Roger Penrose

The Road to Reality

A Complete Guide to the Laws of the Universe

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BY ROGER PENROSE

The Emperor's New Mind:
Concerning Computers, Minds,
and the Laws of Physics

Shadows of the Mind:
A Search for the Missing Science
of Consciousness

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I dedicate this book to the memory of

DENNIS SCIAMA

who showed me the excitement of physics

Preface

THE purpose of this book is to convey to the reader some feeling for what is surely one of the most important and exciting voyages of discovery that humanity has embarked upon. This is the search for the underlying principles that govern the behaviour of our universe. It is a voyage that has lasted for more than two-and-a-half millennia, so it should not surprise us that substantial progress has at last been made. But this journey has proved to be a profoundly difficult one, and real understanding has, for the most part, come but slowly. This inherent difficulty has led us in many false directions; hence we should learn caution. Yet the 20th century has delivered us extraordinary new insights—some so impressive that many scientists of today have voiced the opinion that we may be close to a basic understanding of *all* the underlying principles of physics. In my descriptions of the current fundamental theories, the 20th century having now drawn to its close, I shall try to take a more sober view. Not all my opinions may be welcomed by these 'optimists', but I expect further changes of direction greater even than those of the last century.

The reader will find that in this book I have not shied away from presenting mathematical formulae, despite dire warnings of the severe reduction in readership that this will entail. I have thought seriously about this question, and have come to the conclusion that what I have to say cannot reasonably be conveyed without a certain amount of mathematical notation and the exploration of genuine mathematical concepts. The understanding that we have of the principles that actually underlie the behaviour of our physical world indeed depends upon some appreciation of its mathematics. Some people might take this as a cause for despair, as they will have formed the belief that they have no capacity for mathematics, no matter at how elementary a level. How could it be possible, they might well argue, for them to comprehend the research going on at the cutting edge of physical theory if they cannot even master the manipulation of *fractions*? Well, I certainly see the difficulty.

Yet I am an optimist in matters of conveying understanding. Perhaps I am an incurable optimist. I wonder whether those readers who cannot manipulate fractions—or those who claim that they cannot manipulate fractions—are not deluding themselves at least a little, and that a good proportion of them actually have a potential in this direction that they are not aware of. No doubt there are some who, when confronted with a line of mathematical symbols, however simply presented, can see only the stern face of a parent or teacher who tried to force into them a non-comprehending parrot-like apparent competence—a duty, and a duty alone—and no hint of the magic or beauty of the subject might be allowed to come through. Perhaps for some it is too late; but, as I say, I am an optimist and I believe that there are many out there, even among those who could never master the manipulation of fractions, who have the capacity to catch some glimpse of a wonderful world that I believe must be, to a significant degree, genuinely accessible to them.

One of my mother's closest friends, when she was a young girl, was among those who could not grasp fractions. This lady once told me so herself after she had retired from a successful career as a ballet dancer. I was still young, not yet fully launched in my activities as a mathematician, but was recognized as someone who enjoyed working in that subject. 'It's all that cancelling', she said to me, 'I could just never get the hang of cancelling.' She was an elegant and highly intelligent woman, and there is no doubt in my mind that the mental qualities that are required in comprehending the sophisticated choreography that is central to ballet are in no way inferior to those which must be brought to bear on a mathematical problem. So, grossly overestimating my expositional abilities, I attempted, as others had done before, to explain to her the simplicity and logical nature of the procedure of 'cancelling'.

I believe that my efforts were as unsuccessful as were those of others. (Incidentally, her father had been a prominent scientist, and a Fellow of the Royal Society, so she must have had a background adequate for the comprehension of scientific matters. Perhaps the 'stern face' could have been a factor here, I do not know.) But on reflection, I now wonder whether she, and many others like her, did not have a more rational hang-up—one that with all my mathematical glibness I had not noticed. There is, indeed, a profound issue that one comes up against again and again in mathematics and in mathematical physics, which one first encounters in the seemingly innocent operation of cancelling a common factor from the numerator and denominator of an ordinary numerical fraction.

Those for whom the action of cancelling has become second nature, because of repeated familiarity with such operations, may find themselves insensitive to a difficulty that actually lurks behind this seemingly simple procedure. Perhaps many of those who find cancelling mysterious are seeing a certain profound issue more deeply than those of us who press onwards in a cavalier way, seeming to ignore it. What issue is this? It concerns the very way in which mathematicians can provide an existence to their mathematical entities and how such entities may relate to physical reality.

I recall that when at school, at the age of about 11, I was somewhat taken aback when the teacher asked the class what a fraction (such as $\frac{3}{8}$) actually is! Various suggestions came forth concerning the dividing up of pieces of pie and the like, but these were rejected by the teacher on the (valid) grounds that they merely referred to imprecise physical situations to which the precise mathematical notion of a fraction was to be *applied*; they did not tell us what that clear-cut mathematical notion actually *is*. Other suggestions came forward, such as $\frac{3}{8}$ is 'something with a 3 at the top and an 8 at the bottom with a horizontal line in between' and I was distinctly surprised to find that the teacher seemed to be taking these suggestions seriously! I do not clearly recall how the matter was finally resolved, but with the hindsight gained from my much later experiences as a mathematics undergraduate, I guess my schoolteacher was making a brave attempt at telling us the definition of a fraction in terms of the ubiquitous mathematical notion of an *equivalence class*.

What is this notion? How can it be applied in the case of a fraction and tell us what a fraction actually is? Let us start with my classmate's 'something with a 3 at the top and an 8 on the bottom'. Basically, this is suggesting to us that a fraction is specified by an ordered pair of whole numbers, in this case the numbers 3 and 8. But we clearly cannot regard the fraction as *being* such an ordered pair because, for example, the fraction $\frac{6}{16}$ is the same number as the fraction $\frac{3}{8}$, whereas the pair (6, 16) is certainly not the same as the pair (3, 8). This is only an issue of cancelling; for we can write $\frac{6}{16}$ as $\frac{3\times 2}{8\times 2}$ and then cancel the 2 from the top and the bottom to get $\frac{3}{8}$. Why are we allowed to do this and thereby, in some sense, 'equate' the pair (6, 16) with the pair (3, 8)? The mathematician's answer—which may well sound like a cop-out—has the cancelling rule just built in to the definition of a fraction: a pair of whole numbers $(a \times n,\ b \times n)$ is deemed to represent the same fraction as the pair (a, b) whenever n is any non-zero whole number (and where we should not allow b to be zero either).

But even this does not tell us what a fraction is; it merely tells us something about the way in which we represent fractions. What *is* a fraction, then? According to the mathematician's "equivalence class" notion, the fraction $\frac{3}{8}$, for example, simply is the infinite collection of all pairs

$$(3, 8), (-3, -8), (6, 16), (-6, -16), (9, 24), (-9, -24), (12, 32), \ldots,$$

where each pair can be obtained from each of the other pairs in the list by repeated application of the above cancellation rule.* We also need definitions telling us how to add, subtract, and multiply such infinite collections of pairs of whole numbers, where the normal rules of algebra hold, and how to identify the whole numbers themselves as particular types of fraction.

This definition covers all that we mathematically need of fractions (such as $\frac{1}{2}$ being a number that, when added to itself, gives the number 1, etc.), and the operation of cancelling is, as we have seen, built into the definition. Yet it seems all very formal and we may indeed wonder whether it really captures the intuitive notion of what a fraction is. Although this ubiquitous equivalence class procedure, of which the above illustration is just a particular instance, is very powerful as a pure-mathematical tool for establishing consistency and mathematical existence, it can provide us with very top-heavy-looking entities. It hardly conveys to us the intuitive notion of what $\frac{3}{8}$ is, for example! No wonder my mother's friend was confused.

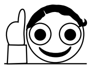
In my descriptions of mathematical notions, I shall try to avoid, as far as I can, the kind of mathematical pedantry that leads us to define a fraction in terms of an 'infinite class of pairs' even though it certainly has its value in mathematical rigour and precision. In my descriptions here I shall be more concerned with conveying the idea—and the beauty and the magic—inherent in many important mathematical notions. The idea of a fraction such as $\frac{3}{8}$ is simply that it is some kind of an entity which has the property that, when added to itself 8 times in all, gives 3. The magic is that the idea of a fraction actually works despite the fact that we do not really directly experience things in the physical world that are exactly quantified by fractions—pieces of pie leading only to approximations. (This is quite unlike the case of natural numbers, such as 1, 2, 3, which do precisely quantify numerous entities of our direct experience.) One way to see that fractions do make consistent sense is, indeed, to use the 'definition' in terms of infinite collections of pairs of integers (whole numbers), as indicated above. But that does not mean that $\frac{3}{8}$ actually *is* such a collection. It is better to think of $\frac{3}{8}$ as being an entity with some kind of (Platonic) existence of its own, and that the infinite collection of pairs is merely one way of our coming to terms with the consistency of this type of entity. With familiarity, we begin to believe that we can easily grasp a notion like $\frac{3}{8}$ as something that has its own kind of existence, and the idea of an 'infinite collection of pairs' is merely a pedantic device—a device that quickly recedes from our imaginations once we have grasped it. Much of mathematics is like that.

* This is called an 'equivalence class' because it actually is a class of entities (the entities, in this particular case, being pairs of whole numbers), each member of which is deemed to be equivalent, in a specified sense, to each of the other members.

To mathematicians (at least to most of them, as far as I can make out), mathematics is not just a cultural activity that we have ourselves created, but it has a life of its own, and much of it finds an amazing harmony with the physical universe. We cannot get any deep understanding of the laws that govern the physical world without entering the world of mathematics. In particular, the above notion of an equivalence class is relevant not only to a great deal of important (but confusing) mathematics, but a great deal of important (and confusing) physics as well, such as Einstein's general theory of relativity and the 'gauge theory' principles that describe the forces of Nature according to modern particle physics. In modern physics, one cannot avoid facing up to the subtleties of much sophisticated mathematics. It is for this reason that I have spent the first 16 chapters of this work directly on the description of mathematical ideas.

What words of advice can I give to the reader for coping with this? There are four different levels at which this book can be read. Perhaps you are a reader, at one end of the scale, who simply turns off whenever a mathematical formula presents itself (and some such readers may have difficulty with coming to terms with fractions). If so, I believe that there is still a good deal that you can gain from this book by simply skipping all the formulae and just reading the words. I guess this would be much like the way I sometimes used to browse through the chess magazines lying scattered in our home when I was growing up. Chess was a big part of the lives of my brothers and parents, but I took very little interest, except that I enjoyed reading about the exploits of those exceptional and often strange characters who devoted themselves to this game. I gained something from reading about the brilliance of moves that they frequently made, even though I did not understand them, and I made no attempt to follow through the notations for the various positions. Yet I found this to be an enjoyable and illuminating activity that could hold my attention. Likewise, I hope that the mathematical accounts I give here may convey something of interest even to some profoundly non-mathematical readers if they, through bravery or curiosity, choose to join me in my journey of investigation of the mathematical and physical ideas that appear to underlie our physical universe. Do not be afraid to skip equations (I do this frequently myself) and, if you wish, whole chapters or parts of chapters, when they begin to get a mite too turgid! There is a great variety in the difficulty and technicality of the material, and something elsewhere may be more to your liking. You may choose merely to dip in and browse. My hope is that the extensive cross-referencing may sufficiently illuminate unfamiliar notions, so it should be possible to track down needed concepts and notation by turning back to earlier unread sections for clarification.

At a second level, you may be a reader who is prepared to peruse mathematical formulae, whenever such is presented, but you may not have the inclination (or the time) to verify for yourself the assertions that I shall be making. The confirmations of many of these assertions constitute the solutions of the exercises that I have scattered about the mathematical portions of the book. I have indicated three levels of difficulty by the icons –

 very straight forward

 needs a bit of thought

 not to be undertaken lightly.

It is perfectly reasonable to take these on trust, if you wish, and there is no loss of continuity if you choose to take this position.

If, on the other hand, you are a reader who does wish to gain a facility with these various (important) mathematical notions, but for whom the ideas that I am describing are not all familiar, I hope that working through these exercises will provide a significant aid towards accumulating such skills. It is always the case, with mathematics, that a little direct experience of thinking over things on your own can provide a much deeper understanding than merely reading about them. (If you need the solutions, see the website www.roadsolutions.ox.ac.uk.)

Finally, perhaps you are already an expert, in which case you should have no difficulty with the mathematics (most of which will be very familiar to you) and you may have no wish to waste time with the exercises. Yet you may find that there is something to be gained from my own perspective on a number of topics, which are likely to be somewhat different (sometimes very different) from the usual ones. You may have some curiosity as to my opinions relating to a number of modern theories (e.g. supersymmetry, inflationary cosmology, the nature of the Big Bang, black holes, string theory or M-theory, loop variables in quantum gravity, twistor theory, and even the very foundations of quantum theory). No doubt you will find much to disagree with me on many of these topics. But controversy is an important part of the development of science, so I have no regrets about presenting views that may be taken to be partly at odds with some of the mainstream activities of modern theoretical physics.

It may be said that this book is really about the relation between mathematics and physics, and how the interplay between the two strongly influences those drives that underlie our searches for a better theory of the universe. In many modern developments, an essential ingredient of these drives comes from the judgement of mathematical beauty, depth, and sophistication. It is clear that such mathematical influences can be vitally important, as with some of the most impressively successful achievements of 20th-century physics: Dirac's equation for the electron, the general framework of quantum mechanics, and Einstein's general relativity. But in all these cases, physical considerations—ultimately observational ones—have provided the overriding criteria for acceptance. In many of the modern ideas for fundamentally advancing our understanding of the laws of the universe, adequate physical criteria—i.e. experimental data, or even the possibility of experimental investigation—are not available. Thus we may question whether the accessible mathematical desiderata are sufficient to enable us to estimate the chances of success of these ideas. The question is a delicate one, and I shall try to raise issues here that I do not believe have been sufficiently discussed elsewhere.

Although, in places, I shall present opinions that may be regarded as contentious, I have taken pains to make it clear to the reader when I am actually taking such liberties. Accordingly, this book may indeed be used as a genuine guide to the central ideas (and wonders) of modern physics. It is appropriate to use it in educational classes as an honest introduction to modern physics—as that subject is understood, as we move forward into the early years of the third millennium.

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Special gratitude is expressed to The M.C. Escher Company, Holland for permission to reproduce Escher works in Figs. 2.11, 2.12, 2.16, and 2.22, and particularly to allow the modifications of Fig. 2.11 that are used in Figs. 2.12 and 2.16, the latter being an explicit mathematical transformation. All the Escher works used in this book are copyright (2004) The M.C. Escher Company. Thanks go also to the Institute of Theoretical Physics, University of Heidelberg and to Charles H. Lineweaver for permission to reproduce the respective graphs in Figs. 27.19 and 28.19.

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Notation

(Not to be read until you are familiar with the concepts, but perhaps find the fonts confusing!)

I have tried to be reasonably consistent in the use of particular fonts in this book, but as not all of this is standard, it may be helpful to the reader to have the major usage that I have adopted made explicit.

Italic lightface (Greek or Latin) letters, such as in $w^2, p^n, \log z$, $\cos\theta$, $\mathrm{e}^{i\theta}$, or e^x are used in the conventional way for mathematical variables which are numerical or scalar quantities; but established numerical constants, such as e, i, or π or established functions such as sin, cos, or log are denoted by upright letters. Standard physical constants such as c, G, h, $\hbar$, g, or k are italic, however.

A vector or tensor quantity, when being thought of in its (abstract) entirety, is denoted by a boldface italic letter, such as $\boldsymbol{R}$ for the Riemann curvature tensor, while its set of components might be written with italic letters (both for the kernel symbol its indices) as R_{abcd}. In accordance with the abstract-index notation, introduced here in §12.8, the quantity R_{abcd} may alternatively stand for the entire tensor $\boldsymbol{R}$, if this interpretation is appropriate, and this should be made clear in the text. Abstract linear transformations are kinds of tensors, and boldface italic letters such as $\boldsymbol{T}$ are used for such entities also. The abstract-index form $T^a{}_b$ is also used here for an abstract linear transformation, where appropriate, the staggering of the indices making clear the precise connection with the ordering of matrix multiplication. Thus, the (abstract-)index expression $S^a{}_bT^b{}_c$ stands for the product $\boldsymbol{ST}$ of linear transformations. As with general tensors, the symbols $S^a{}_b$ and $T^b{}_c$ could alternatively (according to context or explicit specification in the text) stand for the corresoponding arrays of components—these being *matrices*—for which the corresponding bold upright letters **S** and **T** can also be used. In that case, **ST** denotes the corresponding matrix product. This 'ambivalent' interpretation of symbols such as R_{abcd} or $S^a{}_b$ (either standing for the array of components or for the abstract tensor itself) should not cause confusion, as the algebraic (or differential) relations that these symbols are subject to are identical for both interpretations. A third notation for such quantities—the *diagrammatic* notation—is also sometimes used here, and is described in Figs. 12.17, 12.18, 14.6, 14.7, 14.21, 19.1 and elsewhere in the book.

There are places in this book where I need to distinguish the 4-dimensional spacetime entities of relativity theory from the corresponding ordinary 3-dimensional purely spatial entities. Thus, while a boldface italic notation might be used, as above, such as $\boldsymbol{p}$ or $\boldsymbol{x}$, for the 4-momentum or 4-position, respectively, the corresponding 3-dimensional purely spatial entities would be denoted by the corresponding upright bold letters $\mathbf{p}$ or $\mathbf{x}$. By analogy with the notation $\mathbf{T}$ for a matrix, above, as opposed to $\boldsymbol{T}$ for an abstract linear transformation, the quantities $\mathbf{p}$ and $\mathbf{x}$ would tend to be thought of as 'standing for' the three spatial components, in each case, whereas $\boldsymbol{p}$ and $\boldsymbol{x}$ might be viewed as having a more abstract component-free interpretation (although I shall not be particularly strict about this). The Euclidean 'length' of a 3-vector quantity $\mathbf{a} = (a_1,a_2,a_3)$ may be written a, where $a^2 = a_1^2 + a_2^2 + a_3^2$, and the scalar product of $\mathbf{a}$ with $\mathbf{b} = (b_1,b_2,b_3)$, written $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. This 'dot' notation for scalar products applies also in the general n-dimensional context, for the scalar (or inner) product $\boldsymbol{\alpha} \bullet \boldsymbol{\xi}$ of an abstract covector $\boldsymbol{\alpha}$ with a vector $\boldsymbol{\xi}$.

A notational complication arises with quantum mechanics, however, since physical quantities, in that subject, tend to be represented as linear operators. I do not adopt what is a quite standard procedure in this context, of putting 'hats' (circumflexes) on the letters representing the quantum-operator versions of the familiar classical quantities, as I believe that this leads to an unnecessary cluttering of symbols. (Instead, I shall tend to adopt a philosophical standpoint that the classical and quantum entities are really the 'same'—and so it is fair to use the same symbols for each—except that in the classical case one is justified in ignoring quantities of the order of $\hbar$, so that the classical commutation properties $ab = ba$ can hold, whereas in quantum mechanics, ab might differ from ba by something of order $\hbar$.) For consistency with the above, such linear operators would seem to have to be denoted by italic bold letters (like $\boldsymbol{T}$), but that would nullify the philosophy and the distinctions called for in the preceding paragraph. Accordingly, with regard to specific quantities, such as the momentum $\mathbf{p}$ or $\boldsymbol{p}$, or the position $\mathbf{x}$ or $\boldsymbol{x}$, I shall tend to use the same notation as in the classical case, in line with what has been said earlier in this paragraph. But for less specific quantum operators, bold italic letters such as $\boldsymbol{Q}$ will tend to be used.

The shell letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{F}_q$, respectively, for the system of natural numbers (i.e. non-negative integers), integers, real numbers, complex numbers, and the finite field with q elements (q being some power of a prime number, see §16.1), are now standard in mathematics, as are the corresponding $\mathbb{N}^n$, $\mathbb{Z}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{F}_q^n$, for the systems of ordered n-tuples of such numbers. These are canonical mathematical entities in standard use. In this book (as is not all that uncommon), this notation is extended to some other standard mathematical structures such as Euclidean 3-space $\mathbb{E}^3$ or, more generally, Euclidean *n*-space $\mathbb{E}^n$. In frequent use in this book is the standard flat 4-dimensional Minkowski spacetime, which is itself a kind of 'pseudo-' Euclidean space, so I use the shell letter $\mathbb{M}$ for this space (with $\mathbb{M}^n$ to denote the *n*-dimensional version—a 'Lorentzian' spacetime with 1 time and $(n-1)$ space dimensions). Sometimes I use $\mathbb{C}$ as an adjective, to denote 'complexified', so that we might consider the complex Euclidean 4-space, for example, denoted by $\mathbb{CE}^n$. The shell letter $\mathbb{P}$ can also be used as an adjective, to denote 'projective' (see §15.6), or as a noun, with $\mathbb{P}^n$ denoting projective *n*-space (or I use $\mathbb{RP}^n$ or $\mathbb{CP}^n$ if it is to be made clear that we are concerned with real or complex projective *n*-space, respectively). In twistor theory (Chapter 33), there is the complex 4-space $\mathbb{T}$, which is related to $\mathbb{M}$ (or its complexification $\mathbb{CM}$) in a canonical way, and there is also the projective version $\mathbb{PT}$. In this theory, there is also a space $\mathbb{N}$ of *null* twistors (the double duty that this letter serves causing no conflict here), and its projective version $\mathbb{PN}$.

The adjectival role of the shell letter $\mathbb{C}$ should not be confused with that of the lightface sans serif C, which here stands for 'complex conjugate of' (as used in §13.1,2). This is basically similar to another use of C in particle physics, namely *charge conjugation*, which is the operation which interchanges each particle with its antiparticle (see Chapters 24, 25). This operation is usually considered in conjunction with two other basic particle-physics operations, namely P for *parity* which refers to the operation of reflection in a mirror, and T, which refers to *time-reveral*. Sans serif letters which are bold serve a different purpose here, labelling *vector spaces*, the letters $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{H}$, being most frequently used for this purpose. The use of $\mathbf{H}$, is specific to the Hilbert spaces of quantum mechanics, and $\mathbf{H}^n$ would stand for a Hilbert space of *n* complex dimensions. Vector spaces are, in a clear sense, flat. Spaces which are (or could be) *curved* are denoted by script letters, such as $\mathcal{M}$, $\mathcal{S}$, or $\mathcal{T}$, where there is a special use for the particular script font $\mathscr{I}$ to denote *null infinity*. In addition, I follow a fairly common convention to use script letters for Lagrangians ($\mathcal{L}$) and Hamiltonians ($\mathcal{H}$), in view of their very special status in physical theory.

Prologue

AM-TEP was the King's chief craftsman, an artist of consummate skills. It was night, and he lay sleeping on his workshop couch, tired after a handsomely productive evening's work. But his sleep was restless—perhaps from an intangible tension that had seemed to be in the air. Indeed, he was not certain that he was asleep at all when it happened. Daytime had come—quite suddenly—when his bones told him that surely it must still be night.

He stood up abruptly. Something was odd. The dawn's light could not be in the north; yet the red light shone alarmingly through his broad window that looked out northwards over the sea. He moved to the window and stared out, incredulous in amazement. The Sun had never before risen in the north! In his dazed state, it took him a few moments to realize that this could not possibly be the Sun. It was a distant shaft of a deep fiery red light that beamed vertically upwards from the water into the heavens.

As he stood there, a dark cloud became apparent at the head of the beam, giving the whole structure the appearance of a distant giant parasol, glowing evilly, with a smoky flaming staff. The parasol's hood began to spread and darken—a daemon from the underworld. The night had been clear, but now the stars disappeared one by one, swallowed up behind this advancing monstrous creature from Hell.

Though terror must have been his natural reaction, he did not move, transfixed for several minutes by the scene's perfect symmetry and awesome beauty. But then the terrible cloud began to bend slightly to the east, caught up by the prevailing winds. Perhaps he gained some comfort from this and the spell was momentarily broken. But apprehension at once returned to him as he seemed to sense a strange disturbance in the ground beneath, accompanied by ominous-sounding rumblings of a nature quite unfamiliar to him. He began to wonder what it was that could have caused this fury. Never before had he witnessed a God's anger of such magnitude.

His first reaction was to blame himself for the design on the sacrificial cup that he had just completed—he had worried about it at the time. Had his depiction of the Bull-God not been sufficiently fearsome? Had that god been offended? But the absurdity of this thought soon struck him. The fury he had just witnessed could not have been the result of such a trivial action, and was surely not aimed at him specifically. But he knew that there would be trouble at the Great Palace. The Priest-King would waste no time in attempting to appease this Daemon-God. There would be sacrifices. The traditional offerings of fruits or even animals would not suffice to pacify an anger of this magnitude. The sacrifices would have to be human.

Quite suddenly, and to his utter surprise, he was blown backwards across the room by an impulsive blast of air followed by a violent wind. The noise was so extreme that he was momentarily deafened. Many of his beautifully adorned pots were whisked from their shelves and smashed to pieces against the wall behind. As he lay on the floor in a far corner of the room where he had been swept away by the blast, he began to recover his senses, and saw that the room was in turmoil. He was horrified to see one of his favourite great urns shattered to small pieces, and the wonderfully detailed designs, which he had so carefully crafted, reduced to nothing.

Am-tep arose unsteadily from the floor and after a while again approached the window, this time with considerable trepidation, to re-examine that terrible scene across the sea. Now he thought he saw a disturbance, illuminated by that far-off furnace, coming towards him. This appeared to be a vast trough in the water, moving rapidly towards the shore, followed by a clifflike wall of wave. He again became transfixed, watching the approaching wave begin to acquire gigantic proportions. Eventually the disturbance reached the shore and the sea immediately before him drained away, leaving many ships stranded on the newly formed beach. Then the cliff-wave entered the vacated region and struck with a terrible violence. Without exception the ships were shattered, and many nearby houses instantly destroyed. Though the water rose to great heights in the air before him, his own house was spared, for it sat on high ground a good way from the sea.

The Great Palace too was spared. But Am-tep feared that worse might come, and he was right—though he knew not how right he was. He did know, however, that no ordinary human sacrifice of a slave could now be sufficient. Something more would be needed to pacify the tempestuous anger of this terrible God. His thoughts turned to his sons and daughters, and to his newly born grandson. Even they might not be safe.

Am-tep had been right to fear new human sacrifices. A young girl and a youth of good birth had been soon apprehended and taken to a nearby temple, high on the slopes of a mountain. The ensuing ritual was well under way when yet another catastrophe struck. The ground shook with devastating violence, whence the temple roof fell in, instantly killing all the priests and their intended sacrificial victims. As it happened, they would lie there in mid-ritual—entombed for over three-and-a-half millennia!

The devastation was frightful, but not final. Many on the island where Am-tep and his people lived survived the terrible earthquake, though the Great Palace was itself almost totally destroyed. Much would be rebuilt over the years. Even the Palace would recover much of its original splendour, constructed on the ruins of the old. Yet Am-tep had vowed to leave the island. His world had now changed irreparably.

In the world he knew, there had been a thousand years of peace, prosperity, and culture where the Earth-Goddess had reigned. Wonderful art had been allowed to flourish. There was much trade with neighbouring lands. The magnificent Great Palace was a huge luxurious labyrinth, a virtual city in itself, adorned by superb frescoes of animals and flowers. There was running water, excellent drainage, and flushed sewers. War was almost unknown and defences unnecessary. Now, Am-tep perceived the Earth-Goddess overthrown by a Being with entirely different values.

It was some years before Am-tep actually left the island, accompanied by his surviving family, on a ship rebuilt by his youngest son, who was a skilled carpenter and seaman. Am-tep's grandson had developed into an alert child, with an interest in everything in the world around. The voyage took some days, but the weather had been supremely calm. One clear night, Am-tep was explaining to his grandson about the patterns in the stars, when an odd thought overtook him: *The patterns of stars had been disturbed not one iota from what they were before the Catastrophe of the emergence of the terrible daemon.*

Am-tep knew these patterns well, for he had a keen artist's eye. Surely, he thought, those tiny candles of light in the sky should have been blown at least a little from their positions by the violence of that night, just as his pots had been smashed and his great urn shattered. The Moon also had kept her face, just as before, and her route across the star-filled heavens had changed not one whit, as far as Am-tep could tell. For many moons after the Catastrophe, the skies had appeared different. There had been darkness and strange clouds, and the Moon and Sun had sometimes worn unusual colours. But this had now passed, and their motions seemed utterly undisturbed. The tiny stars, likewise, had been quite unmoved.

If the heavens had shown such little concern for the Catastrophe, having a stature far greater even than that terrible Daemon, Am-tep reasoned, why should the forces controlling the Daemon itself show concern for what the little people on the island had been doing, with their foolish rituals and human sacrifice? He felt embarrassed by his *own* foolish thoughts at the time, that the daemon might be concerned by the mere patterns on his pots.

Yet Am-tep was still troubled by the question ‘why?’ What deep forces control the behaviour of the world, and why do they sometimes burst forth in violent and seemingly incomprehensible ways? He shared his questions with his grandson, but there were no answers.

. . .

A century passed by, and then a millennium, and still there were no answers.

. . .

Amphos the craftsman had lived all his life in the same small town as his father and his father before him, and his father’s father before that. He made his living constructing beautifully decorated gold bracelets, earrings, ceremonial cups, and other fine products of his artistic skills. Such work had been the family trade for some forty generations—a line unbroken since Am-tep had settled there eleven hundred years before.

But it was not just artistic skills that had been passed down from generation to generation. Am-tep’s questions troubled Amphos just as they had troubled Am-tep earlier. The great story of the Catastrophe that destroyed an ancient peaceful civilization had been handed down from father to son. Am-tep’s perception of the Catastrophe had also survived with his descendants. Amphos, too, understood that the heavens had a magnitude and stature so great as to be quite unconcerned by that terrible event. Nevertheless, the event had had a catastrophic effect on the little people with their cities and their human sacrifices and insignificant religious rituals. Thus, by comparison, the event itself must have been the result of enormous forces quite unconcerned by those trivial actions of human beings. Yet the nature of those forces was as unknown in Amphos’s day as it was to Am-tep.

Amphos had studied the structure of plants, insects and other small animals, and crystalline rocks. His keen eye for observation had served him well in his decorative designs. He took an interest in agriculture and was fascinated by the growth of wheat and other plants from grain. But none of this told him ‘why?’, and he felt unsatisfied. He believed that there was indeed reason underlying Nature’s patterns, but he was in no way equipped to unravel those reasons.

One clear night, Amphos looked up at the heavens, and tried to make out from the patterns of stars the shapes of those heroes and heroines who formed constellations in the sky. To his humble artist’s eye, those shapes made poor resemblances. He could himself have arranged the stars far more convincingly. He puzzled over why the gods had not organized the stars in a more appropriate way? As they were, the arrangements seemed more like scattered grains randomly sowed by a farmer, rather than the deliberate design of a god. Then an odd thought overtook him: *Do not seek for reasons in the specific patterns of stars, or of other scattered arrangements of objects; look, instead, for a deeper universal order in the way that things behave.*

Amphos reasoned that we find order, after all, not in the patterns that scattered seeds form when they fall to the ground, but in the miraculous way that each of those seeds develops into a living plant having a superb structure, similar in great detail to one another. We would not try to seek the meaning in the precise arrangement of seeds sprinkled on the soil; yet, there must be meaning in the hidden mystery of the inner forces controlling the growth of each seed individually, so that each one follows essentially the same wonderful course. Nature's laws must indeed have a superbly organized precision for this to be possible.

Amphos became convinced that without precision in the underlying laws, there could be no order in the world, whereas much order is indeed perceived in the way that things behave. Moreover, there must be precision in our ways of thinking about these matters if we are not to be led seriously astray.

It so happened that word had reached Amphos of a sage who lived in another part of the land, and whose beliefs appeared to be in sympathy with those of Amphos. According to this sage, one could not rely on the teachings and traditions of the past. To be certain of one's beliefs, it was necessary to form precise conclusions by the use of unchallengeable reason. The nature of this precision had to be mathematical—ultimately dependent on the notion of *number* and its application to geometric forms. Accordingly, it must be number and geometry, not myth and superstition, that governed the behaviour of the world.

As Am-tep had done a century and a millennium before, Amphos took to the sea. He found his way to the city of Croton, where the sage and his brotherhood of 571 wise men and 28 wise women were in search of truth. After some time, Amphos was accepted into the brotherhood. The name of the sage was *Pythagoras*.

1
The roots of science

1.1 The quest for the forces that shape the world

WHAT laws govern our universe? How shall we know them? How may this knowledge help us to comprehend the world and hence guide its actions to our advantage?

Since the dawn of humanity, people have been deeply concerned by questions like these. At first, they had tried to make sense of those influences that do control the world by referring to the kind of understanding that was available from their own lives. They had imagined that whatever or whoever it was that controlled their surroundings would do so as they would themselves strive to control things: originally they had considered their destiny to be under the influence of beings acting very much in accordance with their own various familiar human drives. Such driving forces might be pride, love, ambition, anger, fear, revenge, passion, retribution, loyalty, or artistry. Accordingly, the course of natural events—such as sunshine, rain, storms, famine, illness, or pestilence—was to be understood in terms of the whims of gods or goddesses motivated by such human urges. And the only action perceived as influencing these events would be appeasement of the god-figures.

But gradually patterns of a different kind began to establish their reliability. The precision of the Sun's motion through the sky and its clear relation to the alternation of day with night provided the most obvious example; but also the Sun's positioning in relation to the heavenly orb of stars was seen to be closely associated with the change and relentless regularity of the seasons, and with the attendant clear-cut influence on the weather, and consequently on vegetation and animal behaviour. The motion of the Moon, also, appeared to be tightly controlled, and its phases determined by its geometrical relation to the Sun. At those locations on Earth where open oceans meet land, the tides were noticed to have a regularity closely governed by the position (and phase) of the Moon. Eventually, even the much more complicated apparent motions of the planets began to yield up their secrets, revealing an immense underlying precision and regularity. If the heavens were indeed controlled by the whims of gods, then these gods themselves seemed under the spell of exact mathematical laws.

Likewise, the laws controlling earthly phenomena—such as the daily and yearly changes in temperature, the ebb and flow of the oceans, and the growth of plants—being seen to be influenced by the heavens in this respect at least, shared the mathematical regularity that appeared to guide the gods. But this kind of relationship between heavenly bodies and earthly behaviour would sometimes be exaggerated or misunderstood and would assume an inappropriate importance, leading to the occult and mystical connotations of astrology. It took many centuries before the rigour of scientific understanding enabled the true influences of the heavens to be disentangled from purely suppositional and mystical ones. Yet it had been clear from the earliest times that such influences did indeed exist and that, accordingly, the mathematical laws of the heavens must have relevance also here on Earth.

Seemingly independently of this, there were perceived to be other regularities in the behaviour of earthly objects. One of these was the tendency for all things in one vicinity to move in the same downward direction, according to the influence that we now call *gravity*. Matter was observed to transform, sometimes, from one form into another, such as with the melting of ice or the dissolving of salt, but the total quantity of that matter appeared never to change, which reflects the law that we now refer to as *conservation of mass*. In addition, it was noticed that there are many material bodies with the important property that they retain their shapes, whence the idea of rigid spatial motion arose; and it became possible to understand spatial relationships in terms of a precise, well-defined geometry—the 3-dimensional geometry that we now call *Euclidean*. Moreover, the notion of a 'straight line' in this geometry turned out to be the same as that provided by rays of light (or lines of sight). There was a remarkable precision and beauty to these ideas, which held a considerable fascination for the ancients, just as it does for us today.

Yet, with regard to our everyday lives, the implications of this mathematical precision for the actions of the world often appeared unexciting and limited, despite the fact that the mathematics itself seemed to represent a deep truth. Accordingly, many people in ancient times would allow their imaginations to be carried away by their fascination with the subject and to take them far beyond the scope of what was appropriate. In astrology, for example, geometrical figures also often engendered mystical and occult connotations, such as with the supposed magical powers of pentagrams and heptagrams. And there was an entirely suppositional attempted association between Platonic solids and the basic elementary states of matter (see Fig. 1.1). It would not be for many centuries that the deeper understanding that we presently have, concerning the actual

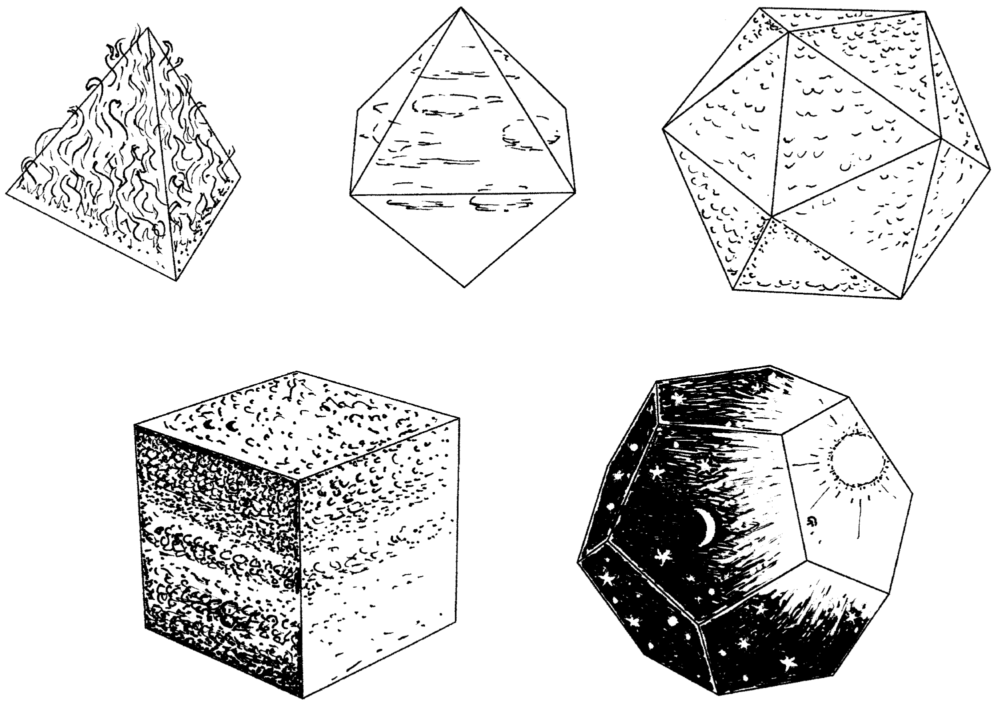


Fig. 1.1 A fanciful association, made by the ancient Greeks, between the five Platonic solids and the four 'elements' (fire, air, water, and earth), together with the heavenly firmament represented by the dodecahedron.

relationships between mass, gravity, geometry, planetary motion, and the behaviour of light, could come about.

1.2 Mathematical truth

The first steps towards an understanding of the real influences controlling Nature required a disentangling of the true from the purely suppositional. But the ancients needed to achieve something else first, before they would be in any position to do this reliably for their understanding of Nature. What they had to do first was to discover how to disentangle the true from the suppositional in *mathematics*. A procedure was required for telling whether a given mathematical assertion is or is not to be trusted as true. Until that preliminary issue could be settled in a reasonable way, there would be little hope of seriously addressing those more difficult problems concerning forces that control the behaviour of the world and whatever their relations might be to mathematical truth. This realization that the key to the understanding of Nature lay within an unassailable mathematics was perhaps the first major breakthrough in science.

Although mathematical truths of various kinds had been surmised since ancient Egyptian and Babylonian times, it was not until the great Greek philosophers Thales of Miletus (*c*.625–547 BC) and

Pythagoras[1*] of Samos (*c*.572–497 BC) began to introduce the notion of *mathematical proof* that the first firm foundation stone of mathematical understanding—and therefore of science itself—was laid. Thales may have been the first to introduce this notion of proof, but it seems to have been the Pythagoreans who first made important use of it to establish things that were not otherwise obvious. Pythagoras also appeared to have a strong vision of the importance of *number*, and of arithmetical concepts, in governing the actions of the physical world. It is said that a big factor in this realization was his noticing that the most beautiful harmonies produced by lyres or flutes corresponded to the simplest fractional ratios between the lengths of vibrating strings or pipes. He is said to have introduced the 'Pythagorean scale', the numerical ratios of what we now know to be frequencies determining the principal intervals on which Western music is essentially based.[2] The famous *Pythagorean theorem*, asserting that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides, perhaps more than anything else, showed that indeed there is a precise relationship between the arithmetic of numbers and the geometry of physical space (see Chapter 2).

He had a considerable band of followers—the *Pythagoreans*—situated in the city of Croton, in what is now southern Italy, but their influence on the outside world was hindered by the fact that the members of the Pythagorean brotherhood were all sworn to secrecy. Accordingly, almost all of their detailed conclusions have been lost. Nonetheless, some of these conclusions were leaked out, with unfortunate consequences for the 'moles'—on at least one occasion, death by drowning!

In the long run, the influence of the Pythagoreans on the progress of human thought has been enormous. For the first time, with mathematical proof, it was possible to make significant assertions of an unassailable nature, so that they would hold just as true even today as at the time that they were made, no matter how our knowledge of the world has progressed since then. The truly timeless nature of mathematics was beginning to be revealed.

But what is a mathematical proof? A proof, in mathematics, is an impeccable argument, using only the methods of pure logical reasoning, which enables one to infer the validity of a given mathematical assertion from the pre-established validity of other mathematical assertions, or from some particular primitive assertions—the *axioms*—whose validity is taken to be self-evident. Once such a mathematical assertion has been established in this way, it is referred to as a *theorem*.

Many of the theorems that the Pythagoreans were concerned with were geometrical in nature; others were assertions simply about numbers. Those

*Notes, indicated in the text by superscript numbers, are gathered at the ends of the chapter (in this case on p. 23).

that were concerned merely with numbers have a perfectly unambiguous validity today, just as they did in the time of Pythagoras. What about the *geometrical* theorems that the Pythagoreans had obtained using their procedures of mathematical proof? They too have a clear validity today, but now there is a complicating issue. It is an issue whose nature is more obvious to us from our modern vantage point than it was at that time of Pythagoras. The ancients knew of only one kind of geometry, namely that which we now refer to as *Euclidean geometry*, but now we know of many other types. Thus, in considering the geometrical theorems of ancient Greek times, it becomes important to specify that the notion of geometry being referred to is indeed Euclid's geometry. (I shall be more explicit about these issues in §2.4, where an important example of non-Euclidean geometry will be given.)

Euclidean geometry is a specific mathematical structure, with its own specific axioms (including some less assured assertions referred to as postulates), which provided an excellent approximation to a particular aspect of the physical world. That was the aspect of reality, well familiar to the ancient Greeks, which referred to the laws governing the geometry of rigid objects and their relations to other rigid objects, as they are moved around in 3-dimensional space. Certain of these properties were so familiar and self-consistent that they tended to become regarded as 'self-evident' mathematical truths and were taken as axioms (or postulates). As we shall be seeing in Chapters 17–19 and §§27.8,11, Einstein's general relativity—and even the Minkowskian spacetime of special relativity—provides geometries for the physical universe that are different from, and yet more accurate than, the geometry of Euclid, despite the fact that the Euclidean geometry of the ancients was already extraordinarily accurate. Thus, we must be careful, when considering geometrical assertions, whether to trust the 'axioms' as being, in any sense, actually *true*.

But what does 'true' mean, in this context? The difficulty was well appreciated by the great ancient Greek philosopher Plato, who lived in Athens from *c*.429 to 347 BC, about a century after Pythagoras. Plato made it clear that the mathematical propositions—the things that could be regarded as unassailably true—referred not to actual physical objects (like the approximate squares, triangles, circles, spheres, and cubes that might be constructed from marks in the sand, or from wood or stone) but to certain idealized entities. He envisaged that these ideal entities inhabited a different world, distinct from the physical world. Today, we might refer to this world as the *Platonic world of mathematical forms*. Physical structures, such as squares, circles, or triangles cut from papyrus, or marked on a flat surface, or perhaps cubes, tetrahedra, or spheres carved from marble, might conform to these ideals very closely, but only approximately. The actual *mathematical* squares, cubes, circles, spheres, triangles, etc., would not be part of the physical world, but would be inhabitants of Plato's idealized mathematical world of forms.

1.3 Is Plato's mathematical world 'real'?

This was an extraordinary idea for its time, and it has turned out to be a very powerful one. But does the Platonic mathematical world actually exist, in any meaningful sense? Many people, including philosophers, might regard such a 'world' as a complete fiction—a product merely of our unrestrained imaginations. Yet the Platonic viewpoint is indeed an immensely valuable one. It tells us to be careful to distinguish the precise mathematical entities from the approximations that we see around us in the world of physical things. Moreover, it provides us with the blueprint according to which modern science has proceeded ever since. Scientists will put forward models of the world—or, rather, of certain aspects of the world—and these models may be tested against previous observation and against the results of carefully designed experiment. The models are deemed to be appropriate if they survive such rigorous examination and if, in addition, they are internally consistent structures. The important point about these models, for our present discussion, is that they are basically purely abstract *mathematical* models. The very question of the internal consistency of a scientific model, in particular, is one that requires that the model be precisely specified. The required precision demands that the model be a mathematical one, for otherwise one cannot be sure that these questions have well-defined answers.

If the model itself is to be assigned any kind of 'existence', then this existence is located within the Platonic world of mathematical forms. Of course, one might take a contrary viewpoint: namely that the model is itself to have existence only within our various *minds*, rather than to take Plato's world to be in any sense absolute and 'real'. Yet, there is something important to be gained in regarding mathematical structures as having a reality of their own. For our individual minds are notoriously imprecise, unreliable, and inconsistent in their judgements. The precision, reliability, and consistency that are required by our scientific theories demand something beyond any one of our individual (untrustworthy) minds. In mathematics, we find a far greater robustness than can be located in any particular mind. Does this not point to something outside ourselves, with a reality that lies beyond what each individual can achieve?

Nevertheless, one might still take the alternative view that the mathematical world has no independent existence, and consists merely of certain ideas which have been distilled from our various minds and which have been found to be totally trustworthy and are agreed by all.

Yet even this viewpoint seems to leave us far short of what is required. Do we mean 'agreed by all', for example, or 'agreed by those who are in their right minds', or 'agreed by all those who have a Ph.D. in mathematics' (not much use in Plato's day) and who have a right to venture an 'authoritative' opinion? There seems to be a danger of circularity here; for to judge whether or not someone is 'in his or her right mind' requires some external standard. So also does the meaning of 'authoritative', unless some standard of an unscientific nature such as 'majority opinion' were to be adopted (and it should be made clear that majority opinion, no matter how important it may be for democratic government, should in no way be used as the criterion for scientific acceptability). Mathematics itself indeed seems to have a robustness that goes far beyond what any individual mathematician is capable of perceiving. Those who work in this subject, whether they are actively engaged in mathematical research or just using results that have been obtained by others, usually feel that they are merely explorers in a world that lies far beyond themselves—a world which possesses an objectivity that transcends mere opinion, be that opinion their own or the surmise of others, no matter how expert those others might be.

It may be helpful if I put the case for the actual existence of the Platonic world in a different form. What I mean by this 'existence' is really just the objectivity of mathematical truth. Platonic existence, as I see it, refers to the existence of an objective external standard that is not dependent upon our individual opinions nor upon our particular culture. Such 'existence' could also refer to things other than mathematics, such as to morality or aesthetics (cf. §1.5), but I am here concerned just with mathematical objectivity, which seems to be a much clearer issue.

Let me illustrate this issue by considering one famous example of a mathematical truth, and relate it to the question of 'objectivity'. In 1637, Pierre de Fermat made his famous assertion now known as 'Fermat's Last Theorem' (that no positive nth power[3] of an integer, i.e. of a whole number, can be the sum of two other positive nth powers if n is an integer greater than 2), which he wrote down in the margin of his copy of the *Arithmetica*, a book written by the 3rd-century Greek mathematician Diophantos. In this margin, Fermat also noted: 'I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.' Fermat's mathematical assertion remained unconfirmed for over 350 years, despite concerted efforts by numerous outstanding mathematicians. A proof was finally published in 1995 by Andrew Wiles (depending on the earlier work of various other mathematicians), and this proof has now been accepted as a valid argument by the mathematical community.

Now, do we take the view that Fermat's assertion was always true, long before Fermat actually made it, or is its validity a purely cultural matter, dependent upon whatever might be the subjective standards of the community of human mathematicians? Let us try to suppose that the validity of the Fermat assertion is in fact a subjective matter. Then it would not be an absurdity for some other mathematician X to have come up with an actual and specific counter-example to the Fermat assertion, so long as X had done this before the date of 1995.[4] In such a circumstance, the mathematical community would have to accept the correctness of X's counter-example. From then on, any effort on the part of Wiles to prove the Fermat assertion would have to be fruitless, for the reason that X had got his argument in first and, as a result, the Fermat assertion would now be false! Moreover, we could ask the further question as to whether, consequent upon the correctness of X's forthcoming counter-example, Fermat himself would necessarily have been mistaken in believing in the soundness of his 'truly marvellous proof', at the time that he wrote his marginal note. On the subjective view of mathematical truth, it could possibly have been the case that Fermat had a valid proof (which would have been accepted as such by his peers at the time, had he revealed it) and that it was Fermat's secretiveness that allowed the possibility of X later obtaining a counter-example! I think that virtually all mathematicians, irrespective of their professed attitudes to 'Platonism', would regard such possibilities as patently absurd.

Of course, it might still be the case that Wiles's argument in fact contains an error and that the Fermat assertion is indeed false. Or there could be a fundamental error in Wiles's argument but the Fermat assertion is true nevertheless. Or it might be that Wiles's argument is correct in its essentials while containing 'non-rigorous steps' that would not be up to the standard of some future rules of mathematical acceptability. But these issues do not address the point that I am getting at here. The issue is the objectivity of the Fermat assertion itself, not whether anyone's particular demonstration of it (or of its negation) might happen to be convincing to the mathematical community of any particular time.

It should perhaps be mentioned that, from the point of view of mathematical logic, the Fermat assertion is actually a mathematical statement of a particularly simple kind,[5] whose objectivity is especially apparent. Only a tiny minority[6] of mathematicians would regard the truth of such assertions as being in any way 'subjective'—although there might be some subjectivity about the types of argument that would be regarded as being convincing. However, there are other kinds of mathematical assertion whose truth could plausibly be regarded as being a 'matter of opinion'. Perhaps the best known of such assertions is the *axiom of choice*. It is not important for us, now, to know what the axiom of choice is. (I shall describe it in §16.3.) It is cited here only as an example. Most mathematicians would probably regard the axiom of choice as 'obviously true', while others may regard it as a somewhat questionable assertion which might even be false (and I am myself inclined, to some extent, towards this second viewpoint). Still others would take it as an assertion whose 'truth' is a mere matter of opinion or, rather, as something which can be taken one way or the other, depending upon which system of axioms and rules of procedure (a 'formal system'; see §16.6) one chooses to adhere to. Mathematicians who support this final viewpoint (but who accept the objectivity of the truth of particularly clear-cut mathematical statements, like the Fermat assertion discussed above) would be relatively weak Platonists. Those who adhere to objectivity with regard to the truth of the axiom of choice would be stronger Platonists.

I shall come back to the axiom of choice in §16.3, since it has some relevance to the mathematics underlying the behaviour of the physical world, despite the fact that it is not addressed much in physical theory. For the moment, it will be appropriate not to worry overly about this issue. If the axiom of choice can be settled one way or the other by some appropriate form of unassailable mathematical reasoning,[7] then its truth is indeed an entirely objective matter, and either it belongs to the Platonic world or its negation does, in the sense that I am interpreting this term 'Platonic world'. If the axiom of choice is, on the other hand, a mere matter of opinion or of arbitrary decision, then the Platonic world of absolute mathematical forms contains neither the axiom of choice nor its negation (although it could contain assertions of the form 'such-and-such follows from the axiom of choice' or 'the axiom of choice is a theorem according to the rules of such-and-such mathematical system').

The mathematical assertions that can belong to Plato's world are precisely those that are objectively true. Indeed, I would regard mathematical objectivity as really what mathematical Platonism is all about. To say that some mathematical assertion has a Platonic existence is merely to say that it is true in an objective sense. A similar comment applies to mathematical *notions*—such as the concept of the number 7, for example, or the rule of multiplication of integers, or the idea that some set contains infinitely many elements—all of which have a Platonic existence because they are objective notions. To my way of thinking, Platonic existence is simply a matter of objectivity and, accordingly, should certainly not be viewed as something 'mystical' or 'unscientific', despite the fact that some people regard it that way.

As with the axiom of choice, however, questions as to whether some particular proposal for a mathematical entity is or is not to be regarded as having objective existence can be delicate and sometimes technical. Despite this, we certainly need not be mathematicians to appreciate the general robustness of many mathematical concepts. In Fig. 1.2, I have depicted various small portions of that famous mathematical entity known

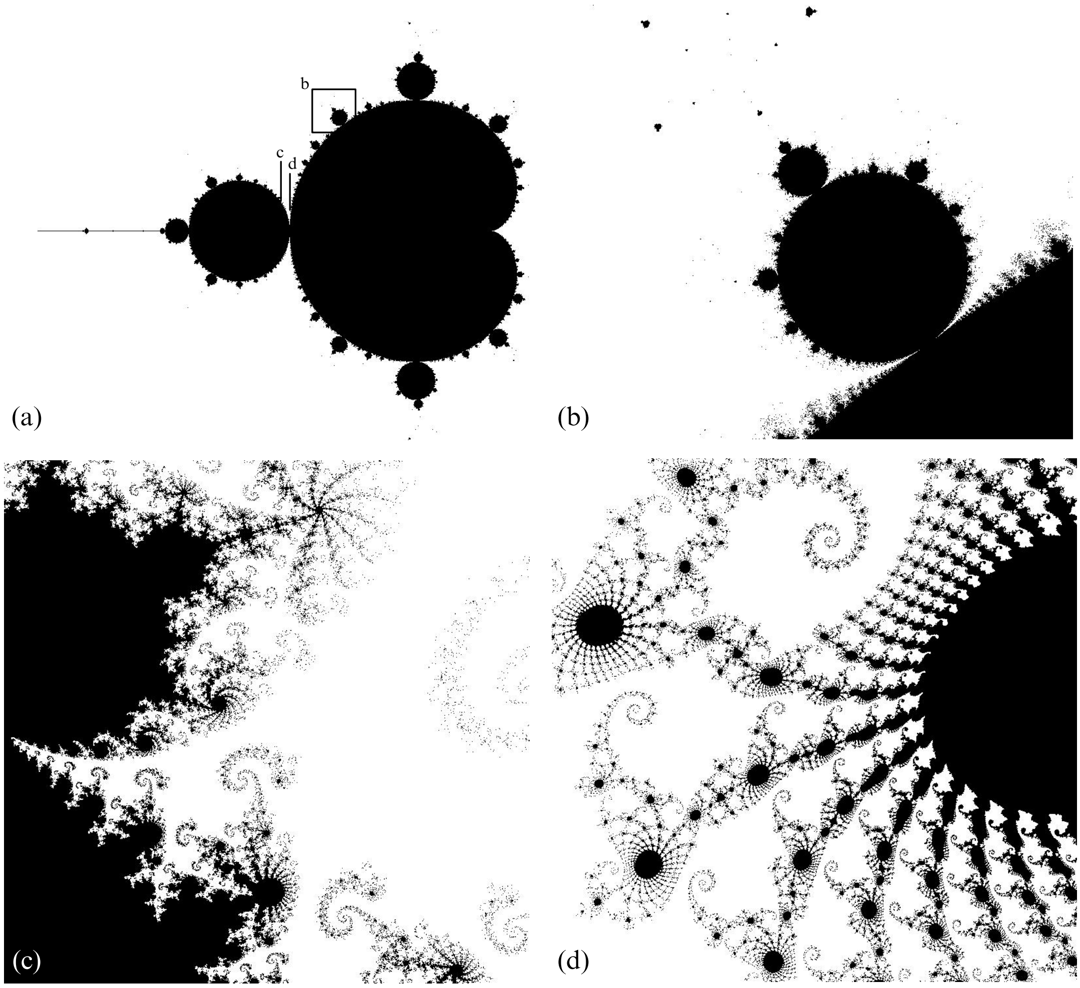


Fig. 1.2 (a) The Mandelbrot set. (b), (c), and (d) Some details, illustrating blow-ups of those regions correspondingly marked in Fig. 1.2a, magnified by respective linear factors 11.6, 168.9, and 1042.

as the *Mandelbrot set*. The set has an extraordinarily elaborate structure, but it is not of any human design. Remarkably, this structure is defined by a mathematical rule of particular simplicity. We shall come to this explicitly in §4.5, but it would distract us from our present purposes if I were to try to provide this rule in detail now.

The point that I wish to make is that no one, not even Benoit Mandelbrot himself when he first caught sight of the incredible complications in the fine details of the set, had any real preconception of the set's extraordinary richness. The Mandelbrot set was certainly no invention of any human mind. The set is just objectively there in the mathematics itself. If it has meaning to assign an actual existence to the Mandelbrot set, then that existence is not within our minds, for no one can fully comprehend the set's endless variety and unlimited complication. Nor can its existence lie within the multitude of computer printouts that begin to capture some of its incredible sophistication and detail, for at best those printouts capture but a shadow of an approximation to the set itself. Yet it has a robustness that is beyond any doubt; for the same structure is revealed—in all its perceivable details, to greater and greater fineness the more closely it is examined—independently of the mathematician or computer that examines it. Its existence can only be within the Platonic world of mathematical forms.

I am aware that there will still be many readers who find difficulty with assigning any kind of actual existence to mathematical structures. Let me make the request of such readers that they merely broaden their notion of what the term 'existence' can mean to them. The mathematical forms of Plato's world clearly do not have the same kind of existence as do ordinary physical objects such as tables and chairs. They do not have spatial locations; nor do they exist in time. Objective mathematical notions must be thought of as timeless entities and are not to be regarded as being conjured into existence at the moment that they are first humanly perceived. The particular swirls of the Mandelbrot set that are depicted in Fig. 1.2c or 1.2d did not attain their existence at the moment that they were first seen on a computer screen or printout. Nor did they come about when the general idea behind the Mandelbrot set was first humanly put forth—not actually first by Mandelbrot, as it happened, but by R. Brooks and J. P. Matelski, in 1981, or perhaps earlier. For certainly neither Brooks nor Matelski, nor initially even Mandelbrot himself, had any real conception of the elaborate detailed designs that we see in Fig. 1.2c and 1.2d. Those designs were already 'in existence' since the beginning of time, in the potential timeless sense that they would necessarily be revealed precisely in the form that we perceive them today, no matter at what time or in what location some perceiving being might have chosen to examine them.

1.4 Three worlds and three deep mysteries

Thus, mathematical existence is different not only from physical existence but also from an existence that is assigned by our mental perceptions. Yet there is a deep and mysterious connection with each of those other two forms of existence: the physical and the mental. In Fig. 1.3, I have schematically indicated all of these three forms of existence—the physical, the mental, and the Platonic mathematical—as entities belonging to three separate 'worlds', drawn schematically as spheres. The mysterious connections between the worlds are also indicated, where in drawing the diagram

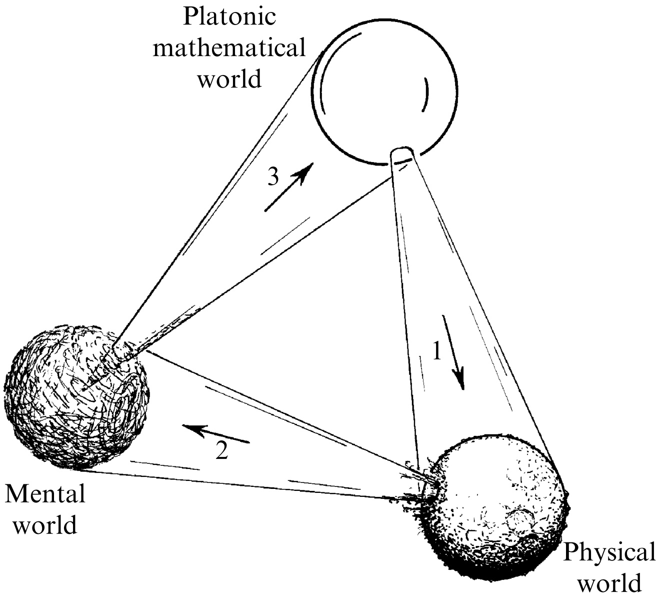


Fig. 1.3 Three 'worlds'—the Platonic mathematical, the physical, and the mental—and the three profound mysteries in the connections between them.

I have imposed upon the reader some of my beliefs, or prejudices, concerning these mysteries.

It may be noted, with regard to the *first* of these mysteries—relating the Platonic mathematical world to the physical world—that I am allowing that only a small part of the world of mathematics need have relevance to the workings of the physical world. It is certainly the case that the vast preponderance of the activities of pure mathematicians today has no obvious connection with physics, nor with any other science (cf. §34.9), although we may be frequently surprised by unexpected important applications. Likewise, in relation to the *second* mystery, whereby mentality comes about in association with certain physical structures (most specifically, healthy, wakeful human brains), I am not insisting that the majority of physical structures need induce mentality. While the brain of a cat may indeed evoke mental qualities, I am not requiring the same for a rock. Finally, for the *third* mystery, I regard it as self-evident that only a small fraction of our mental activity need be concerned with absolute mathematical truth! (More likely we are concerned with the multifarious irritations, pleasures, worries, excitements, and the like, that fill our daily lives.) These three facts are represented in the smallness of the base of the connection of each world with the next, the worlds being taken in a clockwise sense in the diagram. However, it is in the encompassing of each entire world within the scope of its connection with the world preceding it that I am revealing my prejudices.

Thus, according to Fig. 1.3, the entire physical world is depicted as being governed according to mathematical laws. We shall be seeing in later chapters that there is powerful (but incomplete) evidence in support of this contention. On this view, everything in the physical universe is indeed governed in completely precise detail by mathematical principles—perhaps by equations, such as those we shall be learning about in chapters to follow, or perhaps by some future mathematical notions fundamentally different from those which we would today label by the term 'equations'. If this is right, then even our own physical actions would be entirely subject to such ultimate mathematical control, where 'control' might still allow for some random behaviour governed by strict probabilistic principles.

Many people feel uncomfortable with contentions of this kind, and I must confess to having some unease with it myself. Nonetheless, my personal prejudices are indeed to favour a viewpoint of this general nature, since it is hard to see how any line can be drawn to separate physical actions under mathematical control from those which might lie beyond it. In my own view, the unease that many readers may share with me on this issue partly arises from a very limited notion of what 'mathematical control' might entail. Part of the purpose of this book is to touch upon, and to reveal to the reader, some of the extraordinary richness, power, and beauty that can spring forth once the right mathematical notions are hit upon.

In the Mandelbrot set alone, as illustrated in Fig. 1.2, we can begin to catch a glimpse of the scope and beauty inherent in such things. But even these structures inhabit a very limited corner of mathematics as a whole, where behaviour is governed by strict computational control. Beyond this corner is an incredible potential richness. How do I really feel about the possibility that all my actions, and those of my friends, are ultimately governed by mathematical principles of this kind? I can live with that. I would, indeed, prefer to have these actions controlled by something residing in some such aspect of Plato's fabulous mathematical world than to have them be subject to the kind of simplistic base motives, such as pleasure-seeking, personal greed, or aggressive violence, that many would argue to be the implications of a strictly scientific standpoint.

Yet, I can well imagine that a good many readers will still have difficulty in accepting that all actions in the universe could be entirely subject to mathematical laws. Likewise, many might object to two other prejudices of mine that are implicit in Fig. 1.3. They might feel, for example, that I am taking too hard-boiled a scientific attitude by drawing my diagram in a way that implies that all of mentality has its roots in physicality. This is indeed a prejudice, for while it is true that we have no reasonable scientific evidence for the existence of 'minds' that do not have a physical basis, we cannot be completely sure. Moreover, many of a religious persuasion would argue strongly for the possibility of physically independent minds and might appeal to what they regard as powerful evidence of a different kind from that which is revealed by ordinary science.

A further prejudice of mine is reflected in the fact that in Fig. 1.3 I have represented the entire Platonic world to be within the compass of mentality. This is intended to indicate that—at least in principle—there are no mathematical truths that are beyond the scope of reason. Of course, there are mathematical statements (even straightforward arithmetical addition sums) that are so vastly complicated that no one could have the mental fortitude to carry out the necessary reasoning. However, such things would be *potentially* within the scope of (human) mentality and would be consistent with the meaning of Fig. 1.3 as I have intended to represent it. One must, nevertheless, consider that there might be other mathematical statements that lie outside even the potential compass of reason, and these would violate the intention behind Fig. 1.3. (This matter will be considered at greater length in §16.6, where its relation to Gödel's famous incompleteness theorem will be discussed.)[8]

In Fig. 1.4, as a concession to those who do not share all my personal prejudices on these matters, I have redrawn the connections between the three worlds in order to allow for all three of these possible violations of my prejudices. Accordingly, the possibility of physical action beyond the scope of mathematical control is now taken into account. The diagram also allows for the belief that there might be mentality that is not rooted in physical structures. Finally, it permits the existence of true mathematical assertions whose truth is in principle inaccessible to reason and insight.

This extended picture presents further potential mysteries that lie even beyond those which I have allowed for in my own preferred picture of the world, as depicted in Fig. 1.3. In my opinion, the more tightly organized scientific viewpoint of Fig. 1.3 has mysteries enough. These mysteries are not removed by passing to the more relaxed scheme of Fig. 1.4. For it

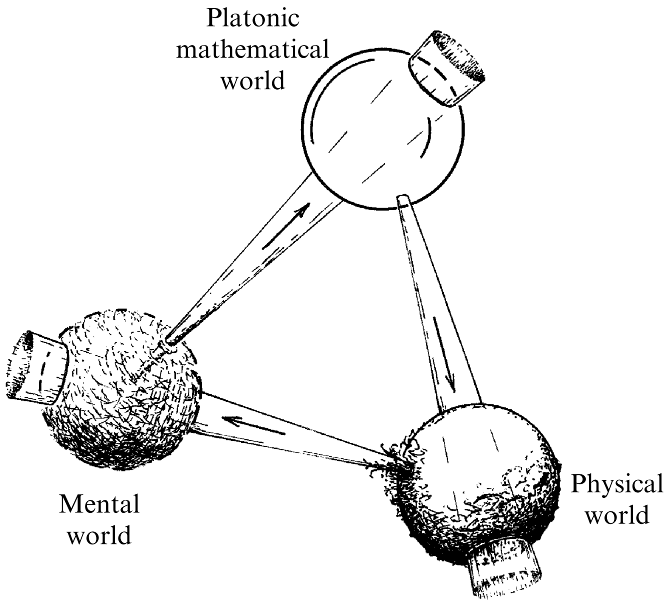


Fig. 1.4 A redrawing of Fig. 1.3 in which violations of three of the prejudices of the author are allowed for.

remains a deep puzzle why mathematical laws should apply to the world with such phenomenal precision. (We shall be glimpsing something of the extraordinary accuracy of the basic physical theories in §19.8, §26.7, and §27.13.) Moreover, it is not just the precision but also the subtle sophistication and mathematical beauty of these successful theories that is profoundly mysterious. There is also an undoubted deep mystery in how it can come to pass that appropriately organized physical material—and here I refer specifically to living human (or animal) brains—can somehow conjure up the mental quality of conscious awareness. Finally, there is also a mystery about how it is that we perceive mathematical truth. It is not just that our brains are programmed to 'calculate' in reliable ways. There is something much more profound than that in the insights that even the humblest among us possess when we appreciate, for example, the actual meanings of the terms 'zero', 'one', 'two', 'three', 'four', etc.[9]

Some of the issues that arise in connection with this third mystery will be our concern in the next chapter (and more explicitly in §§16.5,6) in relation to the notion of *mathematical proof*. But the main thrust of this book has to do with the first of these mysteries: the remarkable relationship between mathematics and the actual behaviour of the physical world. No proper appreciation of the extraordinary power of modern science can be achieved without at least some acquaintance with these mathematical ideas. No doubt, many readers may find themselves daunted by the prospect of having to come to terms with such mathematics in order to arrive at this appreciation. Yet, I have the optimistic belief that they may not find all these things to be so bad as they fear. Moreover, I hope that I may persuade many reader that, despite what she or he may have previously perceived, mathematics can be fun!

I shall not be especially concerned here with the second of the mysteries depicted in Figs. 1.3 and 1.4, namely the issue of how it is that mentality—most particularly conscious awareness—can come about in association with appropriate physical structures (although I shall touch upon this deep question in §34.7). There will be enough to keep us busy in exploring the physical universe and its associated mathematical laws. In addition, the issues concerning mentality are profoundly contentious, and it would distract from the purpose of this book if we were to get embroiled in them. Perhaps one comment will not be amiss here, however. This is that, in my own opinion, there is little chance that any deep understanding of the nature of the mind can come about without our first learning much more about the very basis of physical reality. As will become clear from the discussions that will be presented in later chapters, I believe that major revolutions are required in our physical understanding. Until these revolutions have come to pass, it is, in my view, greatly optimistic to expect that much real progress can be made in understanding the actual nature of mental processes.[10]

1.5 The Good, the True, and the Beautiful

In relation to this, there is a further set of issues raised by Figs. 1.3 and 1.4. I have taken Plato's notion of a 'world of ideal forms' only in the limited sense of mathematical forms. Mathematics is crucially concerned with the particular ideal of *Truth*. Plato himself would have insisted that there are two other fundamental absolute ideals, namely that of the *Beautiful* and of the *Good*. I am not at all averse to admitting to the existence of such ideals, and to allowing the Platonic world to be extended so as to contain absolutes of this nature.

Indeed, we shall later be encountering some of the remarkable interrelations between truth and beauty that both illuminate and confuse the issues of the discovery and acceptance of physical theories (see §§34.2,3,9 particularly; see also Fig. 34.1). Moreover, quite apart from the undoubted (though often ambiguous) role of beauty for the mathematics underlying the workings of the physical world, aesthetic criteria are fundamental to the development of mathematical ideas for their own sake, providing both the drive towards discovery and a powerful guide to truth. I would even surmise that an important element in the mathematician's common conviction that an external Platonic world actually has an existence independent of ourselves comes from the extraordinary unexpected hidden beauty that the ideas themselves so frequently reveal.

Of less obvious relevance here—but of clear importance in the broader context—is the question of an absolute ideal of morality: what is good and what is bad, and how do our minds perceive these values? Morality has a profound connection with the mental world, since it is so intimately related to the values assigned by conscious beings and, more importantly, to the very presence of consciousness itself. It is hard to see what morality might mean in the absence of sentient beings. As science and technology progress, an understanding of the physical circumstances under which mentality is manifested becomes more and more relevant. I believe that it is more important than ever, in today's technological culture, that scientific questions should not be divorced from their moral implications. But these issues would take us too far afield from the immediate scope of this book. We need to address the question of separating true from false before we can adequately attempt to apply such understanding to separate good from bad.

There is, finally, a further mystery concerning Fig. 1.3, which I have left to the last. I have deliberately drawn the figure so as to illustrate a paradox. How can it be that, in accordance with my own prejudices, each world appears to encompass the next one in its entirety? I do not regard this issue as a reason for abandoning my prejudices, but merely for demonstrating the presence of an even deeper mystery that transcends those which I have been pointing to above. There may be a sense in which the three worlds are not separate at all, but merely reflect, individually, aspects of a deeper truth about the world as a whole of which we have little conception at the present time. We have a long way to go before such matters can be properly illuminated.

I have allowed myself to stray too much from the issues that will concern us here. The main purpose of this chapter has been to emphasize the central importance that mathematics has in science, both ancient and modern. Let us now take a glimpse into Plato's world—at least into a relatively small but important part of that world, of particular relevance to the nature of physical reality.

Notes

Section 1.2

1.1. Unfortunately, almost nothing reliable is known about Pythagoras, his life, his followers, or of their work, apart from their very existence and the recognition by Pythagoras of the role of simple ratios in musical harmony. See Burkert (1972). Yet much of great importance is commonly attributed to the Pythagoreans. Accordingly, I shall use the term 'Pythagorean' simply as a label, with no implication intended as to historical accuracy.

1.2. This is the pure 'diatonic scale' in which the frequencies (in inverse proportion to the lengths of the vibrating elements) are in the ratios 24 : 27 : 30 : 36 : 40 : 45 : 48, giving many instances of simple ratios, which underlie harmonies that are pleasing to the ear. The 'white notes' of a modern piano are tuned (according to a compromise between Pythagorean purity of harmony and the facility of key changes) as approximations to these Pythagorean ratios, according to the *equal temperament* scale, with relative frequencies $1{:}\alpha^2{:}\alpha^4{:}\alpha^5{:}\alpha^7{:}\alpha^9{:}\alpha^{11}{:}\alpha^{12}$, where $\alpha = \sqrt[12]{2} = 1.05946\ldots$. (Note: α^5 means the fifth power of α, i.e. $\alpha \times \alpha \times \alpha \times \alpha \times \alpha$. The quantity $\sqrt[12]{2}$ is the twelfth root of 2, which is the number whose twelfth power is 2, i.e. $2^{1/12}$, so that $\alpha^{12} = 2$. See Note 1.3 and §5.2.)

Section 1.3

1.3. Recall from Note 1.2 that the nth power of a number is that number multiplied by itself n times. Thus, the third power of 5 is 125, written $5^3 = 125$; the fourth power of 3 is 81, written $3^4 = 81$; etc.

1.4. In fact, while Wiles was trying to fix a 'gap' in his proof of Fermat's Last Theorem which had become apparent after his initial presentation at Cambridge in June 1993, a rumour spread through the mathematical community that the mathematician Noam Elkies had found a counter-example to Fermat's assertion. Earlier, in 1988, Elkies had found a counter-example to Euler's conjecture—that there are no positive solutions to the equation $x^4 + y^4 + z^4 = w^4$—thereby proving it false. It was not implausible, therefore, that he had proved that Fermat's assertion also was false. However, the e-mail that started the rumour was dated 1 April and was revealed to be a spoof perpetrated by Henri Darmon; see Singh (1997), p. 293.

1.5. Technically it is a Π_1-sentence; see §16.6.

1.6. I realize that, in a sense, I am falling into my own trap by making such an assertion. The issue is not really whether the mathematicians taking such an extreme subjective view happen to constitute a tiny minority or not (and I have certainly not conducted a trustworthy survey among mathematicians on this point); the issue is whether such an extreme position is actually to be taken seriously. I leave it to the reader to judge.

1.7. Some readers may be aware of the results of Gödel and Cohen that the axiom of choice is independent of the more basic standard axioms of set theory (the Zermelo–Frankel axiom system). It should be made clear that the Gödel–Cohen argument does not in itself establish that the axiom of choice will never be settled one way or the other. This kind of point is stressed, for example, in the final section of Paul Cohen's book (Cohen 1966, Chap. 14, §13), except that, there, Cohen is more explicitly concerned with the *continuum hypothesis* than the axiom of choice; see §16.5.

Section 1.4

1.8. There is perhaps an irony here that a fully fledged anti-Platonist, who believes that mathematics is 'all in the mind' must also believe—so it seems—that there are no true mathematical statements that are in principle beyond reason. For example, if Fermat's Last Theorem had been inaccessible (in principle) to reason, then this anti-Platonist view would allow no validity either to its truth or to its falsity, such validity coming only through the mental act of perceiving some proof or disproof.

1.9. See e.g. Penrose (1997b).

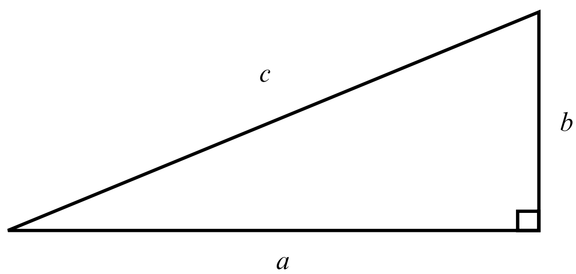
1.10. My own views on the kind of change in our physical world-view that will be needed in order that conscious mentality may be accommodated are expressed in Penrose (1989, 1994, 1996,1997).

2
An ancient theorem and a modern question

2.1 The Pythagorean theorem

LET us consider the issue of geometry. What, indeed, are the different 'kinds of geometry' that were alluded to in the last chapter? To lead up to this issue, we shall return to our encounter with Pythagoras and consider that famous theorem that bears his name:[1] for any right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides (Fig. 2.1). What reasons do we have for believing that this assertion is true? How, indeed, do we 'prove' the Pythagorean theorem? Many arguments are known. I wish to consider two such, chosen for their particular transparency, each of which has a different emphasis.

For the first, consider the pattern illustrated in Fig. 2.2. It is composed entirely of squares of two different sizes. It may be regarded as 'obvious' that this pattern can be continued indefinitely and that the entire plane is thereby covered in this regular repeating way, without gaps or overlaps, by squares of these two sizes. The repeating nature of this pattern is made manifest by the fact that if we mark the centres of the larger squares, they form the vertices of another system of squares, of a somewhat greater size than either, but tilted at an angle to the original ones (Fig. 2.3) and which alone will cover the entire plane. Each of these tilted squares is marked in exactly the same way, so that the markings on these squares fit together to



$a^2 + b^2 = c^2$

Fig. 2.1 The Pythagorean theorem: for any right-angled triangle, the squared length of the hypotenuse c is the sum of the squared lengths of the other two sides a and b.