. Figure 11 shows every possible case, where shading indicates a disk and a dash between two disks indicates that those disks are glued.

Of these, case a is trivial since there are no disks and no intersection. Case b is eliminated by rule 3a, which states that of every two adjacent edges of a disk, at least one must connect to another disk. Cases d and f are ruled out by rule 3b, which maintains that if three disks connect at an intersection, a fourth disk must also connect on two sides at the same intersection.

Case e requires further consideration. Figure 12 shows the only two possible configurations which would produce this case. Both of these imply that C is not immersed with transversal intersections.

This leaves only the cases in Figure 13. Each of these constructions is smooth; therefore we have a smooth manifold.

Having constructed this manifold, we define F as the function which takes each region of the manifold to the region of $\mathbb{R}^2$ of which it is a copy.

We must now show that $F(\partial M) = C$. Our immersion F takes every disk in M to the region of $\mathbb{R}^2$ of which it is a copy. These regions are exactly $\widetilde{C}$. The immersion F must therefore take the boundary of M to the boundary of $\widetilde{C}$, which is exactly C.

Thus, given a curve C with a graph, we can find a smooth manifold M with a smooth boundary and and immersion F, such that F takes M to $\widetilde{C}$ and $F(\partial M) = C$.

Definition. Let $F_1 : M_1 \longrightarrow \mathbb{R}^2$ and $F_2 : M_1 \longrightarrow \mathbb{R}^2$ be immersions of two 2-manifolds M_1 and M_2. We say the immersions are *equivalent* if there exists a diffeomorphism $\phi : M_1 \longrightarrow M_2$ such that $\phi \circ F_2 = F_1$.

Theorem. *Let $F_1 : M_1 \longrightarrow \mathbb{R}^2$ and $F_2 : M_2 \longrightarrow \mathbb{R}^2$ be immersion which are equivalent. Then their graphs are isomorphic.*

Proof. On M_1 we can find the set inverse of C, namely $B_1 = F_1^{-1}(F_1(\partial M))$. With the regions marked out by B_1, we can create a dual graph H_1 on M_1.

Since M_1 and M_2 are diffeomorphic, there exists a diffeomorphism $\phi : M_1 \longrightarrow M_2$ which takes M_1 to M_2. Since ϕ takes a point in M_1 to a

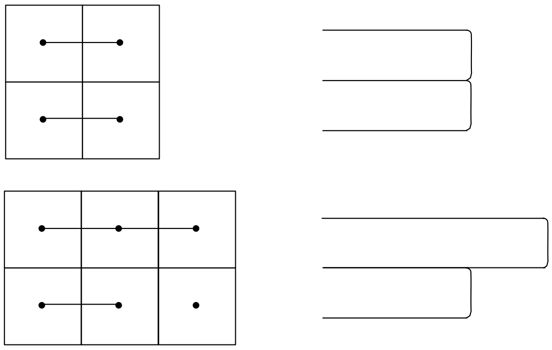

FIGURE 12

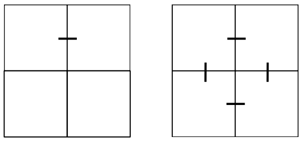

FIGURE 13

corresponding point in M_2, ϕ takes B_1 to B_2. That is,

$$\phi(H_1) = H_2.$$

It follows that

$$F_2\phi(H_1) = F_2(H_2).$$

But we have that $\phi \circ F_2 = F_1$, so

$$F_1(H_1) = F_2(H_2).$$

But $F_1(H_1) = G_1$ and $F_2(H_2) = G_2$, so $G_1 = G_2$. $\square$

3. Examples

The following section consists of examples of curves that we have constructed over the course of our research. Throughout the section we mention numerous properties of these curves, such as the number of immersions, and the genus of the manifolds. Where applicable these properties correspond to the definitions given in the preceeding pro of Although many of these can b e computed by hand, much of the data was gleaned from an invaluable computer program written by Nicholas Schmitt.

The sequence in Figure 14 occurs when you add successive loops inside the curve. Each of the curves in the top row (with an even number of crossings) bounds an immersed disks. These curves make up what we call the family of "C's" The curves in the bottom row do not bound immersed surfaces.

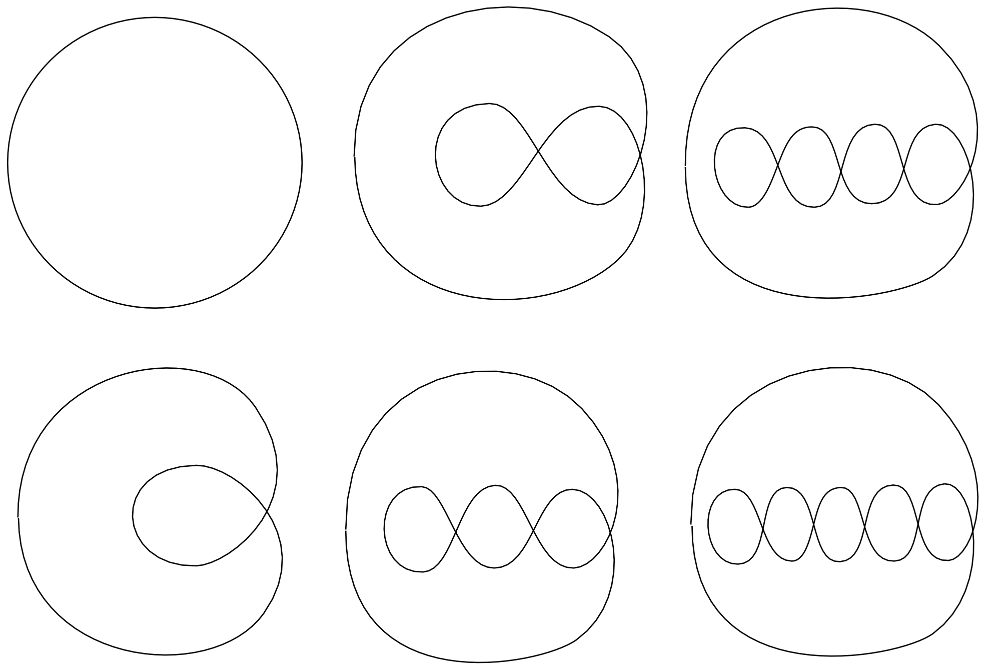

Figure 14

The sequence in Figure 15 is created when you add successive loops on the outside of the curve. The curves with an even number of crossings bound immersed manifolds on the sphere. In fact, as you proceed through the sequence the number of ways each curve bounds an immersed surface increases by approximately a factor of four. The curves with an odd number of crossing do not bound immersed surfaces.

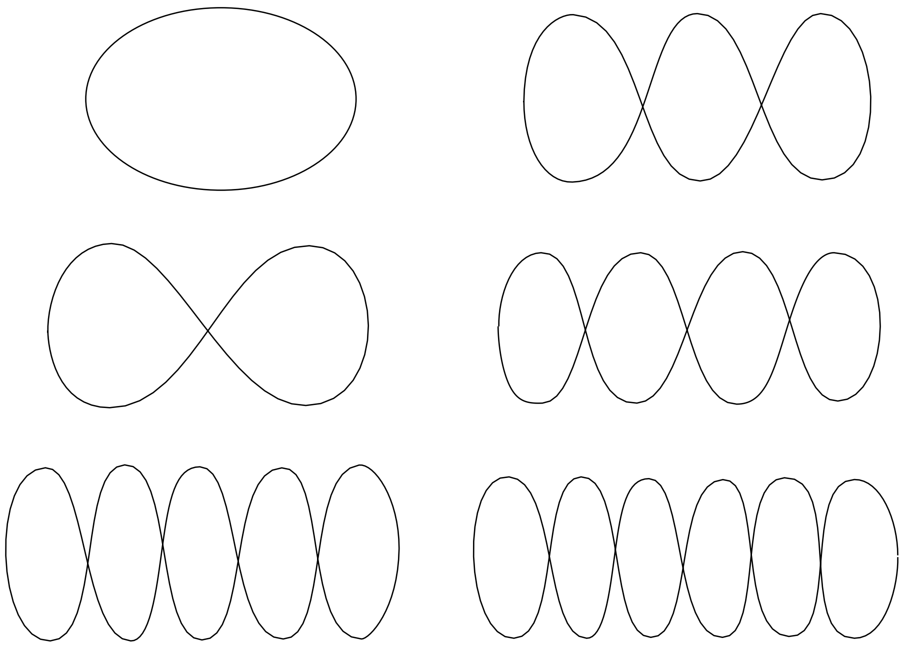

FIGURE 15

The sequence in Figure 16 takes a curve with a single inner loop and adds successive outer lo ops that (inner) loop. The curves in the top row bound nothing. The curves on the bottom bound successively higher genus manifolds. That is, the first bounds a punctured sphere (a disk), the second a punctured torus the third a punctured 2-torus. The sequence continues in this manner. If one considers the curves on the bottom with 1 added to each degree number, not only do these curves bound n-tori but also $(n-1)$-tori in multiple ways. The number of immersions a_n increases according to the recursive formula $a_n = 4a_{n-1} - 1$.

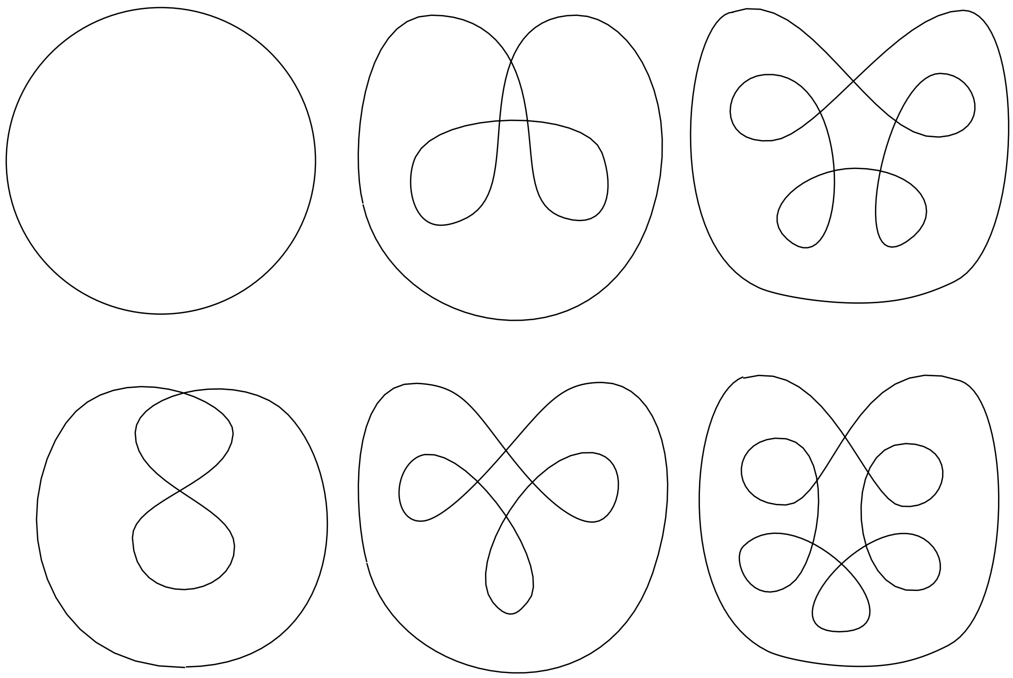

FIGURE 16

The curves in Figure 17 are equivalent (on the sphere) to the curves in Figure 16. Again, they bound n-tori in an increasing number of ways.

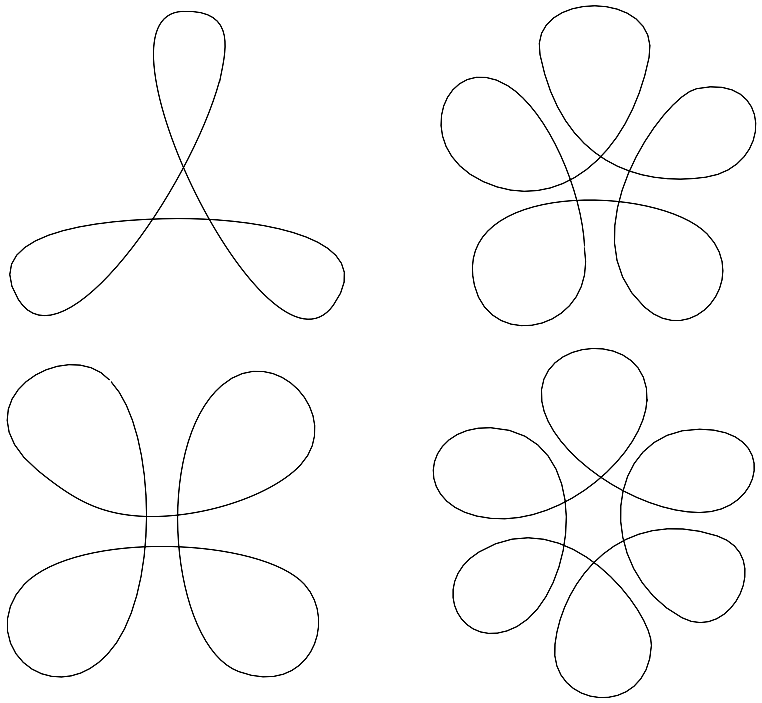

FIGURE 17

The first picture in Figure 18 is the Milnor example which bounds a disk in exactly two ways. The curve below is an extension of the Milnor example which bounds a disk in exactly three ways. If one continues this sequence as in "Extending Immersions and Regular Homotopies in Codimension 1" by Samuel Blank, one may construct a curve that bounds a disk in any number n ways as shown in Figure 19.

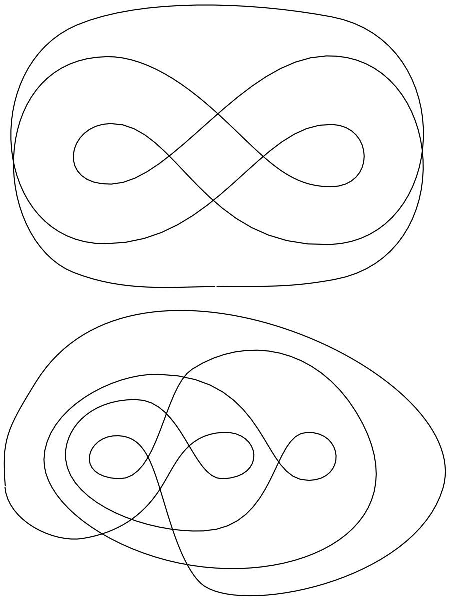

FIGURE 18

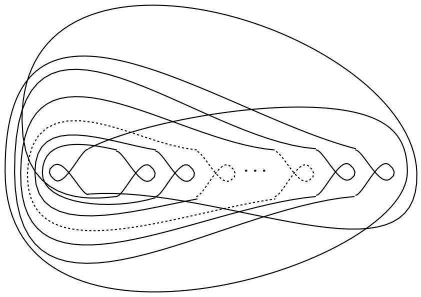

FIGURE 19

Figure 20 clearly shows the two different ways of going from the Milnor example to a disk. As expected there is no diffeomorphism from the two manifolds such that the first immersion is equal to the composition of the diffeomorphism and the second immersion (i.e. $F_1 = \phi(F_2)$).

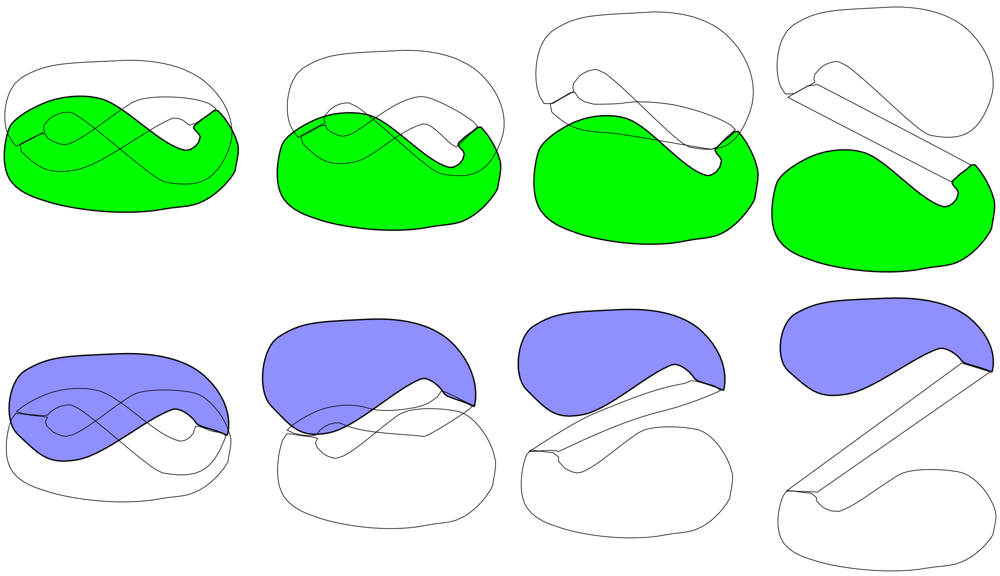

FIGURE 20

Figure 21 shows steps in a regular homotopy from an immersion of a disk (on the sphere) to the curve that we recognize as bounding a punctured torus (with a degree number added to each region). Since, on the sphere, this curve is equivalent to the Milnor example we have shown that this curve bounds a disk in two ways as well as the torus.

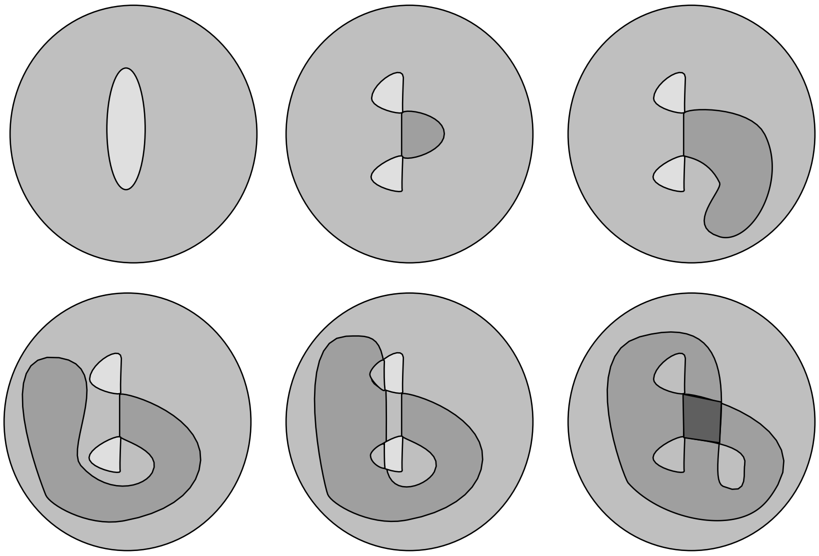

Figure 21

Figure 22 shows that the Milnor example is equivalent to the fifth curve in Figure 16.

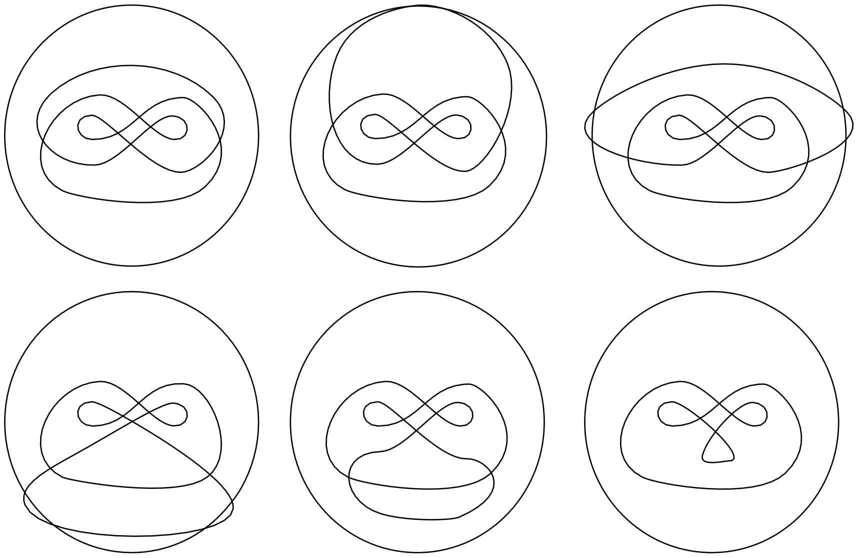

FIGURE 22

In Figure 23 we have one of the dual graphs of the Milnor example.

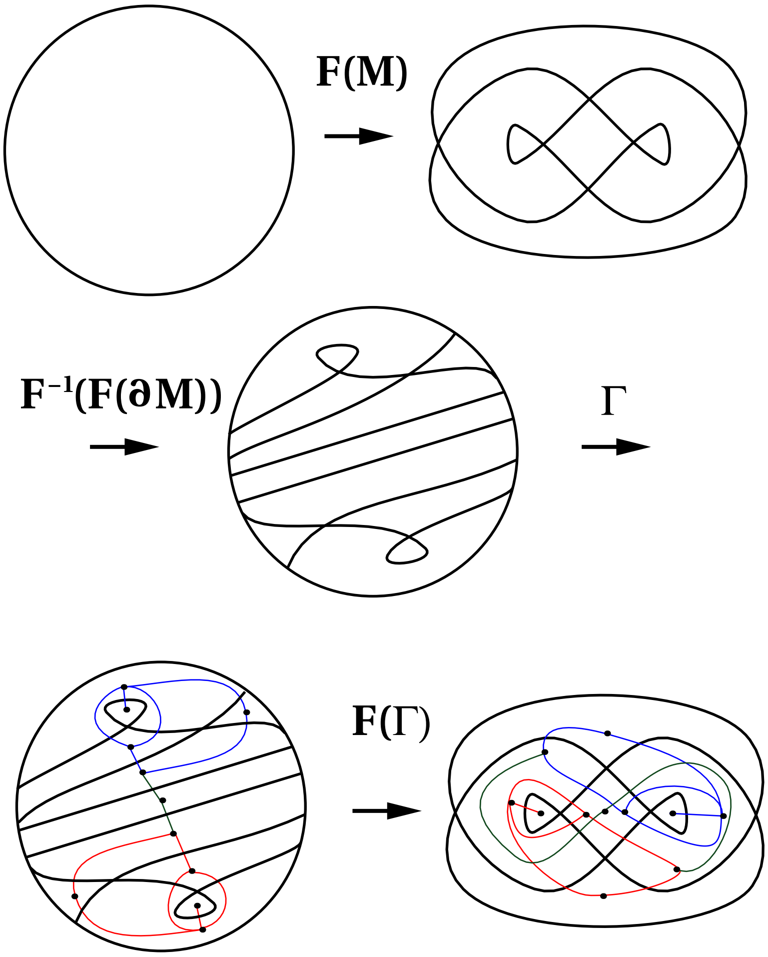

FIGURE 23

Finally, in Figure 24 we have a curve that bounds a punctured torus.

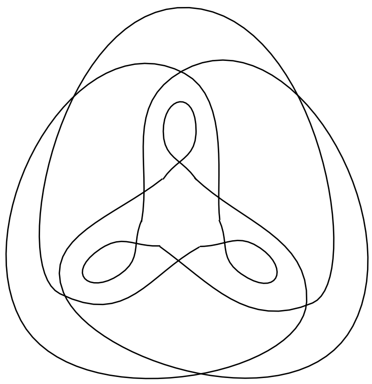

FIGURE 24