CONSTRUCTING CMC TRINOIDS

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ABSTRACT. Constructing constant mean curvature trinoids via DPW. GANG Seminar lecture notes, Spring 2000. Work in progress.

CONTENTS

1. **Introduction**

The CMC trinoid construction has two parts: solving the monodromy problem, and finding potentials with appropriate eigenvalues.

The monodromy problem is solved using known results on representations of the free group with two generators:

1. A choice of three half-traces determines an SL$(2, \mathbb{C})$ representation up to conjugacy class (away from boundary cases) (proposition 2.2).

2. An SL$(2, \mathbb{C})$ representation is conjugate to an SU(2) representation if and only if the half-traces are in $[-1, 1]$ and satisfy a certain polynomial inequality (away from boundary cases) (proposition 2.3).

To construct examples, potentials are produced for which at each end (possibly after a gauge) the residue matrices has "Delaunay eigenvalues"

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(12), and for which the traces of the monodromies satisfy the above-mentioned polynomial inequality for every value of the loop parameter λ.

FIGURE 1. CMC trinoids.

2. Monodromy on the thrice-punctured sphere

2.1. $\mathrm{SL}(2,\,\mathbb{C})$ **and** $\mathrm{SU}(2)$ **representations.** How many monodromy representations on the thrice-punctured sphere are there in each conjugacy class? Counting complex dimensions,

$$\begin{array}{rl} 6: & \text{dimension of } \mathrm{Hom}(\pi_1,\,\mathrm{SL}(2,\,\mathbb{C})) \\ -3: & \text{modulo adjoint action} \\ -3: & \text{specify three traces} \\ \hline 0: & \text{total} \end{array}$$

This indicates that a choice of three traces determine the monodromy representation modulo adjoint action. This is indeed the case and is the content of the following theorem, which is well-known (see [2, 3, 1]) and was essentially known classically.

Throughout the following, an impulse to abuse notation is indulged whereby a conjugacy class $[\rho]$ is conflated with its representative ρ.

Theorem 2.1. *Let* Π *denote the free group with two generators. Let*

$$\langle \gamma_1,\,\gamma_2,\,\gamma_3 \,|\, \gamma_1\gamma_2\gamma_3 = 1\rangle \tag{1}$$

be a fixed presentation of Π. *Let*

$$V_{\mathrm{SL}} := \mathrm{Hom}(\Pi,\,\mathrm{SL}(2,\,\mathbb{C}))/\mathrm{Ad}$$
$$V_{\mathrm{SU}} := \mathrm{Hom}(\Pi,\,\mathrm{SU}(2))/\mathrm{Ad},$$

where the quotients indicate modulo the adjoint actions of $\mathrm{SL}(2,\,\mathbb{C})$ *and* $\mathrm{SU}(2)$ *respectively. Let*

$$\iota : V_{\mathrm{SU}} \longrightarrow V_{\mathrm{SL}}$$

be the canonical injection. Let $\chi : V_{\mathrm{SL}} \longrightarrow \mathbb{C}^3$, *non-canonically depending on the choice* (1), *be defined by*

$$\chi(\rho) := \tfrac{1}{2}\left(\mathrm{tr}\,\rho(\gamma_1),\,\mathrm{tr}\,\rho(\gamma_2),\,\mathrm{tr}\,\rho(\gamma_3)\right).$$

Let $\kappa : \mathbb{C}^3 \longrightarrow \mathbb{C}$ *be defined by*

$$\kappa(t_1, t_2, t_3) := 2(t_1^2 + t_2^2 + t_3^2) - 4t_1t_2t_3 - 1.$$

Let $\rho \in V_{\mathrm{SL}}$*. Then* $\rho \in \iota(V_{\mathrm{SU}})$ *if and only if either*

$$\chi(\rho) \in [-1, 1]^3 \text{ and } \kappa(\chi(\rho)) < 1 \tag{2}$$

or

$$\kappa(\chi(\rho)) = 1 \text{ and } \rho \text{ is diagonalizable.}$$

For $X, Y \in \mathrm{SL}(2, \mathbb{C})$, the polynomial κ is the trace of the commutator $XYX^{-1}Y^{-1}$ in terms of the traces of X, Y and XY:

$$\kappa(\chi(\gamma_1, \gamma_2, \gamma_3)) = \mathrm{tr}\left(\rho(\gamma_1)\rho(\gamma_2)\rho^{-1}(\gamma_1)\rho^{-1}(\gamma_2)\right).$$

The boundary $\kappa = 1$ is shown in figure 2. $\kappa < 1$ is the finite region bounded by the pillow. Contrary to what the low-resolution picture suggests, the pillow is smooth except at the four points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$.

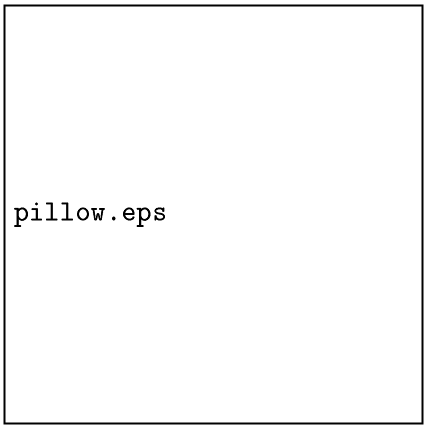

FIGURE 2. The pillow $\kappa = 1$.

Theorem 2.1 follows from the following two propositions.

Proposition 2.2. *Let* Π, γ_1, γ_2, γ_3, V_{SL}, χ, κ *and* $\mathbb{C}^3$ *be as in theorem 2.1. Let* $u \in \mathbb{C}^3$*. Then:*

(i) *if* $\kappa(u) \neq 1$*, then* $\chi^{-1}(u)$ *is a single irreducible element;*
(ii) *if* $\kappa(u) = 1$*, then every element of* $\chi^{-1}(u)$ *is reducible.*

In the case $\kappa(u) = 1$ of proposition 2.2, for some basis of $\mathbb{C}^2$ ρ is upper-triangular:

$$\rho(\gamma_k) = \begin{pmatrix} a_k & b_k \\ 0 & a_k^{-1} \end{pmatrix} \quad (k = 1, 2, 3).$$

The ρ which is diagonalizable ($b_k = 0$) is a non-Hausdorff point of V_{SL}.

Proposition 2.3. *Let* Π, γ_1, γ_2, γ_3, V_{SL}, V_{SU}, ι, χ *and* κ *be as in theorem 2.1. Then*

(i)
$$\chi\iota : V_{\mathrm{SU}} \longrightarrow \big\{ u \in [-1,\, 1]^3 \,\big|\, \kappa(u) \le 1 \big\}$$
is a bijection;

(ii) *this bijection identifies the reducible representations in* V_{SU} *with*
$$\big\{ u \in [-1,\, 1]^3 \,\big|\, \kappa(u) = 1 \big\}\,.$$

2.2. **Eigenvalues of** $\mathrm{SU}(2)$ **representations.** In our application of theorem 2.1, the eigenvalues of the monodromy are $\exp(\pm 2\pi i \mu(\lambda))$, where $\pm\mu(\lambda)$ are the eigenvalues of the residue matrix A_λ. The half-trace of the monodromy is then

$$\cos 2\pi\mu(\lambda). \tag{3}$$

The polynomial κ of theorem 2.1 becomes linear when it is interpreted geometrically in terms of μ:

Proposition 2.4. *Let*
$$t_k = \cos(2\pi\mu_k), \quad k = 1,\, 2,\, 3.$$
The condition
$$\kappa(t_1,\, t_2,\, t_3) = 1 \tag{4}$$
is equivalent to
$$\pm\mu_1 \pm \mu_2 \pm \mu_3 \in \mathbb{Z}. \tag{5}$$

Proof. By an elementary calculation,
$$\kappa(t_1,\, t_2,\, t_3) - 1 = -\tfrac{1}{2} e^{-2\pi i(\mu_1+\mu_2+\mu_3)} \left(e^{2\pi i(\mu_1+\mu_2+\mu_3)} - 1\right) \times \\ \left(e^{2\pi i(-\mu_1+\mu_2+\mu_3)} - 1\right) \left(e^{2\pi i(\mu_1-\mu_2+\mu_3)} - 1\right) \left(e^{2\pi i(\mu_1+\mu_2-\mu_3)} - 1\right).$$
The result follows from the fact that this is zero if and only if one of the exponents in the last four factors is in $2\pi i\mathbb{Z}$. ☐

The set of points $(\mu_1,\, \mu_2,\, \mu_3) \in \mathbb{R}^3$ satisfying (5) consists of four families of parallel planes with respective normals $(1,\, 1,\, 1)$, $(1,\, 1,\, -1)$, $(1,\, -1,\, 1)$, $(-1,\, 1,\, 1)$. The planes in each family are one unit apart as measured along the coordinate axes. The planes partition $\mathbb{R}^3$ into cells, and inequality (2) is satisfied by triples $(\mu_1,\, \mu_2,\, \mu_3) \in \mathbb{R}^3$ in every other cell; that is, the cells in which it is satisfied are connected by paths that pass through even numbers of planes.

These planes partition of $\mathbb{R}^3$ into regular tetrahedrons and octahedrons; (2) is satisfied in and only in the tetrahedrons. Four polyhedra meet at each edge and fourteen at each vertex as shown in figure 3.

FIGURE 3. Part of the tiling of $\mathbb{R}^3$ by tetrahedra and octahedra.

3. Delaunay ends

3.1. Monodromy at simple poles.

Theorem 3.1 (The "$z^A P$" theorem). *Let $\xi_\lambda \in \Lambda^\sigma \mathrm{sl}(2, \mathbb{C})$ be meromorphic $\mathrm{sl}_2(\mathbb{C})$-valued one-form on a Riemann surface M near $p \in M$, defining a flat connection*

$$d\Phi_\lambda = \Phi_\lambda \xi_\lambda, \tag{6}$$

with Φ_λ and ξ_λ depending on a loop parameter $\lambda \in \mathrm{SU}(1)$. If ξ_λ has a simple pole at $z = z_0$, the local behavior of Φ_λ is, under suitable conditions, as follows:

$$\begin{aligned}
&\Phi_\lambda(z) = B_\lambda E_\lambda(z) P_\lambda(z)\\
&B_\lambda \text{ is } z\text{-independent}\\
&E_\lambda(z) = \exp(\log(z - z_0) A_\lambda)\\
&A_\lambda = \mathrm{res}_{z_0} \xi_\lambda\\
&P_\lambda(z_0) = I.
\end{aligned}$$

For a proof, see [4].

Since $A_\lambda \in \Lambda^\sigma \mathrm{su}(2)$, its eigenvalues $\pm\mu(\lambda)$ sum to 0. If the conditions of theorem 3.1 are satisfied, the monodromy around a curve which encloses z_0 and no other pole of ξ_λ is conjugate to $\exp(2\pi i A_\lambda)$ and has eigenvalues $\exp(\pm 2\pi i \mu(\lambda))$.

The "monodromy problem" for CMC surfaces is to find appropriate initial condition that closes the ends.

Theorem 3.2. *For any potential, conditions* (7) *and* (8) *on the monodromy M guarantee that the surface closes:*

$$\begin{aligned}
&M|_{\theta=0} = (-1)^\nu I, \quad m \in \mathbb{Z}\\
&\left.\frac{\partial M}{\partial \theta}\right|_{\theta=0} = 0
\end{aligned} \tag{7}$$

and

$$M(\theta) \in \Lambda^\sigma \mathrm{SU}(2) \text{ for all } \lambda. \tag{8}$$

At a simple pole satisfying the conditions of theorem 3.1, these conditions can be translated to the eigenvalues $\pm\mu(\lambda)$ of the residue matrix A_λ.

Let $\pm\mu$ be the eigenvalues of A_λ. The surface closes at $z = 0$ if

$$\begin{aligned} &\mu|_{\theta=0} = m/2, \quad m \in \mathbb{Z} \\ &\left.\frac{\partial\mu}{\partial\theta}\right|_{\theta=0} = 0 \end{aligned} \tag{9}$$

The integer $\nu = 2\mu|_{\theta=0}$ is the end's wrapping number and the end is embedded if and only if $\nu = \pm 1$.

The monodromy M_λ at a simple pole is conjugate to a matrix in $\Lambda^\sigma\mathrm{SU}(2)$ if and only its eigenvalues are in $\mathrm{SU}(1)$, if and only if

$$\mu(\lambda) \in \mathbb{R} \text{ for all } \lambda \in \mathrm{SU}(1). \tag{10}$$

3.2. **The Delaunay base case.** Constructing CMC surfaces whose ends are asymptotic to Delaunay surfaces entails understanding the Delaunay base case of which the k-noids are local perturbations.

With Riemann surface M, point $p \in M$ and ξ_λ as above, assume ξ_λ at p has a simple pole, with principle part

$$A_\lambda \frac{dz}{z} \tag{11}$$

where A_λ is z-independent.

Proposition 3.3. *Let* $A_\lambda \in \Lambda^\sigma\mathrm{sl}(2, \mathbb{C})$ *and assume that the eigenvalues of* A_λ *satisfy* (9) *and* (10), *and that* $\det(A)$ *has a finite series in* λ. *Then*

$$\mu(\lambda) = \pm\frac{1}{2}\sqrt{\nu^2 - \omega\sin^2\theta}, \quad \lambda = e^{i\theta} \tag{12}$$

for $\nu \in \mathbb{Z}$ *and* $\omega \in (-\infty, 1]$.

Proof. The finiteness condition implies that $\mu(\lambda)$ must be of the form

$$\pm\sqrt{c_0 + c_2\lambda^2 + \overline{c}_2\lambda^{-2}}$$

for some $c_0 \in \mathbb{R}$, $c_2 \in \mathbb{C}$. Conditions (9) become respectively

$$\begin{aligned} &c_0 + c_2 + \overline{c}_2 = \nu^2/2 \quad m \in \mathbb{Z} \\ &c_2 = \overline{c}_2. \end{aligned} \tag{13}$$

With $\omega := 16c_2$,

$$\mu(\lambda) = \pm\frac{1}{2}\sqrt{\nu^2 + \frac{\omega}{4}(\lambda - \lambda^{-1})^2}.$$

and (12) follows. ∎

In the embedded case ($\nu^2 = 1$) if we define

$$\text{end weight} := \omega.$$

it can be shown that for embedded surfaces, the end weight is proportional to the product of the neck and bulge radii, normalized to be 1 for the round cylinder. The term "weight" makes sense in an extended metaphor involving force balancing and torque.

The asymptotic neck and bulge radii expressed in terms of $\mu(\lambda)$ are

$$\begin{aligned} \text{neck radius} &= \frac{1}{H}\left(\frac{1}{2} - \mu(i)\right) \\ \text{bulge radius} &= \frac{1}{H}\left(\frac{1}{2} + \mu(i)\right). \end{aligned} \tag{14}$$

where the non-negative square root is taken in (12) and $H \in \mathbb{R}^+$ is the constant mean curvature. The neck and bulge radii and end weights (assuming mean curvature H=1) of the two kinds of Delaunay surfaces together with their limiting cases, are tabulated in 15 and shown in figure 3.2.

(15)

surface	neck radius	bulge	end weight
round cylinder	$1/2$	$1/2$	1
Delaunay unduloid	$r \in (0, 1/2)$	$1 - r$	$4r(1 - r) \in (0, 1)$
round sphere	0	1	0
Delaunay nodoid	$r \in (-\infty, 0)$	$1 - r$	$4r(1 - r) \in (-\infty, 0)$

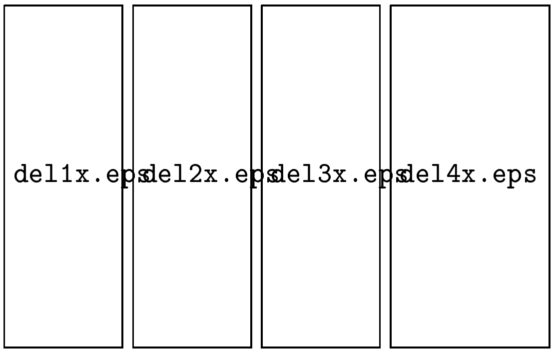

FIGURE 4. Cylinder, unduloid, sphere and nodoid in order of decreasing weight.

3.3. **The Delaunay potential with Hermitian residue.** With ξ_λ as in (11), if we impose the condition that A is Hermitian for all $\lambda \in \mathrm{SU}(2)$, then if A_λ has finite powers in λ near $\lambda = 0$, it is of the form

$$\begin{aligned} A_\lambda &:= \begin{pmatrix} c & a\lambda^{-1} + \bar{b}\lambda \\ b\lambda^{-1} + \bar{a}\lambda & -c \end{pmatrix} \\ &a, b \in \mathbb{C}; \quad c \in \mathbb{R}. \end{aligned} \tag{16}$$

The eigenvalues of A_λ are

$$\mu(\lambda) = \pm\sqrt{\left(a\lambda^{-1} + \overline{b}\lambda\right)\left(b\lambda^{-1} + \overline{a}\lambda\right) + c^2}. \tag{17}$$

The conditions (9) become

$$\begin{aligned} &\left|a + \overline{b}\right|^2 + c^2 = \nu^2/4 \\ &ab \in \mathbb{R}. \end{aligned} \tag{18}$$

Under these conditions, the eigenvalues $\mu(\lambda)$ are (12) with $\omega = 16ab$.

The moduli space of Delaunay surfaces is one-dimensional parametrized by the neck radius, or alternately, the end weight. The space of matrices A_λ of the form (16) satisfying (18) has real dimension 3. Geometrically, the parameters specify the neck radius together with two Euclidean motions. By the DPW construction, for a fixed basepoint and initial condition, the initial point in $\mathbb{R}^3$ and the tangent plane of the surface there is fixed, so the Euclidean motions move the surface in such a way as to fix this data.

4. **Moduli of Trinoids potentials**

One is now prepared to bring sections 2 and 3 of the paper together, applying the result 2.1 on representations to Delaunay eigenvalues (12).

4.1. **Order zero branch points.** The connection one-form of a CMC trinoid with once-wrapped ends cannot have three simple poles (and no others). For the monodromy satisfies $M(1) = -I$ at each end (7). The product of the monodromies then satisfies

$$M_1(1)M_2(1)M_3(1) = -I,$$

contradicting the fact that product of all the monodromies of a connection must be I.

The one-form for a CMC trinoid must then have at least four poles, either all simple or some of higher order. In this paper I use a configuration of four simple poles.

Recall that the Hopf differential is (up to constant multiple) the determinant of the coefficient of λ^{-1} of ξ.

Theorem 4.1. *Let A_λ as in* (16) *be the residue of ξ at $z = 0$. Suppose a, b, c satisfy the closing condition* (18) *and further that $a = 0$. Suppose that the Hopf differential has a zero of order* 0 *at $z = 0$. Let*

$$\frac{\xi}{dz} = \frac{A}{z} + \sum_{k=0}^{\infty} B_k z^k$$

be the local series for ξ at $z = 0$. Suppose that

$$[A,\, B_0] = 0. \tag{19}$$

Then for any initial condition, z is a regular point of the CMC immersion induced by ξ.

Proof. With $a = 0$, the eigenvalues (17) of A_λ become

$$\pm\sqrt{|b|^2 + c^2}.$$

By (18), these eigenvalues are half-integers $\pm\sqrt{|b|^2 + c^2} = \nu/2$. The condition (19) insures that the monodromy at $z = 0$ is $\exp(2\pi i A_\lambda) = -I$. Thus (8) is satisfied at $z = 0$, and the surface closes □

Theorem 2.1 is adapted to this situation of the four-punctured sphere with monodromy $-I$ at one puncture as follows:

Theorem 4.2. *Let $M = \mathbb{CP}^1 \setminus \{p_1, p_2, p_3, p_4\}$ be a four-punctured sphere, and let*

$$\langle \delta_1, \delta_2, \delta_3, \delta_4 \mid \delta_1\delta_2\delta_3\delta_4 = 1\rangle$$

be a presentation of the fundamental group $\pi_1(M, p)$, Let $\rho : \pi_1(M, p) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ be a monodromy representation. Assume that $\rho(\delta_4) = -I$. Let V_{SU} be as in theorem 2.1, and let

$$s_k = \tfrac{1}{2}\,\mathrm{tr}\,\rho(\delta_k), \quad k = 1, 2, 3.$$

Then $\rho \in \iota(V_{\mathrm{SU}})$ if and only if either

$$s_1, s_2, s_3 \in [-1, 1] \text{ and } \kappa(s_1, s_2, -s_3) < 1 \tag{20}$$

or

$$\kappa(s_1, s_2, -s_3) = 1 \text{ and } \rho \text{ is diagonalizable.}$$

Proof. If you take the presentation (1) defined by

$$\gamma_1 = \delta_1, \quad \gamma_2 = \delta_2, \quad \gamma_3 = \delta_3\delta_4,$$

you know the corresponding half-traces: they are

$$t_1 = \tfrac{1}{2}\,\mathrm{tr}\,\rho(\gamma_1) = s_1, \quad t_2 = \tfrac{1}{2}\,\mathrm{tr}\,\rho(\gamma_2) = s_2, \quad t_3 = \tfrac{1}{2}\,\mathrm{tr}\,\rho(\gamma_3) = -s_3.$$

Apply theorem 2.1 to this presentation. □

The condition (5) is adapted to the four-punctured situation by a shift of 1/2 as follows: if μ_k are the eigenvalues of A_k ($k = 1, 2, 3$), and $t_k = cos 2\pi\mu_k$ the half-traces of the corresponding monodromies, then the condition $\kappa(t_1, t_2, -t_3) = 1$ is equivalent to

$$\pm\mu_1 \pm \mu_2 \pm \mu_3 \in \tfrac{1}{2} + \mathbb{Z}. \tag{21}$$

4.2. **Moduli of eigenvalues.** [To be revised.] The neck-radii or weights of CMC trinoids produced by this method are precisely those which satisfy the inequalities (22) and (22). The resulting trinoids are of four types according to whether they have 3, 2, 1 or 0 unduloidal ends (the remaining ends are nodoidal).

Theorem 4.3. *Let*

$$\mu_k(\theta) := \frac{1}{2}\sqrt{1-\omega_k\sin^2\theta}, \quad (k=1,\,2,\,3)$$

as in (12) *and* t_k *as in* (3) *(*$k=1,\,2,\,3$*). Let*

$$n_k := \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\omega_k} = \tfrac{1}{2} - \mu_k(\pi/2) \quad (k=1,\,2,\,3).$$

Then t_k *satisfy inequality* (20) *for all* $\theta\in\mathbb{R}$ *if and only if*

$$\begin{aligned} &|n_1|+|n_2|+|n_3|\le 1, \\ &|n_i|\le|n_j|+|n_k|, \quad \{i,\,j,\,k\}=\{1,\,2,\,3\} \end{aligned} \tag{22}$$

$$|\omega_i|\le|\omega_j|+|\omega_k|, \quad \{i,\,j,\,k\}=\{1,\,2,\,3\}. \tag{23}$$

Proof. Proof of (23)*:* The inequality on the weights (23) is satisfied if and only if t_k satisfy inequality (20) near $(-1,\,-1,\,-1)$. Let $f(\theta)=\kappa(\theta)-1$. The first eight derivatives of f at $\theta=0$ are

$$\begin{aligned} &f^{(n)}(0)=0 \quad (n=1,\ldots,7) \\ &f^{(8)}(0) = -1260\pi^4(\omega_1+\omega_2+\omega_3)(-\omega_1+\omega_2+\omega_3)\times \\ &\qquad\qquad(\omega_1-\omega_2+\omega_3)(\omega_1+\omega_2-\omega_3). \end{aligned}$$

Hence $f(\theta)\le 0$ near $\theta=0$ if and only if $f^{(8)}(0)\le 0$. This condition is equivalent to (23).

Proof of (22)*:* [To be completed.] □

4.3. **The k-noid one-forms.** [To be revised.] We construct one-forms for CMC trinoids. We make the assumption that the potential ξ_λ has simple poles whose principle parts are Hermitian Delaunay potentials (16). That is, ξ_λ looks like

$$\xi_\lambda := \sum_{k=1}^{n}\frac{A_k dz}{z-z_k} =: \begin{pmatrix} \gamma_\lambda & \alpha_\lambda\lambda^{-1}+\widetilde{\beta}_\lambda\lambda \\ \beta_\lambda\lambda^{-1}+\widetilde{\alpha}_\lambda\lambda & -\gamma_\lambda \end{pmatrix}.$$

with each A_k of the form (16) satisfying the conditions (18). (If $\infty\in\mathbb{CP}^1$ is a pole of ξ_λ, then the $\operatorname{res}_\infty\xi_\lambda = -\sum A_k$ must likewise be of Delaunay form.)

The orders of the zeros and poles of the trinoid one-form are given in figure 4.3. The trinoid one-form is determined by its poles and residues. The residues are give in in table 4.3. The parameters satisfy

$$\begin{aligned} &z_1,\,z_2,\,z_3\in\mathbb{R}, \\ &z_1+z_2+z_3=0, \\ &x_1^2+y_1^2=1/4, \\ &x_2^2+y_2^2=1/4. \end{aligned}$$

	α	β	γ
end 1	-1	-1	-1
end 2	-1	-1	-1
end 3	-1	-1	-1
umbilic 1	0	1	0
umbilic 2	0	1	0
extra point	1	-1	1

FIGURE 5. Orders of zeros and poles of the trinoid one-form ξ.

	α	β	γ
end 1	z_1	$x_1 - z_1$	y_1
end 2	z_2	$-x_1 - z_2$	$-y_1$
end 3	z_3	$x_2 - z_3$	y_2
extra point	0	$-x_2$	$-y_2$

FIGURE 6. Residues of the trinoid one-form ξ.

5. **Concluding Remarks**

Further questions:

(i) Are these all the CMC trinoids? Are these all the CMC trinoids of a certain type? How to exclude bubbletons from the classification.
(ii) Are these trinoids "embedded" in an appropriate sense of the term?
(iii) Extension to k-noids with special symmetries.
(iv) The general case of k-noids with four or more ends.
(v) "Pair-of-pants" construction via DPW.

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