Basic facts about Lie groups, Lie algebras, group actions etc. in the context of this course

The first exercises are standard material which every grad student should be able to do efficiently in their sleep (it is like knowing how to calculate the derivative of $x^2$ if you are an undergrad).

Group actions.

Problem 1. Here some basic calculation exercises which you may want to go over in case you haven’t already. You should assume here standard results of (sub)manifold theory. An important result is a Lie group action $G \times M \to M$ is free and proper (the map $G \times M \to M \times M$ sending $(g, p) \mapsto (p, gp)$ is proper), then the orbit space $M/G$ has a unique manifold structure so that $\pi : G \to M/G$ is a (surjective) submersion. This implies in particular that any smooth, $G$-invariant map $f : M \to N$ factors through a smooth, injective map $\bar{f} : M/G \to N$, that is, $\bar{f} \circ \pi = f$.

(i) Verify that if a Lie group acts on a manifold $G \times M \to M$ from the left, then it also acts on the tangent bundle $G \times TM \to TM$ via $gv := d(\lambda_g)_p(v)$ if $v \in T_p M$ (note that $gv \in T_{gp} M$). The action on the cotangent bundle from the left is given by $g\beta = \beta \circ d(\lambda^{-1}_g)_{gp}$. Therefore, by taking tensor products, $G$ acts on all tensor bundles $TM^k \otimes (TM^*)^l$. If the action on $M$ is free and proper, so is the action on $TM$ and $TM^*$ (and on all tensor bundles).

If $g \in M$ is a fixed point of the action on $M$, we get a linear action of $G$ on $T_g M$, or equivalently, a linear representation (Lie group homomorphism) $\rho : G \to GL(T_g M)$. The action on $T_g M^*$ is then just the contragradient representation.

Make sure you understand that if $pg$ is a right action, then $gp := pg^{-1}$ defines a left action and vice versa. So we usually state results only for left actions, but they all have analogs for right actions.

(ii) If $M = G$, then left multiplication $H \times G \to G$ by any Lie subgroup $H \subset G$ is a left action of $H$ on $G$ which is free and proper. In particular we can
take $H = G$. Then $\text{Int}_g(h) := ghg^{-1}$ is an action of $G$ on itself, moreover $\text{Int}_g$ is a group automorphisms (invertible group homomorphism), hence

$$\text{Int}: G \to \text{Aut}(G).$$

This action has $e = 1_G$ as a fixed point, thus the induced action of $G$ on $TG$ gives a linear action of $G$ on $T_eG$, or equivalently, a representation, the adjoint representation

$$\text{Ad}: G \to \text{GL}(T_eG).$$

Taking derivative at $e \in G$ of $\text{Ad}$ gives the linear map

$$\text{ad}: T_eG \to \text{End}(T_eG).$$

which gives rise to the bilinear map $B: T_eG \times T_eG \to T_eG$ via $B(\xi, \eta) := \text{ad}_e(\eta)$. Verify that $B$ is skew and satisfies the Jacobi identity. Thus, $T_eG$ with $B$ becomes a Lie algebra, which we denote by $\mathfrak{g}$. Carry this out explicitly for the case when $G = \text{GL}_n(\mathbb{R})$ and verify that $B$ is just the usual matrix commutator.

(iii) If $\mu: G \times M \to M$ denotes the left action, then $\lambda_g := \mu(g, -) \in \text{Diff}(M)$, and $\beta_p: G \to M$ is the so called orbit map, since its image $O_p := Gp \subset M$ is the orbit of $G$ through $p \in M$. Check that $\beta_p$ has constant rank $\text{rank}(d\beta_p)_g = \text{rank}(d\beta_p)_h$ for all $g, h \in G$. Use that $\lambda_g \circ \beta_p = \beta_p \circ L_g$, where we denote by $L_g$ and $R_g$ the left, resp. right, multiplication by $g \in G$ on $G$ (thus $L_g = \lambda_g$ for the special case when $M = G$). Apply the constant rank theorem and show that the stabilizer subgroup

$$G_p = \{g \in G; gp = p\}$$

is a Lie subgroup (submanifold and subgroup) of $G$. Deduce that $G/G_p$ is a manifold and that $\beta_p$ induces a smooth, injective map

$$\bar{\beta}_p: G/G_p \to M$$

of constant rank. Now argue that for an injective smooth map $f: N \to M$ between any two manifolds the subset of points $q \in N$ where $\text{rank}_q f = \dim M$ is non-empty. Conclude that $\beta_p$ induces a smooth injective immersion $\bar{\beta}_p: G/G_p \to M$ whose image is the orbit $O_p \subset M$. In general, $O_p \subset M$ is not a submanifold (the standard example is the free action of $\mathbb{R}$ on a 2-torus of irrational slope). But if $G$ is compact (or if $\beta_p$ is a proper map), then $O_p \subset M$ is a submanifold which, via $\bar{\beta}_p$, is diffeomorphic to the homogenous space $G/G_p$.

(iv) For a Lie group $G$ we call a vector field $X \in \Gamma(TG)$ left (right) invariant, if $L_g^*X = X$ (resp. $R_g^*X = X$) for all $g \in G$. Verify that the linear map $X: T_eG \to \Gamma_L(TG)$ given by $\xi X(g) := g\xi$ is a linear isomorphism whose inverse is evaluation at $e \in G$, where $\Gamma_L(TG)$ denotes the subspace of left invariant vector fields. Likewise, there is a linear bijection $Y: T_eG \to \Gamma_R(TG)$ via $Y\xi(g) = \xi g$. Moreover, $\Gamma_L(TG)$ and $\Gamma_R(TG)$ are
Lie subalgebras of $\Gamma(TG)$ under the vector field Lie bracket. This gives us two more Lie algebra structures, $[,]_L$ and $[,]_R$ on $TG$, induced by the maps $X$ and $Y$ above. Verify now that we have $B = [,]_L = -[,]_R$ with $B$ from (ii). From now on we denote $B = [,]_g = [,]$. Thus, $Ad_g$ is a Lie algebra isomorphism for $g \in G$ and $ad_\xi = [\xi,\cdot]$.

(v) Show that if $f: H \to G$ is a Lie group homomorphism, then $df_e: \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism. In particular, if $H \subset G$ is a Lie subgroup, then $[,]_h$ is the restriction of $[,]_g$ to $\mathfrak{h}$.

(vi) For $G \times M \to G$ we define a linear map $Y: \mathfrak{g} \to \Gamma(TM)$ by $Y^\xi(p) := (d\beta_p)_e(\xi)$. The vector field $Y^\xi$ is called the fundamental vector field of the $G$ action on $M$. Show that $Y$ is a Lie algebra (anti) homomorphism

$$Y^{[\xi,\eta]} = - [Y^\xi, Y^\eta]_{C^\infty}$$

and that $Y^{Ad_g \xi} = gY^\xi g^{-1}$.

Verify that $Y^\xi = 0$ if and only if $\xi \in \mathfrak{g}_p$ is in the Lie algebra of the Stabilizer subgroup $G_p \subset G$. Thus (assuming that the orbit $O_p \subset M$ is a submanifold, $Y$ induces a linear isomorphism $\mathfrak{g}/\mathfrak{g}_p \to T_p O_p$.

(vii) Verify that the fundamental vector fields for the left action of $G$ on itself are the right invariant vector fields on $G$.

**Lie groups and algebras.**

**Problem 2.** Let $G$ be Lie group. Then

$$TG \to G \times \mathfrak{g}: v_g \mapsto (g, g^{-1}v_g)$$

is called the left trivialization of the tangent bundle of $G$. Likewise $v_g \mapsto (g, v_g g^{-1})$ is the right trivialization. The Lie algebra valued 1-form $\theta_L \in \Omega^1(G, \mathfrak{g})$ defined by the left trivialization $(\theta_L)_g(v) = g^{-1}v$ is called the left Maurer-Cartan form. Likewise, $(\theta_R)_g(v) = v g^{-1}$ defines the right Maurer-Cartan form $\theta^R \in \Omega^1(G, \mathfrak{g})$.

Verify that $L^*_g \theta^L = \theta^R$, i.e. $\theta^L$ is a left-invariant form. Likewise, $R^*_g \theta^R = \theta^R$ and $\theta^R_g = Ad_g \theta^L_g$.

Verify the zero curvature/Maurer-Cartan equations

$$d\theta^L + \frac{1}{2}[\theta^L \wedge \theta^L] = 0, \quad d\theta^R - \frac{1}{2}[\theta^R \wedge \theta^R] = 0,$$

where we define $[\alpha \wedge \beta](v, w) := [\alpha(v), \beta(w)] - [\alpha(w), \beta(v)]$ for Lie algebra valued 1-forms $\alpha, \beta \in \Omega^1(G, \mathfrak{g})$.

**Problem 3.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Show that evaluation at $e \in G$ is a linear isomorphism between left invariant $k$-forms $\Omega^k_L(G, \mathbb{R})$ on $G$ and $\Lambda^k \mathfrak{g}^*$. Verify that $\Omega^k_L(G, \mathbb{R})$ is closed under exterior derivative and wedge product. Thus, evaluation there is an isomorphism of exterior algebras

$$\Omega^*(G, \mathbb{R}) \cong \Lambda \mathfrak{g}^*$$
commuting with exterior derivative and the differential $d_g$ defining Lie algebra cohomology (with values in $\mathbb{R}$).

**Problem 4.** Let $G$ be a Lie group with left Maurer-Cartan form $\theta \in \Omega^1(G, \mathfrak{g})$. Choosing a basis $\xi_i \in \mathfrak{g}$, we can write $\theta = \sum_{i=1}^m \xi_i \sigma^i$. Show that $\sigma^i \in \Omega^1(G, \mathbb{R})$ are left invariant 1-forms. Thus, $\det_g := \sigma^1 \wedge \cdots \wedge \sigma^m$ defines a left invariant volume form on $G$. Now check that $R_g^* \det_g = \det \text{Ad}_{g^{-1}} \det_g$ and conclude that for compact (connected) $G$ the volume form $\det_g$ is also right invariant, that is, the representation $G \to \mathbb{R}^\times$ given by $g \mapsto \det \text{Ad}_g$ is trivial. Normalizing the volume of $G$ to be one, gives a unique left and right, i.e. bi-, invariant volume form on a compact (connected) Lie group.

**Problem 5.** Averaging over a compact group: Let $G$ be a compact Lie group and $\rho: G \to \text{GL}(V)$ a linear representation. Show that $V$ has an Euclidean inner product $(.,.)$ invariant under $\rho$, that is, $(\rho_g(v), \rho_g(w)) = (v, w)$ for all $g \in G$, $v, w \in V$. If $G$ is connected, this means that $\rho(G) \subset \text{SO}(V)$. Conclude from that that any (linear) representation of a compact Lie group is reducible, that is, a direct sum of irreducible representations.

As a special case show that every Lie algebra of a compact Lie group has an $\text{Ad}$-invariant Euclidean inner product:

$$(\text{Ad}_g \xi, \text{Ad}_g \eta) = (\xi, \eta), \quad \text{and thus also } \text{ad}$-$\text{invariant } ([\xi, \eta], \zeta) + (\eta, [\xi, \zeta]) = 0.$$

**Problem 6.** Use the averaging idea and the last couple of exercises to show that for a compact, connected Lie group $G$ its de Rham cohomology is isomorphic to the Lie algebra cohomology. $\textbf{Hint}$: every closed form can be averaged to a left invariant closed form.

**Problem 7.** The Killing form: every Lie algebra $\mathfrak{g}$ (whether it comes from a Lie group or not) has an inner product:

$$(\xi, \eta) := \text{tr}(\text{ad}_\xi \circ \text{ad}_\eta)$$

where $\text{ad}_\xi = [\xi, \cdot]$. Verify that the Killing form is ad-invariant, and also $\text{Ad}$-invariant in case $\mathfrak{g}$ is the Lie algebra of a Lie group. Moreover, verify that if $G$ is compact, then the Killing form is negative definite.

Calculate the Killing form for $\text{gl}_n(\mathbb{R})$ and $\text{so}_n(\mathbb{R})$.

**Problem 8.** A Lie algebra $\mathfrak{g}$ is called semi-simple, if it is the direct sum $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ of simple ideals $\mathfrak{g}_i \subset \mathfrak{g}$. Show that a Lie algebra is semi-simple if and only if its Killing form is non-degenerate. $\textbf{Hint}$: if $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then also $\mathfrak{a}^\perp \subset \mathfrak{g}$ is an ideal.

Use this to prove the Whitehead Lemmas: for a semi-simple Lie algebra the first and second Lie algebra cohomology groups are trivial.
Unfinished/sloppily done class material to finish up.

Problem 9. Complete the proof of the equivariance of the moment map, that is
\[ \mu(gp) = \text{Ad}_g^* \mu(p) \]

Problem 10. In the example of the cotangent bundle reduction of a Lie group, show that the induced symplectic form \( \bar{\omega} \) on the symplectic reduction
\[ \mu^{-1}(\alpha)/G_\alpha \cong G/G_\alpha \cong O^*_\alpha \]
is, under the above diffeomorphisms, induced by the 2-form \( \langle \alpha, [\theta \wedge \theta] \rangle \) (up to sign and a factor 1/2) on \( G \).

Problem 11. Show that a Poisson structure \( \{ \cdot, \cdot \} \) on a manifold \( M \) gives rise to a skew-symmetric bilinear form \( \Omega : TM \times TM \to \mathbb{R} \). How is the Jacobi identity of the Poisson structure reflected in \( \Omega \)? Show that imposing this condition on a skew form \( \Omega \) defines a Poisson structure. Furthermore, \( \Omega \) is non-degenerate, if and only if \( \{ f, \cdot \} = 0 \) implies \( f = \text{constant} \). In this case the Poisson structure comes from a symplectic structure (which one?).

Problem 12. Show that \( \{ f, g \}(\alpha) := \langle \alpha, [df_\alpha, dg_\alpha] \rangle \) defines a Poisson bracket on smooth functions on the dual \( g^* \) of a Lie algebra \( g \). Verify that \( \{ f, \cdot \} = 0 \) if and only if \( f \) is constant along coadjoint orbits. Conclude that the coadjoint orbits are symplectic.