Infinite dimensional integrable systems

CLASSICAL INTEGRABLE SYSTEMS FROM PHYSICS

Problem 1. Show that the geodesic flow on a surface of revolution $M \subset \mathbb{R}^3$ is completely integrable. What is the phase space? Find the independent and Poisson commuting integrals of motion.

Problem 2. Show that the geodesic flow on an open set $M \subset \mathbb{R}^2$ with Riemannian metric $ds^2 = (f(x) + g(y))(dx^2 + dy^2)$ is completely integrable.

Problem 3. Show that the spherical pendulum (the pendulum is not constrained to a vertical plane) is completely integrable. What is the phase space? What are independent and Poisson commuting integrals of motion?

Problem 4. Consider the Kepler problem, that is, motion in a central force field. The Hamiltonian $h: \mathbb{R}^{2n} \to \mathbb{R}$ is given by

$$h(x, v) = \frac{1}{2}||v||^2 + U(|x|^2)$$

for $U: \mathbb{R}^n \to \mathbb{R}$ a potential depending only on the distance (squared, to avoid differentiability problems at $x = 0$ if $U$ is defined there, which it isn’t for the classical $1/r$ potential).

(i) Show that the action $(x, v) \mapsto (gx, gv)$ by the orthogonal group $g \in \text{O}_n(\mathbb{R})$ leaves the Hamiltonian $h$ and the standard symplectic structure $\omega = dx \wedge dv$ invariant.

(ii) Deduce that for each $\xi \in \text{so}_n(\mathbb{R})$ the vector field $X_\xi(x, v) := (\xi x, \xi v)$ is a Hamiltonian vector field. What is a corresponding Hamiltonian function $f_\xi$, i.e. $\omega(X_\xi, -) = df_\xi$? Note that $f_\xi$ is not unique, as we could add a $\xi$-dependent “constant”. Maybe useful: one can identify $\Lambda^2(\mathbb{R}^n) = \text{so}_n(\mathbb{R})$ via $(u \wedge v)(w) = \langle u, w \rangle v - \langle v, w \rangle u$.

(iii) Verify that there is a smooth map $f: \mathbb{R}^{2n} \to \text{so}_n(\mathbb{R})^*$ given by $\langle f(z), \xi \rangle = f_\xi(z)$ for $z = (x, v)$, $\xi \in \text{so}_n(\mathbb{R})$, where the angle brackets denote the evaluation pairing of dual vectors on vectors.

(iv) Show that there is a plane in $\mathbb{R}^n$ so that the solution $x(t)$ to the equation of motions for $h$ is contained in that plane. Conclude that one can now treat the Kepler problem in a 4-dim phase space.
(v) Find two independent and Poisson commuting integrals of motion, thus verifying that the Kepler problem is completely integrable. Note that this is independent of the special form of the radial force.

(vi) Verify Kepler’s Law that “equal areas are swept out over equal time intervals” by the solution curve \( x(t) \).

(vii) Write the solution curves in polar coordinates \( x(t) = r(t)e^{i \theta(t)} \) and relate \( r, \theta \) and \( t \) by integrals, thereby essentially producing explicit formulas for the solution. Specialize to the case of Newton’s gravitation law.

Problem 5. An alternative, perhaps more slick, way to understand the action-angle variable theorem: let’s assume we have a symplectic manifold \( (M^{2n}, \omega) \) and a Lagrangian fibration \( f: M \to B \) with compact, connected fibers onto a base manifold \( B^n \). We want to show that \( M \) is what is called a \( T^n \)-torsor, which in our case means that \( M = \cup_{b \in B} f^{-1}(b) \) is acted upon fiberwise transitively by a \( T^n \) bundle over \( B \). To fix notations: \( V = \ker df \subset TM \) is called the vertical subbundle, its fibers are the tangent spaces to the Lagrangian fibers \( f^{-1}(b) \).

(i) Show that there is a bundle isomorphism 
\[
 f^*TB^* \cong V
\]
given as follows: \( f^*TB^* \subset TM^* \) and then use \( \omega \) to identify \( TM^* \cong TM \).

(ii) Taking sections of the bundles above gives an injective \( C^\infty \)-linear map 
\[
 \Omega^1(B) \to \Gamma(TM): \alpha \mapsto X^\alpha
\]
The vector fields \( X^\alpha \) are tangent to the fibers \( f^{-1}(b) \) and thus complete. Moreover, if \( \alpha = dh \) with \( h \in C^\infty(B) \), then \( X^\alpha = X_{hof} \) is the Hamiltonian vector field to the pullback of \( h \) by the fibration \( f \).

(iii) The restriction of \( X^\alpha \) to a fiber \( f^{-1}(b) \) only depends on the value of \( \alpha_b \in T_bB^* \). Taking the time 1 flow \( \Phi^\alpha := \Phi^{X_\alpha}_1 \in \text{Diff}(f^{-1}(b)) \) of \( X^\alpha \) on \( f^{-1}(b) \), show that this gives a transitive and locally free group action 
\[
 T_bB^* \times f^{-1}(b) \to f^{-1}(b): (\alpha, p) \mapsto \Phi^\alpha(p)
\]
of the abelian group (vector space) \( T_bB^* \) on \( f^{-1}(b) \). Collecting those actions for all \( b \in B \) gives the fiberwise action 
\[
 \Phi: TB^* \times_B M \to M: (\alpha, p) \mapsto \Phi^\alpha(p)
\]

(iv) From class we know that the stabilizer \( \Gamma_b \subset T_bB^* \) of this action is a full rank lattice and \( T_b = T_bB^*/\Gamma_b \cong T^n \) is an \( n \)-torus. Let \( \Gamma \subset TB^* \) denote the union of \( \Gamma_b \) over \( b \in B \). Taking the fiberwise quotients \( T = TB^*/\Gamma \), we get the induced (fiberwise) transitive action 
\[
 T \times_B M \to M: (t, p) \to \Phi^\alpha(p) := t \cdot p, \quad t = \alpha + T
\]
showing \( M \) as a \( T^n \)-torsor. Furthermore, the canonical symplectic structure on \( TB^* \) descends to a symplectic structure on \( TB^*/\Gamma \) and \( T \) becomes a symplectic manifold.
(v) For $\alpha \in \Omega^1(B)$ the diffeomorphism $\Phi^\alpha \in \text{Diff}(M)$ preserves the Lagrangian fibers $f^{-1}(b)$ and thus induces the identity map on $B$. Verify that those diffeomorphisms pull back the symplectic form by 

$$(\Phi^\alpha)^*\omega = \omega + f^*d\alpha$$

Conclude that if $\alpha$ is a section of $\Gamma$, that is, $\alpha_b \in \Gamma_b \subset T_bB^*$, then $\Phi^\alpha = Id_M$, and thus $d\alpha = 0$. But this implies that $\Gamma \subset TB^*$ is Lagrangian and that $T = TB^*/\Gamma \to B$ is a Lagrangian fibration.

(vi) Around every point $b_0 \in B$ there is an open neighborhood $b_0 \in U \subset B$ and coordinates $x: U \to \mathbb{R}^n$ so that $dx_1, \ldots, dx_n$ trivialize $\Gamma_U$. Conclude that $T_U \cong U \times T^n$ and $x \circ f$ and $y \in T^n$ are action-angle variables.

(vii) Verify that if the symplectic form on $f^{-1}(U)$ is exact, that is, $\omega = d\eta$, then one can construct the action variables via integration as follows: let $\gamma_i(b): S^1 \to f^{-1}(b)$, $i = 1, \ldots, n$ be loops which generate the homology of $f^{-1}(b) \cong T^n$ and depend smoothly on $b$. Then

$$p \to \int_{\gamma_i(f(p))} \eta$$

gives action coordinates on $f^{-1}(U)$. 