SPHERICAL POLYGONS AND UNITARIZATION

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Abstract. (1) We find a set of inequalities on n numbers $\nu_1, \ldots \nu_n \in [0, \frac{1}{2}]$ (the n-gon inequalities) which are equivalent to the existence of n-gons with ν_k as sides. (2) Interpreting ν_k as logarithms of eigenvalues, we show in two ways (by elementary analytic and geometric proofs) that the n-gon inequalities are necessary conditions for the simultaneous unitarizability of n individually unitarizable matrices in $\mathrm{SL}_2(\mathbb{C})$ whose product is I. (3) We give a necessary condition for the simultaneous unitarizability of a set of matrices in $\mathrm{SL}_2(\mathbb{C})$ in terms of the cross ratios of their eigenlines.

Introduction

A *spherical polygon* is a loop of geodesic segments on a 2-sphere, each of whose side lengths is between 0 and the semicircumference inclusively.

We prove that the side lengths of a spherical polygon satisfy the *spherical polygon inequalities*, and conversely, given any lengths which satisfy the spherical polygon inequalities, there exists a spherical polygon whose sides have these lengths (theorem 2.5). For example, on a sphere with circumference 1, the spherical triangle inequalities are

$$\begin{aligned} &\nu_1 + \nu_2 + \nu_3 \leq 1 \\ &\nu_i \leq \nu_j + \nu_k, \quad \{i,\, j,\, k\} \in \{1,\, 2,\, 3\}. \end{aligned}$$

and the spherical 4-gon inequalities are

$$\begin{aligned} &\nu_i \leq \nu_j + \nu_k + \nu_l \quad \{i,\, j,\, k,\, l\} \in \{1,\, 2,\, 3,\, 4\} \\ &\nu_i + \nu_j + \nu_k \leq \nu_l + 1 \quad \{i,\, j,\, k,\, l\} \in \{1,\, 2,\, 3,\, 4\}. \end{aligned}$$

Given n unitary matrices $M_k \in \mathrm{SU}_2$, with $M_1 \ldots M_n = \mathrm{I}$, the spherical n-gon inequalities are necessary conditions for the simultaneous unitarizability of $M_1, \ldots, M_n$ (theorem 3.8).

Conversely, if $M_1, M_2, M_3 \in \mathrm{SL}_2(\mathbb{C})$ with $M_1 M_2 M_3 = \mathrm{I}$ are individually unitarizable and irreducible (i.e. cannot be simultaneously

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conjugated to upper triangular matrices), and $\nu_k \in [0, \frac{1}{2}]$ are defined by $\cos 2\pi\nu_k = \frac{1}{2}\operatorname{tr} M_k$, then the spherical triangle inequalities imply that M_1, M_2, M_3 are simultaneously unitarizable (theorem 3.7). This converse is false for $n > 3$.

This problem has application to the construction of the moduli of constant mean curvature genus zero surfaces with Delaunay ends (k-noids).

The hyperbolic space model of this problem (section 4) is due to Rob Kusner.

1. Planar n-gon

As a preface to geodesic n-gons on $\mathbb{S}^2$, we start with n-gons in the plane $\mathbb{R}^2$. The proof of theorem 1.1 has the same flavor as the analogous theorem in the case of the sphere (theorem 2.5).

A planar n-gon is a loop of straight line segments in $\mathbb{R}^2$ with lengths in $[0, \infty)$. No further constraints are put on a planar n-gon; in particular it may be non-convex, self-intersecting, or fail to bound an immersed disk.

Let $n \geq 2$ and $\nu_1, \dots, \nu_n \in [0, \infty)$. The planar n-gon inequalities are:

$$\nu_{i_k} \leq \sum_{i \neq i_k} \nu_i, \quad k \in \{1, \dots, n\}. \tag{1.1}$$

Theorem 1.1. *Let $n \geq 2$ and $\nu_1, \dots, \nu_n \in [0, \infty)$. The following are equivalent:*

(i) *there exists an n-gon on $\mathbb{R}^2$ whose sides have lengths $\nu_1, \dots, \nu_n$.*
(ii) *$\nu_1, \dots, \nu_n$ satisfy the n-gon inequalities* (1.1)*.*

Proof. The case $n = 2$ is immediate.

That (i) implies (ii) follows from the fact that in $\mathbb{R}^2$, a straight line is the shortest path between two points.

Assume (ii). The case $n = 3$ is a standard theorem in Euclidean geometry. The proof for $n > 3$ is by induction. Suppose the theorem is true for $1, \dots, n-1$.

Let $Q_1, Q_2 \subset \{1, \dots, n\}$ with $Q_1 \sqcup Q_2 = \{1, \dots, n\}$ and $|Q_1| \geq 2$, $|Q_2| \geq 2$.

Let

$$\begin{aligned} A &= \max_{k\in Q_1}\left\{\nu_k - \textstyle\sum_{i\in Q_1\setminus\{k\}}\nu_i\right\}\\ B &= \textstyle\sum_{i\in Q_1\setminus\{k\}}\nu_i\\ C &= \max_{k\in Q_2}\left\{\nu_k - \textstyle\sum_{i\in Q_2\setminus\{k\}}\nu_i\right\}\\ D &= \textstyle\sum_{i\in Q_2\setminus\{k\}}\nu_i. \end{aligned}$$

It is immediate that $A \le B$ and $C \le D$. The n-gon inequalities on $\{\nu_1, \dots, \nu_n\}$ imply that that $A \le D$ and $B \le C$. Hence $\max\{A, B\} \le \min\{C, D\}$, so there exists α such that

$$\max\{A, B\} \le \alpha \le \min\{C, D\}.$$

By the construction of α, $\{\nu_i \,|\, i \in Q_1\} \cup \{\alpha\}$ and $\{\nu_i \,|\, i \in Q_1\} \cup \{\alpha\}$ satisfy the $(|Q_1| + 1)$-gon and $(|Q_2| - 1)$-gon inequalities respectively. Hence by the theorem for $(|Q_1|+1)$ and $(|Q_2|-1)$, there exist $(|Q_1|+1)$- and $(|Q_2| + 1)$-gons whose sides have lengths $\{\nu_i \,|\, i \in Q_1\} \cup \{\alpha\}$ and $\{\nu_i \,|\, i \in Q_1\} \cup \{\alpha\}$ respectively. Since each of the two polygons has a side of length α, they can be glued together to make an n-gon with sides with lengths $\{1, \dots, n\}$. □

Theorem 1.2. *If there exists an n-gon on $\mathbb{R}^2$ with side lengths $\nu_1, \dots, \nu_n \in [0, \infty)$, then there exists an n-gon on $\mathbb{R}^2$ with side lengths $\nu_1, \dots, \nu_n$ in any order.*

Proof. Any subsequence of k sides can be reversed by flipping the $(k+1)$-gon with these as sides, together with the diagonal connecting them These reversals generate the permutation group. □

2. Spherical n-gons

A spherical n-gon is a loop of n geodesic segments on $\mathbb{S}^2(r)$ with lengths in $[0, \pi r]$. No further constraints are put on a spherical n-gon; in particular it may be non-convex, self-intersecting, or fail to bound an immersed disk. These inequalities were found by [1].

2.1. **Spherical triangles.**

Remark 2.1. The spherical triangle inequalities are

$$\begin{aligned} \nu_1 &\le \nu_2 + \nu_3\\ \nu_2 &\le \nu_1 + \nu_3\\ \nu_3 &\le \nu_1 + \nu_2\\ \nu_1 + \nu_2 + \nu_3 &\le 1. \end{aligned} \tag{2.1}$$

Theorem 2.1 (Spherical triangle theorem)**.** *Let* $\nu_1,\ \nu_2,\ \nu_3 \in [0,\ \frac{1}{2}]$*. The following are equivalent:*

(i) $\nu_1,\ \nu_2,\ \nu_3$ *satisfy the spherical triangle inequalities* (2.1)*.*
(ii) *there exists a (possibly degenerate) spherical triangle on* $\mathbb{S}^2(r)$ *whose sides have lengths* $(2\pi r)(\nu_1,\ \nu_2,\ \nu_3)$*.*

First, two lemmas.

Lemma 2.2. *Let* $\nu_1,\ \nu_2,\ \nu_3 \in [0,\ \frac{1}{2}]$ *and let* $t_k = \cos(2\pi\nu_k)$*. Let*

$$f(t_1,\ t_2,\ t_3) = 1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3. \tag{2.2}$$

Then the following are equivalent:

(i) $\nu_1,\ \nu_2,\ \nu_3$ *satisfy the spherical triangle inequalities* (2.1)*.*
(ii) $f(t_1,\ t_2,\ t_3) \geq 0$*;*

Moreover, $f(t_1,\ t_2,\ t_3) = 0$ *iff equality holds in at least one of the inequalities* (2.1)*.*

Proof. Since $t_1,\ t_2,\ t_3$ are related to $\nu_1,\ \nu_2,\ \nu_3$ by $t_k = \cos(2\pi\nu_k)$, we can write f as a function of $\nu_1,\ \nu_2,\ \nu_3$. This expression factors as

$$f = \tfrac{1}{4}e^{2\pi ri(\nu_1+\nu_2+\nu_3)}\big(e^{2\pi ri(1-\nu_1-\nu_2-\nu_3)} - 1\big)\big(e^{2\pi ri(-\nu_1+\nu_2+\nu_3)} - 1\big)\times \\ \big(e^{2\pi ri(\nu_1-\nu_2+\nu_3)} - 1\big)\big(e^{2\pi ri(\nu_1+\nu_2-\nu_3)} - 1\big).$$

Then

$$\{f = 0\} = \{(\nu_1,\ \nu_2,\ \nu_3) \in \mathbb{R} \mid \pm\nu_1 \pm \nu_2 \pm \nu_3 \in \mathbb{Z}\}.$$

and the result follows. □

Lemma 2.3. *Let* $t_1,\ t_2,\ t_3 \in [-1,\ 1]$*. The following are equivalent:*

(i) $f(t_1,\ t_2,\ t_3) \geq 0$*;*
(ii) *There exist* $X_1,\ X_2,\ X_3 \in \mathbb{S}^2(r)$ *such that* $t_k = (X_i \cdot X_j)/r^2$*.*

Moreover, $f(t_1,\ t_2,\ t_3) = 0$ *iff* $X_1,\ X_2,\ X_3$ *are coplanar.*

Proof. First assume (ii). By a change of basis we may assume that

$$\begin{aligned} X_1 &= r(1,\ 0,\ 0) \\ X_2 &= r(x_2,\ y_2,\ 0) \\ X_3 &= r(x_3,\ y_3,\ z_3). \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned} t_1 &= x_2x_3 + y_2y_3 \\ t_2 &= x_3 \\ t_3 &= x_2. \end{aligned} \tag{2.4}$$

A calculation shows that

$$f(t_1,\ t_2,\ t_3) = y_2^2z_3^2. \tag{2.5}$$

Hence $f \geq 0$. Moreover, $f = 0$, iff either $y_2 = 0$ or $z_3 = 0$, iff X_1, X_2, X_3 are coplanar.

Conversely, assume (i). In case $t_3 = \pm 1$, then

$$f(t_1, t_2, t_3) = -(t_1 \mp t_2)^2 \geq 0,$$

so $f = 0$ and $t_1 = \pm t_2$. In this case the choice

$$\begin{aligned} X_1 &= r(1, 0, 0) \\ X_2 &= r(1, 0, 0) \\ X_3 &= r(\pm t_1, 0, 0). \end{aligned}$$

are in $\mathbb{S}^2(r)$ and satisfy (ii).

Assume then that the degenerate case $t_3^2 \neq 1$ does not occur. A calculation shows that

$$\begin{aligned} X_1 &= r(1, 0, 0) \\ X_2 &= r\left(t_3, \sqrt{1 - t_3^2}, 0\right) \\ X_3 &= r\left(t_2, \frac{t_1 - t_2 t_3}{\sqrt{1 - t_3^2}}, \frac{\sqrt{f(t_1, t_2, t_3)}}{\sqrt{1 - t_3^2}}\right) \end{aligned}$$

are in $\mathbb{S}^2(r)$ and satisfy (ii). □

Proof of the triangle theorem 2.1. Let

$$t_k = \cos(2\pi\nu_k) \tag{2.6}$$

and f as in lemma 2.2. By lemma 2.2, the triangle inequalities are equivalent to $f(t_1, t_2, t_3) \geq 0$. By lemma 2.3, this is equivalent to the existence of $X_1, X_2, X_3 \in \mathbb{S}^2(r)$ such that $t_k = (X_i \cdot X_j)/r^2$.

Given such X_k, the triangle with vertices X_k has sides with lengths $\nu_k(2\pi r)$. Conversely, fix a spherical triangle on $\mathbb{S}^2(r)$ whose sides have lengths $(2\pi r)\nu_k$, and let X_k be its vertices. Then

$$t_k = \cos(2\pi\nu_i) = \frac{X_j \cdot X_k}{r^2}.$$

Hence the existence of $X_1, X_2, X_3 \in \mathbb{S}^2(r)$ such that $t_k = (X_i \cdot X_j)/r^2$ is equivalent to the existence of a spherical triangle on $\mathbb{S}^2(r)$ whose sides have lengths $(2\pi r)\nu_k$. □

Remark 2.2. Given a spherical triangle, there exists a spherical triangle with the same side lengths but with the opposite orientation.

2.2. **The spherical n-gon inequalities.**

Definition 2.4. Let $n \geq 2$ and $\nu_1, \ldots, \nu_n \in [0, \frac{1}{2}]$. The spherical n-gon inequalities are as follows. Let $P \subseteq \{1, \ldots, n\}$ with $|P|$ odd and let $P' = \{1, \ldots, n\} \setminus P$.

$$\sum_{i \in P} \nu_i - \sum_{i \in P'} \nu_i - \frac{|P| - 1}{2} \leq 0.$$

The n-gon inequalities for $n = 2$ to $n = 6$ are listed below. Here, $(i_1, \ldots, i_n)$ ranges over the permutations of $(1, \ldots, n)$.

$n = 2$:

$$(\nu_{i_1}) - (\nu_{i_2}) \leq 0.$$

$n = 3$:

$$\begin{aligned}(\nu_{i_1}) - (\nu_{i_2} + \nu_{i_3}) &\leq 0\\ (\nu_{i_1} + \nu_{i_2} + \nu_{i_3}) &\leq 1.\end{aligned}$$

$n = 4$:

$$\begin{aligned}(\nu_{i_1}) - (\nu_{i_2} + \nu_{i_3} + \nu_{i_4}) &\leq 0\\ (\nu_{i_1} + \nu_{i_2} + \nu_{i_3}) - (\nu_{i_4}) &\leq 1.\end{aligned}$$

$n = 5$:

$$\begin{aligned}(\nu_{i_1}) - (\nu_{i_2} + \nu_{i_3} + \nu_{i_4} + \nu_{i_5}) &\leq 0\\ (\nu_{i_1} + \nu_{i_2} + \nu_{i_3}) - (\nu_{i_4} + \nu_{i_5}) &\leq 1\\ (\nu_{i_1} + \nu_{i_2} + \nu_{i_3} + \nu_{i_4} + \nu_{i_5}) &\leq 2.\end{aligned}$$

$n = 6$:

$$\begin{aligned}(\nu_{i_1}) - (\nu_{i_2} + \nu_{i_3} + \nu_{i_4} + \nu_{i_5} + \nu_{i_6}) &\leq 0\\ (\nu_{i_1} + \nu_{i_2} + \nu_{i_3}) - (\nu_{i_4} + \nu_{i_5} + \nu_{i_6}) &\leq 1\\ (\nu_{i_1} + \nu_{i_2} + \nu_{i_3} + \nu_{i_4} + \nu_{i_5}) - (\nu_{i_6}) &\leq 2.\end{aligned}$$

2.3. **The spherical n-gon theorem.**

Theorem 2.5 (Spherical n-gon theorem)**.** *Let $n \geq 2$ and $\nu_1, \ldots, \nu_n \in [0, \frac{1}{2}]$. The following are equivalent:*

(i) *there exists an n-gon on $\mathbb{S}^2(r)$ whose sides have lengths $(2\pi r)(\nu_1, \ldots, \nu_n)$;*
(ii) *$\nu_1, \ldots, \nu_n$ satisfy the n-gon inequalities* (2.4)*.*

Proof. The case $n = 2$ is immediate. The proof is by induction on n. The base case for the induction, $n = 3$, is the content of theorem 2.1.

Fix $n \geq 4$, and assume the theorem is true for $\{3, \ldots, n - 1\}$.

First assume (i). Fix $P \subseteq \{1, \ldots, n\}$ with $|P|$ odd.

Let Q_1, $Q_2 \subset \{1, \ldots, n\}$ be a partition $Q_1 \sqcup Q_2 = \{1, \ldots, n\}$ such that $|Q_1| \geq 2$, $|Q_2| \geq 2$, and Q_1, Q_2 each index consecutive pieces of the n-gon.

Let $P_1 = Q_1 \cap P$, $P_2 = Q_2 \cap P$, so that $P = P_1 \sqcup P_2$. Since $|P|$ is odd, one of $|P_1|$, $|P_2|$ is odd and the other even. Renumber if necessary so that $|P_1|$ is odd and $|P_2|$ is even. Let $P_1' = Q_1 \setminus P_1$ and $P_2' = Q_2 \setminus P_2$.

Let $(2\pi r)\nu_0$ be the length of the diagonal dividing the n-gon into the sides indexed by Q_1 and those indexed by Q_2.

Applying the theorem to the values $\{\nu_i \,|\, i \in Q_1\} \cup \{\nu_0\}$,

$$\sum_{i \in P_1} \nu_i \leq \sum_{i \in P_1{}'} \nu_i + \nu_0 + \frac{|P_1| - 1}{2}.$$

Again, applying the theorem to the values $\{\nu_i \,|\, i \in Q_2\} \cup \{\nu_0\}$,

$$\sum_{i \in P_2} \nu_i + \nu_0 \leq \sum_{i \in P_2{}'} \nu_i + \frac{|P_2|}{2}.$$

Adding,

$$\sum_{i \in P_1 \cup P_2} \nu_i \leq \sum_{i \in P_1{}' \cup P_2{}'} \nu_i + \frac{|P_1 \cup P_2| - 1}{2},$$

that is,

$$\sum_{i \in P} \nu_i \leq \sum_{i \in P'} \nu_i + \frac{|P| - 1}{2}.$$

This proves that (i) implies (ii).

Conversely, assume (ii). As before, let Q_1, $Q_2 \subset \{1, \ldots, n\}$ be a partition $Q_1 \sqcup Q_2 = \{1, \ldots, n\}$ such that $|Q_1| \geq 2$, $|Q_2| \geq 2$. Let

$$A = \max_{S_1 \subseteq Q_1,\, |S_1| \text{ odd}} \left(\sum_{i \in S_1} \nu_i - \sum_{i \in S_1{}'} \nu_i - \frac{|S_1| - 1}{2} \right)$$

$$B = \min_{S_2 \subseteq Q_2,\, |S_2| \text{ even}} \left(-\sum_{i \in S_2} \nu_i + \sum_{i \in S_2{}'} \nu_i - \frac{|S_2|}{2}. \right)$$

$$C = \max_{T_2 \subseteq Q_2,\, |T_1| \text{ odd}} \left(\sum_{i \in T_2} \nu_i - \sum_{i \in T_2{}'} \nu_i - \frac{|T_2| - 1}{2} \right)$$

$$D = \min_{T_1 \subseteq Q_1,\, |T_1| \text{ even}} \left(-\sum_{i \in T_1} \nu_i + \sum_{i \in T_1{}'} \nu_i - \frac{|T_1|}{2} \right).$$

It follows from the converse above that $A \leq B$ and $C \leq D$. We want to show that $A \leq D$ and $C \leq B$.

Let $S_1 \subseteq Q_1$ with $|S_1|$ odd, and $S_2 \subseteq Q_2$ with $|S_2|$ even. Let $S_1' = Q_1 \setminus S_1$ and $S_2' = Q_2 \setminus S_2$. Then

$$\sum_{i \in S_1 \cup S_2} \nu_i \leq \sum_{i \in {S_1}' \cup {S_2}'} \nu_i + \frac{|S_1 \cup S_2| - 1}{2},$$

from which it follows that

$$\sum_{i \in S_1} \nu_i - \sum_{i \in {S_1}'} \nu_i - \frac{|S_1| - 1}{2} \leq - \sum_{i \in S_2} \nu_i + \sum_{i \in {S_2}'} \nu_i - \frac{|S_2|}{2}.$$

Hence $A \leq D$.

Again, let $T_1 \subseteq Q_1$ with $|T_1|$ even, and $T_2 \subseteq Q_2$ with $|T_2|$ odd. Let $T_1' = Q_1 \setminus T_1$ and $T_2' = Q_2 \setminus T_2$. Then similarly,

$$\sum_{i \in T_2} \nu_i - \sum_{i \in {T_2}'} \nu_i - \frac{|T_2| - 1}{2} \leq - \sum_{i \in T_1} \nu_i + \sum_{i \in {T_1}'} \nu_i - \frac{|T_1|}{2}.$$

Hence $C \leq B$.

Hence $\max\{A,\, C\} \leq \min\{B,\, D\}$. Take $\nu_0 \in [0,\, \frac{1}{2}]$ such that

$$\max\{A,\, C\} \leq \nu_0 \leq \min\{B,\, D\}.$$

Then $\{\nu_i \,|\, i \in Q_1\} \cup \{\nu_0\}$ and $\{\nu_i \,|\, i \in Q_2\} \cup \{\nu_0\}$ satisfy the $(|Q_1|+1)-$ and $(|Q_2|+1)$-gon inequalities respectively, so there exist spherical $(|Q_1|+1)-$ and $(|Q_2|+1)$-gons, each having a side with length $2\pi r \nu_0$. These can be glued together along this side to form an n-gon whose sides have lengths $(2\pi r)(\nu_1, \ldots, \nu_n)$. □

Remark 2.3. The n-gon inequalities obtained with $|P| = 1$ also follow from the fact that each side of a spherical n-gon (whose sides have lengths at most half the circumference of the sphere) is the shortest curve connecting its endpoints.

Lemma 2.6. *Let there be a spherical n-gon on $\mathbb{S}^2(r)$ whose sides have lengths $(2\pi r)(\nu_1, \ldots, \nu_n)$ in order. Then there exists a spherical n-gon on $\mathbb{S}^2(r)$ whose sides have lengths $\nu_{i_1}, \ldots, \nu_{i_n}$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.*

Proof. Any subsequence of k sides can be reversed by flipping the $(k+1)$-gon with these as sides, together with the diagonal connecting them These reversals generate the permutation group. □

3. SU_2

3.1. **Preliminary.**

Lemma 3.1 (QR-decomposition)**.** *Let $M \in \mathrm{SL}_2(\mathbb{C})$. Then there exists a unique pair (U, T), with $U \in \mathrm{SU}_2$ and $T \in \mathrm{SL}_2(\mathbb{C})$ upper-triangular with diagonal elements in $\mathbb{R}^+$.*

Lemma 3.2. *Let $M \in \mathrm{SU}_2$. Then there exists $P \in \mathrm{SU}_2$ such that PMP^{-1} is diagonal and unitary.*

Lemma 3.3. *Let $M \in \mathrm{SU}_2$ and $P \in \mathrm{SL}_2(\mathbb{C})$. Then the following are equivalent:*

(i) $PMP^{-1} \in \mathrm{SU}_2$;
(ii) *there exists (U, C) such that $U \in \mathrm{SU}_2$ and $C \in \mathrm{SL}_2(\mathbb{C})$ with $[M, C] = 0$.*

Proof. That (ii) implies (i) is immediate.

To show that (i) implies (ii), first take the case in which M is diagonal. Let $P \in \mathrm{SL}_2(\mathbb{C})$ such that $PMP^{-1} \in \mathrm{SU}_2$. By lemma 3.1, $P = UT$ with $U \in \mathrm{SU}_2$ and T upper triangular. Then $TMT^{-1} \in \mathrm{SU}_2$. A calculation shows that this implies that T is diagonal, hence $[T, M] = 0$.

For the general case, let $P \in \mathrm{SL}_2(\mathbb{C})$ such that $PMP^{-1} \in \mathrm{SU}_2$. By lemma 3.2, there exists $B \in \mathrm{SU}_2$ such that $B^{-1}MB = D$ is diagonal. Then $PMP^{-1} = (PB)D(PB)^{-1}$. By the diagonal case above, $PB = UC$ with U unitary and C diagonal. Then $P = (UB^{-1})(BCB^{-1})$, and UB^{-1} is unitary, and $[BCB^{-1}, BDB^{-1}] = 0$. □

Lemma 3.4. *Let $M_1, M_2 \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm \mathrm{I}\}$. Then the following are equivalent:*

(i) $\operatorname{tr} M_1 = \operatorname{tr} M_2$
(ii) *M_1 and M_2 are conjugate.*

Proof. That (ii) implies (i) is immediate from the fact that $\operatorname{tr} X = \operatorname{tr} PCP^{-1}$.

To show (i) implies (ii), suppose (i). First assume M_1 and M_2 are diagonal. Then M_1 and M_2 are conjugate iff $M_2 = M_1$ or $M_2 = M_1^{-1}$. In either case, $\operatorname{tr} M_1 = \operatorname{tr} M_2$.

If M_1 and M_2 are diagonalizable, then each is conjugate to a diagonal matrix, so the above case shows $\operatorname{tr} M_1 = \operatorname{tr} M_2$.

If M_1 is not diagonalizable, then $\frac{1}{2} \operatorname{tr} M_1 = \pm 1$, so M_1 and M_2 are upper or lower diagonal. In this case they are again conjugate. □

Lemma 3.5. *Let $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm \mathrm{I}\}$. Then M is unitarizable iff $\frac{1}{2} \operatorname{tr} M \in (-1, 1)$.*

3.2. **Unitarization.**

Notation 3.6. A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ is *unitary* if $A \in \mathrm{SU}_2$.

A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ is *unitarizable* if there exists $P \in \mathrm{SL}_2(\mathbb{C})$ such that $PAP^{-1} \in \mathrm{SU}_2$. The matrix P is a *unitarizer* of A.

Matrices $A_1, \dots, A_n \in \mathrm{SL}_2(\mathbb{C})$ are *individually unitarizable* iff each A_k is unitarizable $(k = 1, \dots, n)$.

Matrices $A_1, \dots, A_n \in \mathrm{SL}_2(\mathbb{C})$ are *simultaneously unitarizable* iff there exists $P \in \mathrm{SL}_2(\mathbb{C})$ such that $PA_kP^{-1} \in \mathrm{SU}_2$ $(k = 1, \dots, n)$. The matrix P is a *unitarizer* of $A_1, \dots, A_n$.

3.3. **Spherical triangle inequalities and** SU_2**.** Theorem 3.7 gives an elementary proof of a necessary and sufficient condition for the simultaneous unitarizability of three matrices whose product is I. This condition is found in [2].

Theorem 3.7. *Let A_1, A_2, $A_3 \in \mathrm{SL}_2(\mathbb{C})$ be individually unitarizable, with $A_1A_2A_3 = \mathrm{I}$. and suppose A_1, A_2, A_3 are irreducible (i.e., cannot be simultaneously conjugated to upper triangular matrices). Let ν_k be defined by*

$$\tfrac{1}{2}\operatorname{tr} A_k = \cos 2\pi\nu_k.$$

Then the following are equivalent:

(i) *ν_1, ν_2, ν_3 satisfy the triangle inequalities* (2.1).
(ii) *A_1, A_2, A_3 are simultaneously unitarizable.*

Analytic proof. A_1, A_2, A_3 are simultaneously unitarizable iff

$$CA_1C^{-1},\ CA_2C^{-1},\ CA_3C^{-1}$$

are simultaneously unitarizable for some $C \in \mathrm{SL}_2(\mathbb{C})$. Hence we can assume without loss of generality that A_1 is diagonal and unitary.

In the degenerate case that $A_1 = \pm I$, then $A_3 = \pm A_2^{-1}$, so $t_1 = \pm 1$, $t_3 = \pm t_2$, and $f(t_1, t_2, t_3) = 0$, so in this case the theorem is true. So assume none of A_1, A_2, A_3 is $\pm I$.

A_1, A_2, A_3 are simultaneously unitarizable iff A_1, A_2 are simultaneously unitarizable iff A_2 is unitarizable by a diagonal matrix.

Let

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$$

where $\alpha = x_1 + iy_1$ and $x_1^2 + y_1^2 = 1$. Let

$$A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a = x_2 + iy_2$, $d = x_2' - iy_2$. Then

$$A_3^{-1} = \begin{pmatrix} \alpha a & \alpha b \\ \bar{\alpha} c & \bar{\alpha} d \end{pmatrix}.$$

The half traces of A_1, A_2, A_3 are

$$t_1 = x_1$$
$$t_2 = \tfrac{1}{2}(x_2 + x_2')$$
$$t_3 = \tfrac{1}{2}x_1(x_2 + x_2') - y_1 y_2 + \tfrac{1}{2}i(x_2 - x_2')y_1.$$

Since t_3 is real, $(x_2 - x_2')y_1 = 0$. But by assumption, $A_1 \neq \pm I$, so $y_1 \neq 0$, and so $x_2' = x_2$, and $d = \overline{a}$, so bc is real (since $ad - bc = 1$). And $t_3 = x_1 x_2 - y_1 y_2$, so

$$\begin{aligned} f(t_1, t_2, t_3) &= (1 - x_1^2)(1 - x_2^2) - y_1^2 y_2^2 = y_1^2(1 - x_2^2 - y_2^2) \\ &= y_1^2(1 - ad) = y_1^2(-bc). \end{aligned}$$

Thus $f \geq 0$ iff $ad \geq 1$ iff bc is nonpositive.

A_2 is diagonal iff $bc = 0$ iff $f = 0$, so from which it follows that $f = 0$ iff A_1, A_2, A_3 are reducible.

So assume $f \neq 0$. Then there is a diagonal matrix which unitarizes A_2 iff there is a solution to $|x|^4 = -\bar{c}/b$ iff $-\bar{c}/b$ is positive iff bc is negative. This proves the theorem. ☐

3.4. **Spherical n-gon inequalities and SU_2.**

Theorem 3.8. *Let $A_1, \dots, A_n \in \mathrm{SU}_2$ with $\prod A_k = \mathrm{I}$. Let ν_k be defined by*

$$\tfrac{1}{2} \operatorname{tr} A_k = \cos 2\pi\nu_k.$$

Then $\nu_1, \dots, \nu_n$ satisfy the n-gon inequalities (2.4).

Analytic proof. The case $n = 2$ is immediate. The proof is by induction on n. The base case for the induction $n = 3$ is part of theorem 3.7.

The proof for $n \geq 4$ is similar to that of theorem 2.5. Fix $n \geq 4$, and assume the theorem is true for $\{3, \dots, n-1\}$. Fix $P \subseteq \{1, \dots, n\}$ with $|P|$ odd.

Let $Q_1, Q_2 \subset \{1, \dots, n\}$ be a partition $Q_1 \sqcup Q_2 = \{1, \dots, n\}$ such that $|Q_1| \geq 2$, $|Q_2| \geq 2$, and Q_1, Q_2 each index consecutive matrices, that is, $Q_1 = \{1, \dots, m\}$, $Q_2 = \{m+1, \dots, n\}$. Let $B^{-1} = \prod_{k=1}^m A_k$, so

$$A_1 \cdot \dots \cdot A_m B = \mathrm{I}$$
$$B^{-1} A_{m+1} \cdot \dots \cdot A_n = \mathrm{I}\,.$$

Let $\nu_B \in [0, \tfrac{1}{2}]$ with

$$\tfrac{1}{2} \operatorname{tr} B = \cos 2\pi\nu_B.$$

Let $P_1 = Q_1 \cap P$, $P_2 = Q_2 \cap P$, so that $P = P_1 \sqcap P_2$. Since $|P|$ is odd, one of $|P_1|$, $|P_2|$ is odd and the other even. Renumber if necessary so that $|P_1|$ is odd and $|P_2|$ is even. Let $P_1' = Q_1 \setminus P_1$ and $P_2' = Q_2 \setminus P_2$.

Applying the theorem to the values $\{\nu_i \,|\, i \in Q_1\} \cup \{\nu_B\}$,

$$\sum_{i\in P_1} \nu_i \le \sum_{i\in {P_1}'} \nu_i + \nu_B + \frac{|P_1| - 1}{2}.$$

Again, applying the theorem to the values $\{\nu_i \,|\, i \in Q_2\} \cup \{\nu_B\}$,

$$\sum_{i\in P_2} \nu_i + \nu_B \le \sum_{i\in {P_2}'} \nu_i + \frac{|P_2|}{2}.$$

Adding,

$$\sum_{i\in P_1\cup P_2} \nu_i \le \sum_{i\in {P_1}'\cup {P_2}'} \nu_i + \frac{|P_1 \cup P_2| - 1}{2},$$

that is,

$$\sum_{i\in P} \nu_i \le \sum_{i\in P'} \nu_i + \frac{|P| - 1}{2}.$$

This proves the theorem. □

Corollary 3.9. *Let $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{C})$ with $\prod A_k = \mathrm{I}$. Suppose that $A_1, \ldots, A_n$ are simultaneously unitarizable. Let ν_k be defined by*

$$\tfrac{1}{2} \operatorname{tr} A_k = \cos 2\pi\nu_k.$$

Then $\nu_1, \ldots, \nu_n$ satisfy the n-gon inequalities (2.4).

Proof. Let P be a unitarizer of $A_1, \ldots, A_n$, and let $B_k = PA_kP^{-1}$. The trace is invariant under conjugation of a matrix, so

$$\tfrac{1}{2} \operatorname{tr} B_k = \cos 2\pi\nu_k.$$

But the B_k are unitary, so by theorem 3.8, the $\nu_1, \ldots, \nu_n$ satisfy the n-gon inequalities. □

3.5. **The axial triangle.** Any $M \in \mathrm{SU}_2$ can be written uniquely as

$$M = x\,\mathrm{I} + yA,$$

with $x \in [0,\, 1]$, $y \in [0, 1]$ and $A \in \mathrm{su}_2$. A is the *axis* of M and $x = \frac{1}{2} \operatorname{tr} M$. If $\nu \in [0,\, \frac{1}{2}]$ is defined by $x = \cos 2\pi\nu$, then M is a rotation about A by an angle of 2ν. Also,

$$M^{-1} = x\,\mathrm{I} - yA.$$

Theorem 3.10. *Let M_1, M_2, $M_3 \in \mathrm{SU}_2(\mathbb{C})$ such that $M_1M_2M_3 = \mathrm{I}$. Let A_k be the axes of M_k, $x_k = \frac{1}{2} \operatorname{tr} M_k$, and $t_k = \cos 2\pi\nu_k$. Let P_k the planes perpendicular to A_k through the center of $\mathbb{S}^2(r)$. Then the length of the sides of the triangle formed by P_{ij} with angles $\frac{1}{2} \operatorname{tr} A_iA_j$ are n_k.*

Proof. Write $M_k = x_k \mathrm{I} + y_k A_k$. Then $M_i^{-1} = M_j M_k$, so

$$x_i \mathrm{I} - y_i A_i = (x_j \mathrm{I} + y_j A_j)(x_k \mathrm{I} + y_k A_k).$$

Multiplying,

$$x_i \mathrm{I} - y_i A_i = x_j x_k \mathrm{I} + y_j x_k A_j + x_j y_k A_k + y_j y_k A_j A_k.$$

Taking the half-trace,

$$x_i = x_j x_k + y_j y_k \tfrac{1}{2} \operatorname{tr} A_j A_k.$$

This is the spherical law of cosines, hence the triangle with angles cosines $\frac{1}{2} \operatorname{tr} A_j A_k$ has side cosines x_1, x_2, x_3. □

Remark 3.1. Theorem 3.10 provides an alternate proof of theorem 3.7. For since ν_1, ν_2, ν_3 are the sides of a triangle, they satisfy the spherical triangle inequalities by theorem 2.1.

Remark 3.2. Theorem 3.10 does not extend to $n > 3$. Let $M_1, \dots, M_n \in \mathrm{SU}_2(\mathbb{C})$ such that $\prod M_i = \mathrm{I}$. Let A_k be the axes of M_k, $x_k = \frac{1}{2} \operatorname{tr} M_k$, and $t_k = \cos 2\pi\nu_k$. Let P_k the planes perpendicular to A_k through the center of $\mathbb{S}^2(r)$. Then the length of the sides of the n-gon formed by P_{ij} with angles $\frac{1}{2} \operatorname{tr} A_i A_j$ need not be n_k.

In particular, let $n = 4$ and let $M_1 M_2 M_3 M_4 = \mathrm{I}$. Then for any P which commutes with $M_1 M_2$ (and thus with $M_3 M_4$) we have that the product of the four matrices M_1, M_2, $P M_3 P^{-1}$, M_4 is I. The planes P_1, P_2 remain fixed, but the planes P_3, P_4 get rotated.

Theorem 3.10 provides the first step in an alternate inductive proof of theorem 3.8.

Geometric proof of theorem 3.8. The theorem is true for $n = 3$ by remark 3.1. Assume the theorem is true for $1, \dots, n-1$. Given $M_1, \dots, M_n \in \mathrm{SU}_2$ with $M_1 \dots M_n = \mathrm{I}$. Let $\nu_k \in [0, \frac{1}{2}]$ be defined by $\cos 2\pi\nu_k = \frac{1}{2} \operatorname{tr} M_k$. Choose $k \in 2, \dots, n-2$ and let $A^{-1} = M_1 \dots M_k$, so

$$M_1 \dots M_k A = \mathrm{I}$$
$$A^{-1} M_{k+1} \dots M_n = \mathrm{I}\,.$$

Let $\alpha \in [0, \frac{1}{2}]$ be defined by $\cos 2\pi\alpha = \frac{1}{2} \operatorname{tr} M_k$. By the induction hypothesis, $\nu_1, \dots, \nu_k$, α satisfy the spherical $(k+1)$-gon inequalities, and $\nu_{k+1}, \dots, \nu_n$, α satisfy the spherical $(n-k+1)$-gon inequalities. Hence by theorem 2.5 there exists a spherical $(k+1)$-gon P_1 with side lengths $\nu_1, \dots, \nu_k$, α, and a spherical $(n-k+1)$-gon P_2 with side lengths $\nu_{k+1}, \dots, \nu_n$, α. Since the polygons P_1 and P_2 each has a side with length α, they can be glued together along this side to form an n-gon with sides $\nu_1, \dots, \nu_n$. □

3.6. **Computing the Unitarizer.**

Lemma 3.11. *Let $M_1, \dots, M_n \in \mathrm{SL}_2(\mathbb{C})$. The following are equivalent: (1) $P \in \mathrm{SL}_2(\mathbb{C})$ simultaneously unitarizes $M_1, \dots, M_n \in \mathrm{SL}_2(\mathbb{C})$; (2) P^*P is in the kernel of the linear operator defined by*

$$XM_k - M_k^{*-1}X.$$

Remark 3.3. Thus to construct the simultaneous unitarizer of $M_1, \dots, M_n \in \mathrm{SL}_2(\mathbb{C})$, let X be a Hermitian positive-definite element in the kernel. Then X factors into $X = P^*P$, and P is a simultaneous unitarizer.

4. Simultaneous Unitarizability and Hyperbolic Space

The problem of simultaneous unitarizability can be visualized in the ball model of hyperbolic 3-space. In this model, a unitary matrix is a rotation of H^3 about an axis which is a geodesic through the center of the ball. The axes of a set of unitary matrices then all intersect at the center of the ball. Simultaneous conjugation of this set moves the axes to geodesics which do not necessarily pass through the center of the ball, but still intersect at a common point. Conversely, if the axes of a set of unitarizable matrixes intersect at a common point, then the matrices are simultaneously unitarizable (theorem 4.2).

4.1. **Hyperbolic geometry.**

First we give a known description of the group of isometries of hyperbolic space. This is constructed synthetically from the action of the Möbius group on the 2-sphere.

The Möbius group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. This action can be extended synthetically to an action on the half-space or ball in a conformally, making $\mathrm{PSL}_2(\mathbb{C})$ the group of isometries of hyperbolic 3-space.

This action is defined as follows. Let p be a point in the halfspace. There exist three hemispheres through p perpendicular to the plane, intersecting the plane in three circles. An element of $\mathrm{PSL}_2(\mathbb{C})$ takes these three circles to three circles. The three hemispheres on these circles intersect in a point q in the halfspace. The action is defined to take p to q (figure 1).

In definition 4.1, the isometries of H^3 are classified into types analogous to the rotations, translations and screw motions of $\mathbb{R}^3$.

Definition 4.1. Let $M \in \mathrm{PSL}_2(\mathbb{C}) \setminus \{\mathrm{I}\}$.

(i) M is *parabolic* iff $\frac{1}{2}\operatorname{tr} M \in \{\pm 1\}$.
(ii) M is *elliptic* iff $\frac{1}{2}\operatorname{tr} M \in (-1,\, 1)$.
(iii) M is *hyperbolic* iff $\frac{1}{2}\operatorname{tr} M \in \mathbb{R} \setminus [-1,\, 1]$.
(iv) M is *loxodromic* iff M is not parabolic or elliptic.

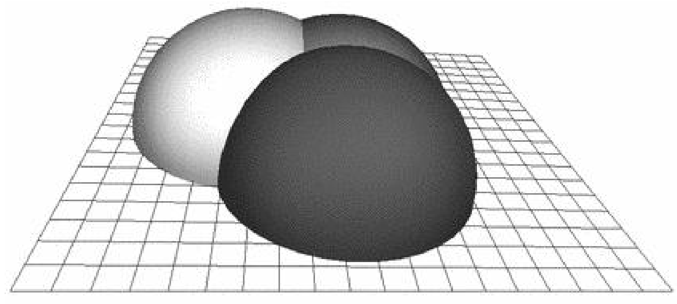

FIGURE 1. Three hemispheres intersecting at a point.

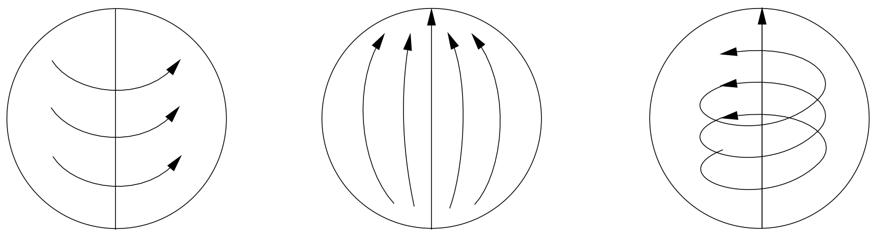

FIGURE 2. The three kinds of isometries of H^3 (ball model): elliptic, hyperbolic and loxodromic.

If $M \in \mathrm{PSL}_2(\mathbb{C}) \setminus \{\mathrm{I}\}$ is parabolic, it has one fixed point; otherwise it has two, and there is a geodesic γ joining the fixed points which is setwise fixed by M. In this case, elliptic elements are "rotations" around γ, hyperbolic elements are "translations" along γ, and elements which are neither parabolic, elliptic or hyperbolic are "screw motions" along γ.

4.2. **Simultaneous Unitarization in Hyperbolic 3-Space.** Elliptic elements of $\mathrm{PSL}_2(\mathbb{C})$ are simultaneously unitarizable iff their axes intersect.

Theorem 4.2. *Let $M_1, \ldots, M_n \in \mathrm{SL}_2(\mathbb{C})$ be elliptic elements in $\mathrm{PSL}_2(\mathbb{C})$. Let $\alpha_1, \ldots, \alpha_n$ be their axes in H^3. Then $M_1, \ldots, M_n \in \mathrm{SL}_2(\mathbb{C})$ are simultaneously unitarizable if and only if $\alpha_1, \ldots, \alpha_n$ intersect at a common point $p \in H^3$.*

Proof. For any point $p \in H^3$ there exists an isometry of H^3 taking p to 0 (ball model). But an isometry of H^3 is conjugation by $P \in \mathrm{SL}_2(\mathbb{C})$. Hence $\alpha_1, \ldots, \alpha_n$ have a common point p iff $M_1, \ldots, M_n \in \mathrm{SL}_2(\mathbb{C})$ can be simultaneously conjugated to elliptic elements of $\mathrm{PSL}_2(\mathbb{C})$ whose axes pass through 0. Such elements are precisely the unitary matrices. □

4.3. **Unitarization and cross ratios.** Theorem 4.2 reduces the unitarization problem to the problem of knowing when geodesics in H^3 intersect. If two geodesics in H^3 intersect, their endpoints must lie on a circle. This property can be measured by the reality of the cross ratio of the endpoints (lemma 4.5). However, if the endpoints lie on a circle, the geodesics do not necessarily intersect (figure 3). They intersect iff their real cross ratio is in a certain interval.

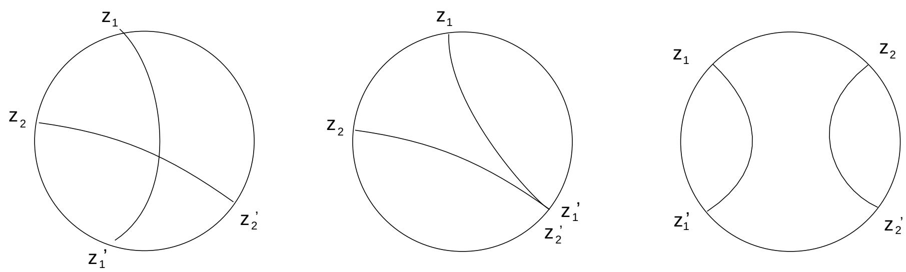

FIGURE 3. The three cases in which circle arcs intersect, meet on the boundary and do not intersect. These correspond to the cases $\phi \in (0, 1)$, $\phi \in \{0, 1\}$ and $\phi \notin [0, 1]$ respectively.

Before proving these results, some elementary properties of the cross ratio are listed.

There are six ways to define the cross ratio. We choose the unique permutation for which $[0, 1, \infty, z] = z$.

Definition 4.3. Let $z_1, z_2, z_3, z_4 \in \mathbb{P}^1$. The *cross ratio* is

$$[z_1, z_2, z_3, z_4] = \frac{(z_4 - z_1)}{(z_2 - z_1)}\frac{(z_2 - z_3)}{(z_4 - z_3)}$$

Remark 4.1. Special cases of the cross ratio: 1. If z_1, z_2, z_3, z_4 are distinct, then

$$\begin{aligned}
[\infty, z_2, z_3, z_4] &= \frac{z_2 - z_3}{z_4 - z_3} \\
[z_1, \infty, z_3, z_4] &= \frac{z_4 - z_1}{z_4 - z_3} \\
[z_1, z_2, \infty, z_4] &= \frac{z_4 - z_1}{z_2 - z_1} \\
[z_1, z_2, z_3, \infty] &= \frac{z_2 - z_3}{z_2 - z_1}.
\end{aligned}$$

2. If no three of z_1, z_2, z_3, z_4 are equal, then

$$\begin{aligned}[z_1, z, z, z_4] &= [z, z_2, z_3, z] = 0\\ [z_1, z, z_3, z] &= [z, z_2, z, z_4] = 1\\ [z_1, z_2, z, z] &= [z, z, z_3, z_4] = \infty\end{aligned}$$

3. If three of z_1, z_2, z_3, z_4 are equal, then the cross ratio $[z_1, z_2, z_3, z_4]$ is undefined.

Lemma 4.4 (Symmetries of the cross ratio)**.** *Let z_1, z_2, z_3, $z_4 \in \mathbb{P}^1$ with no three equal. Let*

$$\phi = [z_1, z_2, z_3, z_4].$$

Then

$$\begin{aligned}\phi &= [z_1, z_2, z_3, z_4] = [z_2, z_1, z_4, z_3] = [z_3, z_4, z_1, z_2]\\ 1 - \phi &= [z_2, z_1, z_3, z_4]\\ 1/\phi &= [z_3, z_2, z_1, z_4]\end{aligned}$$

Lemma 4.5. *Let z_1, z_2, z_3, $z_4 \in \mathbb{P}^1$ with $z_1 \neq z_2$ and $z_3 \neq z_4$. Suppose that $\phi = [z_1, z_2, z_3, z_4] \in \mathbb{R}$, so z_1, z_2, z_3, z_4 lie on a circle or straight line c. Let α, β be the circles from z_1 to z_2 and from z_3 to z_4 respectively which are perpendicular to c. Then*

(i) *α, β intersect off c iff $\phi \in (0, 1)$.*
(ii) *α, β intersect on c iff $\phi \in \{0, 1\}$.*

Proof. The proof is by cases, using the symmetries of the cross ratio. □

Theorems 4.6–4.7 brings together the previous lemmas to give a necessary and sufficient condition for the simultaneous unitarizability of 2 and 3 unitarizable matrices.

Theorem 4.6. *Let M_1, $M_2 \in \mathrm{SL}_2(\mathbb{C})$ be individually unitarizable, and suppose $[M_1, M_2] \neq 0$. Let z_k, z'_k be the eigenlines of M_k and suppose z_1, z'_1, z_2, z'_2 are distinct. Let $\phi = [z_1, z'_2, z_1, z'_2]$. M_1, M_2 are simultaneously unitarizable iff $\phi \in (0, 1)$.*

Proof. Since M_1, M_2 are individually unitarizable, they are elliptic elements of $\mathrm{PSL}_2(\mathbb{C})$. Let α_1, α_2 be their axes in H^3. By lemma 4.5, α_1, α_2 intersect at a point $p \in H^3$ iff $\phi \in (0, 1)$. By lemma 4.2, M_1, M_2 are simultaneously unitarizable. □

The following theorem provides a criterion for the simultaneous unitarizability of three matrices.

Theorem 4.7. *Let M_1, M_2, $M_3 \in \mathrm{SL}_2(\mathbb{C})$ be individually unitarizable, and suppose $[M_i,\, M_j] \neq 0$. Let z_k, z'_k be the eigenlines of M_k and suppose $z_1,\, z'_1, \ldots, z_3,\, z'_3$ are distinct. Let $\phi_{ij} = [z_i,\, z'_i,\, z_j,\, z'_j]$. If (1) $\phi_{ij} \in (0,\, 1)$, and (2) $z_1,\, z'_1,\, z_2,\, z'_2,\, z_3,\, z'_3$ do not lie on a circle, then M_1, M_2, M_3 are simultaneously unitarizable.*

Proof. Since M_k are individually unitarizable, they are elliptic elements of $\mathrm{PSL}_2(\mathbb{C})$. Let α_k be their respective axes in H^3. Condition (1) insures that the α_k intersect pairwise. Condition (2) insures that the α_k do not lie in a common geodesic hemisphere. Since $\phi_{12} \in \mathbb{R}$, α_1, α_2 lie in a common hemisphere Σ and intersect at a point $p \in \Sigma$. But α_3 does not lie in Σ, so α_3 intersects Σ in at most one point. But α_3 intersects both α_1 and α_2. intersects both α_1 and α_2 at p. Hence α_1, α_2, α_3 have a common intersection point. By lemma 4.2, M_1, M_2 are simultaneously unitarizable. □

4.4. **Simultaneous Unitarization in Hyperbolic 3-Space.** The following theorem provides a criterion for the simultaneous unitarizability of n matrices in terms of certain triples.

Theorem 4.8. *Let $M_1, \ldots, M_n \in \mathrm{SL}_2(\mathbb{C})$ be individually unitarizable. Let $\mathcal{T}$ graph of each of whose nodes is a triple of numbers taken from the set $\{1, \ldots, n\}$, and such that each number in $\{1, \ldots, n\}$ is in at least one element of $\mathcal{T}$. The nodes of $\mathcal{T}$ are connected which have two numbers in common. Suppose*

(i) *$\mathcal{T}$ is a connected graph;*
(ii) *for each node $(i,\, j,\, k) \in \mathcal{T}$, the matrices $(M_i,\, M_j,\, M_k)$ are simultaneously unitarizable;*
(iii) *for any pair of connected nodes $(i,\, j,\, k)$ and $(j,\, k,\, l) \in \mathcal{T}$, $[M_j,\, M_k] \neq 0$.*

Then $M_1, \ldots, M_n$ are simultaneously unitarizable.

Proof. For each $k \in (1, \ldots, n)$, let α_k be the axes of M_k. The axes are distinct, since the matrices do not pairwise commute. We prove by induction that the axes $\alpha_1, \ldots, \alpha_n$ intersect at a point p. It then follows by theorem 4.2 that $M_1, \ldots, M_n$ are simultaneously unitarizable.

Let $t_1, \ldots, t_S$ be the nodes of $\mathcal{T}$ arranged so that for each R, t_R is connected to at least one of $t_1, \ldots, t_{R-1}$. For the base case of the induction, let $t_1 = (i,\, j,\, k)$. Since by hypothesis, $(M_i,\, M_j,\, M_k)$ are simultaneously unitarizable, by theorem 4.2, $\alpha_i \cap \alpha_j \cap \alpha_k \neq \emptyset$. Since α_i, α_j, α_k are distinct geodesics in H^3, $\alpha_i \cap \alpha_j \cap \alpha_k$ is a single point p.

Now suppose that for all the numbers i in $t_1, \ldots, t_R$, we have that $p \in \alpha_i$. Since t_{R+1} is connected to some t_J $(1 \leq J \leq R)$, at least two of the numbers in t_R is in one of $t_1, \ldots, t_R$. Let $t_{R+1} = (i,\, j,\, k)$ and suppose i and j are in one of $t_1, \ldots, t_R$. We want to show that $p \in \alpha_k$. Since by hypothesis, (M_i, M_j, M_k) are simultaneously unitarizable, by theorem 4.2, $\alpha_i \cap \alpha_j \cap \alpha_k \neq \emptyset$. Since α_i and α_j are distinct geodesics in H^3, their intersection is a single point, which is p. Therefore $p \in \alpha_k$. □

Corollary 4.9. *Let $M_1, \ldots, M_n \in \mathrm{SL}_2(\mathbb{C})$ be individually unitarizable, and suppose $[M_i, M_{i+1}] \neq 0$ for $2 \leq i \leq n-1$. If the triples*

$$(M_1, M_2, M_3),\ (M_2, M_3, M_4), \ldots, (M_{n-2}, M_{n-1}, M_n)$$

are each simultaneously unitarizable, then $M_1, \ldots, M_n$ are simultaneously unitarizable.

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