Homework 2, M 523

Problem 1. Show that every nonempty subset $A \subset \mathbb{N}$ of the natural numbers has a smallest element $n_0 \in A$, i.e., $n \geq n_0$ for all $n \in A$.

Problem 2. Show that arbitrarily close to any rational number there is a real (non-rational) number. In other words, show that to each real $\epsilon > 0$ and each rational $r \in \mathbb{Q}$ there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ with $|x - r| < \epsilon$.

Problem 3. Show that for a sequence (x_n) the following are true:

(i) $\lim x_n = 0$ if and only if $\lim |x_n| = 0$.
(ii) $\lim x_n = L$ implies $\lim |x_n| = |L|$. Is the converse true? Prove or give a counterexample.
(iii) $\lim x_n = L$ if and only if $\lim |x_n - L| = 0$.

Problem 4. Let $a > 0$ be a real number and consider the sequence (x_n) given by

$$x_{n+1} = \frac{x_n^2 + a}{2x_n}$$

which is suggested by Newton's iteration (see last problem on homework sheet 1) to find a positive root of the quadratic polynomial $f(x) = x^2 - a$. Our aim is to show that this sequence converges and to calculate its limit. Show the following:

(i) if $0 \neq x_1 \in \mathbb{Q}$ and a are rational so are all x_n. In this case our sequence is a sequence of rational numbers.
(ii) Show that for any starting value $x_1 > 0$ all sequence elements are positive.
(iii) Show that for any choice of $x_1 \neq 0$ the sequence satisfies $x_n^2 > a$ for all $n \geq 2$.
(iv) Show that if $x_1 > 0$ the sequence is monotone decreasing for $n \geq 2$ and thus (why?) convergent by a Theorem we proved in class.
(v) Show that $(\lim x_n)^2 = a$ and therefore $\lim x_n$ is a positive square root of a (provided we started with a positive $x_1 > 0$).

Notice that we have shown the existence of all square roots of positive real numbers and moreover those square roots can be obtained as limits of sequences of rational numbers.

Problem 5. Let $a > 0$ be a real number. Show that $\pm\sqrt{a}$ are the only zeros of $x^2 - a = 0$.

Problem 6. Since we now know that every real number $a \geq 0$ has a unique non-negative square root $\sqrt{a}$, show that the square root is monotone: $0 \leq a \leq b$ if and only if $\sqrt{a} \leq \sqrt{b}$.

Problem 7. Let (x_n) be a sequence of *non-negative* real numbers. Show that $\lim x_n = L$ if and only if $\lim(x_n^2) = L^2$. Is this also true for arbitrary convergent sequences? Which implication holds in general?

Problem 8. Show that for $0 \leq q < 1$ the sequence $x_n = q^n$ converges. Calculate its limit.

Problem 9. Consider the sequence $x_n := \sum_{k=0}^{n} q^k$ for a real number q. For which numbers q does (x_n) converge/diverge? Calculate the limit in the converging case.

Problem 10. Let (x_n) be a sequence such that for some $N \in \mathbb{N}$ we have $|x_{n+1} - x_n| < (1/2)^n$ for all $n \geq N$. Show that (x_n) is Cauchy.

Problem 11. Approach Problem 4 differently by showing that the sequence defined by $x_{n+1} = \frac{x_n^2+a}{2x_n}$ is in fact a Cauchy sequence (rather than showing it is a monotone and bounded sequence) and therefore convergent.

Problem 12. Let (x_n) be a sequence of real numbers satisfying $x_n < M$ for all $n \in \mathbb{N}$. Assuming (x_n) converges, show that $\lim x_n \leq M$. Can you give an example where under our assumptions $\lim x_n = M$? So despite $x_n < M$ for all $n \in \mathbb{N}$ the limit of (x_n) can be equal to M.