Homework 1, Advanced Calculus
due 2/1/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Let $x, y \in \mathbb{R}^n$ be linearly independent vectors and consider the quadrilateral given by the 4 points $0, x, y, x+y$.

(i) Prove that those four points lie in a 2-dimensional plane and form a parallelogram (opposite sides are parallel and of equal length).

(ii) Proof the *parallelogram identity*: the sum of the squares of the lengths of the four sides are equal to the sum of the squares of the lengths of the two diagonals in the parallelogram (draw a picture, translate this sentence into a mathematical formula, and verify the formula). *Hint:* try to calculate $||x+y||^2 + ||x-y||^2 = \dots$ using $||x||^2 = < x, x >$ and the various properties of $< , >$.

Problem 2. Show that the "closed" ball $\overline{B}_r(x_0) = \{x \in \mathbb{R}^n \,;\, ||x - x_0|| \leq r\}$ in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ of radius r is closed (in the sense that its complement is open).

Problem 3. Verify the following:

(i) The union of any number of open sets is open;

(ii) The intersection of finitely many open sets is open (can you give an example in $\mathbb{R}$ for which the intersection of infinitely many open sets is closed?)

Write down the corresponding statements for closed sets and proof them (recall that a set A is closed, if its complement $\mathbb{R}^n \setminus A$ is open; what is the complement of a union of sets? Of the intersection of sets?)

Problem 4. Let $\{x_k\}$ and $\{y_k\}$ be sequences in $\mathbb{R}^n$ which converge to L_1 and L_2 respectively. Verify the limit rules:

(i) The sum/difference sequence $\{x_k \pm y_k\}$ converges to $L_1 \pm L_2$.

(ii) The product sequence $\{x_k y_k\}$ converges to $L_1 L_2$.

Problem 5. Determine the limits of the following sequences as $k \to \infty$:

(i) $x_k = (\cos(1/k), e^{-k})$ in $\mathbb{R}^2$

(ii) $x_k = (1/k, \log(1 + 1/k), 1/k + 7)$ in $\mathbb{R}^3$

(iii) $x_k = (\frac{k}{k+1}, \frac{k}{k+2}, \dots, \frac{k}{k+n})$ in $\mathbb{R}^n$

Problem 6. Verify that the n-dimensional sphere

$$S_r^n(x_0) = \{x \in \mathbb{R}^{n+1} \,;\, ||x - x_0|| = r\}$$

of radius $r > 0$ and center $x_0 \in \mathbb{R}^{n+1}$ is a closed (in fact compact) subset of $\mathbb{R}^{n+1}$.

Problem 7. Calculate the closures of the sets $(0,1] \cup \{2\}$, $\mathbb{R}^2 \setminus \{0\}$, and $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,;\, x_3^2 > 1\}$ and provide a proof of your answer.

Problem 8. Verify that the closed interval $[a, b] \subset \mathbb{R}$ is compact.